Focusing of light

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Tight focusing of light

- Microscopy
- Laser micromachining and microprocessing
- Optical data storage
- Optical lithography
- Laser trapping and cooling
- Physics of light/atom interactions
- Cavity QED
Overview

• Complete spherical focusing
• Bessel beams
• Gaussian beams
• High numerical aperture focusing
• Pupil masks (super-resolving filters)
• Polarization in focusing
• 4Pi geometry
• Moments
Complete spherical focusing
Complete spherical $(4\pi)$ scalar focusing

From scalar form of Richards and Wolf

$$I(0, \nu) = \left[ \frac{\sin(kr)}{kr} \right]^2$$

Same as field of a point source and a point sink

Imaging in high-aperture optical systems

C. J. R. Sheppard and H. J. Matthews

A plane-polarized wave after focusing

- \( \mathbf{p}_x \) electric dipole along \( x \) axis
- \( \mathbf{m}_y \) magnetic dipole along \( y \) axis
- \( C \) is nearly linear polarization
- \( A \) is polarization singularity of order 2

Richard & Wolf (Ignatovsky) polarization

A plane polarized wave after focusing:
Polarization on reference sphere

direction of propagation

- Polarization is same as that of
  - \( \mathbf{p}_x \) (electric dipole along \( x \) axis)
  - \( \mathbf{m}_y \) (magnetic dipole along \( y \) axis)
- \( C \) is nearly linear polarization
- Richards & Wolf polarization
Bessel beams
Bessel Beam

Annular mask
(Linfoot & Wolf, 1953)

Axicon
(McLeod, 1954)

Diffractive axicon
(Dyson, 1958)
Bessel beams

$J_0$ beam propagates without spreading:

The radial distribution of amplitude for a δ ring is given by a zero-order Bessel function in any plane (in the region of validity) perpendicular to the optic axis. That this is so is not surprising because such a wave is the circularly symmetric mode of free space. We are acquainted with modes of this form in circular waveguides, and we can consider free space as the limiting case of a waveguide of very large diameter. Such an overmoded waveguide has an infinity of circularly symmetric modes, that is the scale of the Bessel functions may be chosen at will. A wave with zero-order Bessel-function radial distribution propagates without change.


Also higher order beams $J_n(ν) \exp (iνφ)$ with a phase singularity (vortex) Sometimes called “diffraction-free” beams

Bessel-Gauss beam

\[ U = \frac{\exp(ikz)}{1 + i \tan \psi} \exp \left[ -\frac{v^2a^2}{2(1 + i \tan \psi)} \right] \times \exp \left( -\frac{i}{2a^2} \frac{\tan \psi}{1 + i \tan \psi} \right) J_0 \left( \frac{v}{1 + i \tan \psi} \right) \]

\[ \tan \psi = \frac{z}{z_0} \]

\[ \frac{2\psi}{\pi} = \text{fractional Fourier (Hankel) order} \]


Bessel-Gauss beam

annular beam

\[ a = 0.1 \]

Non-paraxial Bessel beam (plane polarized illumination)

Time-averaged electric energy density:

\[
\langle w_e \rangle = C\{[J_0(v)]^2 + 2 \tan^2 \frac{\alpha}{2} [J_1(v)]^2 + \tan^4 \frac{\alpha}{2} [J_2(v)]^2 + 2 \cos 2\phi \tan^2 \frac{\alpha}{2} ([J_1(v)]^2 + J_0(v)J_2(v))\}
\]

(2)

Fig. 2 Contours of constant time-averaged electric energy density in the focal plane of an annular lens or mirror system
Semiangle of convergence is (a) \( \pi/6 \), (b) \( \pi/3 \), (c) \( \pi/2 \), (d) \( 2\pi/3 \)
$x$-polarized illumination:
Intensity along $x, y$ axes: $\text{NA} = 1.4$

Circular pupil

Annular pupil

Broad along $x$ axis because of longitudinal field component

Annular pupils, radial polarization, and superresolution

Colin J. R. Shepard and Aniruddh Choudhury
Radial polarization TM0
Annulus at high NA:
circular polarization or TM0

- High NA, circular polarized annulus: ~same width as Airy
- High NA, TM0 annulus (radially polarized illumination):
similar to paraxial (Dorn, Quabis and Leuchs, *PRL* 91, 233901, 2003)
Widths of Bessel beams

<table>
<thead>
<tr>
<th>System</th>
<th>FWHM (ν)</th>
<th>FWHM (nm) for λ = 488 nm (1.4 NA)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Paraxial Airy disk theory</td>
<td>3.24</td>
<td>180</td>
</tr>
<tr>
<td>Paraxial annulus</td>
<td>2.26</td>
<td>125</td>
</tr>
<tr>
<td>Circular polarized</td>
<td>3.68</td>
<td>204</td>
</tr>
<tr>
<td>Circular polarized annulus</td>
<td>3.30</td>
<td>185</td>
</tr>
<tr>
<td>TM annulus</td>
<td>2.36</td>
<td>131</td>
</tr>
<tr>
<td>Confocal, two circular pupils</td>
<td>2.63</td>
<td>146</td>
</tr>
<tr>
<td>Confocal, one circularly polarized annulus</td>
<td>2.44</td>
<td>135</td>
</tr>
<tr>
<td>Confocal, one TM annulus</td>
<td>2.02</td>
<td>112</td>
</tr>
</tbody>
</table>

*Measured in terms of ν and in nanometers for λ = 488 nm. TM, transverse-magnetic.

Solid immersion lens

<table>
<thead>
<tr>
<th>Media</th>
<th>Refractive Indices</th>
<th>NA</th>
<th>FWHM (ν)</th>
<th>FWHM (nm)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Diamond–air (λ = 488 μm)</td>
<td>2.38/1.00</td>
<td>2.21</td>
<td>3.04</td>
<td>107</td>
</tr>
<tr>
<td>Diamond–water (λ = 488 μm)</td>
<td>2.38/1.33</td>
<td>2.21</td>
<td>2.84</td>
<td>100</td>
</tr>
<tr>
<td>As2Se3–air (λ = 488 μm)</td>
<td>2.77/1.00</td>
<td>2.57</td>
<td>3.12</td>
<td>94</td>
</tr>
<tr>
<td>As2Se3–water (λ = 488 μm)</td>
<td>2.77/1.33</td>
<td>2.57</td>
<td>2.96</td>
<td>89</td>
</tr>
<tr>
<td>Silicon–air (λ = 1.3 μm)</td>
<td>3.5/1.00</td>
<td>3.25</td>
<td>3.20</td>
<td>204</td>
</tr>
<tr>
<td>Silicon–water (λ = 1.3 μm)</td>
<td>3.5/1.23</td>
<td>3.25</td>
<td>3.30</td>
<td>198</td>
</tr>
</tbody>
</table>

*With a cone of semiangle 68°, corresponding to an NA in air of 0.927.
Gaussian beams
Highly convergent Gaussian beams, Complex source/sink theory: Electric + magnetic dipoles at complex location

Complex source point model:
Amplitude is the same as that for a source at the point \( z = iz_0 \), where \( z_0 \) is the confocal parameter

Deschamps *El. Lett.* 7, 684 (1971)

\[
E = \left\{ g(kr) + \left[ f(kr) - g(kr) \right] \frac{x^2}{r^2} + \frac{i}{2} f(kr)x \right\} i + \left[ f(kr) - g(kr) \right] \frac{xy}{r^2} j + \left[ f(kr) - g(kr) \right] \frac{xz}{r^2} - \frac{i}{2} f(kr)x \right\} \mathbf{k},
\]

\[
R = (x^2 + y^2 + z^2 - 2izz_0 - z_0^2)^{1/2}.
\]

\[
f(kr) = j_0(kr) + j_2(kr) = -3 \left[ \frac{\cos kr}{(kr)^2} - \frac{\sin kr}{(kr)^3} \right],
\]

\[
g(kr) = j_0(kr) - \frac{1}{2} j_2(kr) = \frac{3}{2} \left[ \frac{\sin kr}{kr} + \frac{\cos kr}{(kr)^2} - \frac{\sin kr}{(kr)^3} \right],
\]
Intensity in waist, $\text{LP}_{01}$

Time-averaged electric energy density: 

$$W_E = \frac{\epsilon}{2} \langle E^2 \rangle$$

- Double-spot
- Caused by magnetic dipole component
Intensity and phase along axis, $LP_{01}$

Gouy phase shift
Far field

Electric field:

\[ E = -\frac{3i}{4kr} \exp(ikr) \exp(kz_0 \cos \theta) \left[ (1 + \cos \theta \right. \]

\[ - \sin^2 \theta \cos^2 \phi i - \sin^2 \theta \sin \phi \cos \phi j \]

\[ - \sin \theta(1 + \cos \theta) \cos \phi k \], \]

Amplitude:

\[ a(\theta) = \frac{1 + \cos \theta}{2} \exp[-kz_0(1 - \cos \theta)] \]

\[ = \cos^2 \frac{\theta}{2} \exp \left( -2kz_0 \sin^2 \frac{\theta}{2} \right). \]

• axially symmetric

• more directional than the scalar case

Radiation pattern in far field

Fig. 7. Radiation pattern showing the normalized far-field intensity for different values of $kz_0$.

- Directional even for $kz_0 = 0$
Complex source/sink Gaussian beam, \( \text{TM}_{01} \) and \( \text{TE}_{01} \) modes

Transverse magnetic (axial electric dipole, radial illumination):

\[
E = [f(kR) - g(kR)] \frac{(z - iz_0)(xi + yj)}{R^2} + \left[ f(kR) \frac{(z - iz_0)^2}{R^2} + g(kR) \frac{(x^2 + y^2)}{R^2} \right] k,
\]

After focusing, not radial, as axial component

Transverse electric (axial magnetic dipole, azimuthal):

\[
E = -\frac{i}{2} f(kR)(kyi - kxj), \quad f(kr) = j_0(kr) + j_2(kr) = -3 \left[ \frac{\cos kr}{(kr)^2} - \frac{\sin kr}{(kr)^3} \right],
\]

\[
g(kr) = j_0(kr) - \frac{1}{2} j_2(kr) = \frac{3}{2} \left[ \frac{\sin kr}{kr} + \frac{\cos kr}{(kr)^2} - \frac{\sin kr}{(kr)^3} \right],
\]

• Surface-emitting semiconductor lasers
• also in gas, solid state and dye lasers
• components of \( \text{TEM}_{01}^* \) (doughnut mode)
Intensity in waist, TM$_{01}$ and TE$_{01}$ modes

Transverse magnetic (axial electric dipole)

Transverse electric (axial magnetic dipole)

non-zero on axis (longitudinal field)

Radial + longitudinal component

zero on axis

Azimuthal

C. J. R. Sheppard  S. Saghaei
1999 / Vol. 24, No. 22 / OPTICS LETTERS 1543
Highly convergent focusing
Model for focusing by high numerical aperture lens (Debye approximation)

Front focal plane

Black Box

\[ E_1(\rho, \phi) \]

\[ E(r) \]

\[ \alpha \]

Equivalent refractive locus (sphere for aplanatic system)
Angular spectrum of plane waves

\[
\begin{align*}
\epsilon_x(P) &= -iA(I_0 + I_2 \cos 2\phi_P), \\
\epsilon_y(P) &= -iAI_2 \sin 2\phi_P, \\
\epsilon_z(P) &= -2AI_1 \cos \phi_P,
\end{align*}
\]

where

\[
\begin{align*}
I_0 &= I_0(kr_P, \theta_P, \alpha) = \int_0^\alpha \cos^\frac{1}{2} \theta \sin \theta (1 + \cos \theta) J_0(kr_P \sin \theta \sin \theta_P) e^{ikr_P \cos \theta \cos \theta_P} d\theta, \\
I_1 &= I_1(kr_P, \theta_P, \alpha) = \int_0^\alpha \cos^\frac{1}{2} \theta \sin^2 \theta J_1(kr_P \sin \theta \sin \theta_P) e^{ikr_P \cos \theta \cos \theta_P} d\theta, \\
I_2 &= I_2(kr_P, \theta_P, \alpha) = \int_0^\alpha \cos^\frac{1}{2} \theta \sin \theta (1 - \cos \theta) J_2(kr_P \sin \theta \sin \theta_P) e^{ikr_P \cos \theta \cos \theta_P} d\theta.
\end{align*}
\]

Aplanatic factor

\[I_2: \text{cross-polarization component (}\epsilon_y\text{)}\]

\[I_1: \text{longitudinally-polarized component}\]
Focal field as integral over angular spectrum:

$$I_3: \text{cross-polarization component}$$

$$I_1: \text{longitudinally-polarized component}$$
Pupil masks
Performance parameters, paraxial

- Calculate performance directly from pupil

For real-valued pupils:

\[ U(v, u) = 2 \int_0^1 P(\rho)J_0(\nu \rho)\exp(j\nu \rho^2/2)\rho d\rho. \]

\[ t = \rho^2; \]

\[ I_n = \int_0^1 Q(t)t^n dt \quad \text{Moments of pupil} \]

\[ \bar{t} = I_1/I_0, \quad \text{Centre of Gravity} \]

\[ \bar{t}^2 = I_2/I_0, \quad \text{Radius of Gyration squared} \]

\[ G_T = 2\bar{t}, \quad \text{Transverse gain} \]

\[ G_A = 12(\bar{t}^2 - \bar{t}^2). \quad \text{Axial gain (Radius of Gyration}^2 \text{about Centre of Gravity}) \]

By analogy with earlier work\(^9\) we also introduce the following properties. The Strehl ratio \( S \) is the ratio of the intensity at the focal point to that for an unobstructed pupil. Hence

\[ S = I_0^2/(P_{\text{max}})^2. \quad (19) \]

The energy in the diffraction pattern compared with that in the unobstructed case is given by

\[ E = \int_0^1 Q^2(t)dt/(P_{\text{max}})^2, \quad (20) \]

and finally the ratio \( S/E \) is a measure of the intensity of the focus compared with the total power:

\[ F = S/E = I_0^2 \int_0^1 Q^2(t)dt. \quad (21) \]

Thus \( S, E, \) and \( F \) are all unity for the unobstructed pupil.

Axial behavior of pupil-plane filters

C. J. R. Sheppard and Z. S. Hegedus

Division of Applied Physics, Commonwealth Scientific and Industrial Research Organization, National Measurement Laboratory, Lindfield, Australia 2070

High numerical aperture scalar systems

\[ U(\rho, z) = -ikf \int_0^\pi P(\theta)J_0(k\rho \sin \theta) \exp(ikz \cos \theta) \sin \theta \ d\theta \]

\[ c = \cos \theta \]

\[ F = \frac{I_0^2}{2\int_{-1}^1 |Q(c)|^2 \ dc} \]

\[ F_I = \frac{2I_0^2}{\int_0^1 |Q(c)|^2 \ dc} \]

Total power or integrated intensity in axial sidelobes

\[ G_T = \frac{3}{2} \left( 1 - \frac{I_2}{I_0} \right) \]

\[ G_A = 3 \left[ \frac{I_2}{I_0} - \left( \frac{I_1}{I_0} \right)^2 \right] \]

\[ G_P = \frac{1}{3} \left( 2G_T + G_A \right) = 1 - \left( \frac{I_1}{I_0} \right)^2 \]

3D polar gain

Filter performance parameters for high-aperture focusing

Interpretation in terms of Moment of Inertia (2nd moment) of generalized 3D pupil (cap of sphere)
Vectorial electromagnetic case

Richards and Wolf (equivalent form):

\[ I_n = \int_{-1}^{1} Q(c) \left( \frac{1-c}{1+c} \right)^{n/2} J_n \left( k \rho \sqrt{1-c^2} \right) \exp(ikzc) dc. \quad c = \cos \theta \]

Aplanatic (sine condition):

\[ Q(c) = c^{1/2} (1+c) \]

\[ q_n = \int_{-1}^{1} Q(c)c^n dc \]

\[ G_x = \frac{3}{4} \left( \frac{10q_0q_1 - 3q_0q_2 - 3q_0^2 - 4q_i^2}{q_0^2} \right) \quad G_y = \frac{3}{4} \left( \frac{3q_0^2 - 2q_0q_1 - q_0q_2}{q_0^2} \right) \]

Circular polarized or unpolarized:

\[ G_T = \frac{3}{4} \left( \frac{4q_0q_1 - 2q_0q_2 - 2q_i^2}{q_0^2} \right) \quad G_A = 3 \left( \frac{q_0q_2 - q_1^2}{q_0^2} \right) \quad G_P = \frac{(G_x + G_y + G_A)}{3} \]

\[ G_P = \frac{2q_i(q_0 - q_i)}{q_0^2} \]

Only zero and first moment (Centre of Gravity)
Gains for electromagnetic case, plane polarized input

Filter performance parameters for vectorial high-aperture wave fields

Colin J. R. Sheppard and M. Martinez-Corral
Intensity at the focus

Mixed dipole apodization (p + m) gives greatest intensity at the focus

Figure 2. A polar plot of the illumination power incident on the focus O, normalized to unity for $\theta = 0$, for various conditions. In particular, $p + m$ refer to the case of an electric dipole field, oriented along the $x$ axis, and a magnetic dipole field, oriented along the $y$ axis (mixed-dipole field).

Optimal concentration of electromagnetic radiation

C. J. R. SHEPPARD and K. G. LARKIN

JOURNAL OF MODERN OPTICS, 1994, VOL. 41, NO. 7, 1495–1505
Electric dipole polarization
Electric dipole wave: Ratio of focal intensity to power input

Radial polarization (TM0): polarization on reference sphere

direction of propagation

red: electric field
blue: magnetic field
Radial polarization with phase mask

Figure 2 Schematic of the set-up. Radially polarized beam, phase-modulation optical element and focusing lens.

Figure 4 Contour plots for the electric and magnetic density distributions in the yz-plane after additional phase modulation. a, Total energy density distribution. b, Radial component. c, Longitudinal component. d, Magnetic energy density.

Electric dipole: Polarization on reference sphere

Mixed = ED + MD

red: electric field
blue: magnetic field

direction of propagation
Polarization on reference sphere: TE1, TM1
Polarization of input wave
Bessel beams: TE1 polarization
High NA: Intensity at the focus for different polarizations

Performance parameters for highly-focused electromagnetic waves

Colin J.R. Sheppard \textsuperscript{a,b,c}, Naveen K. Balla\textsuperscript{a,c}, Shakil Rehman\textsuperscript{a}

Optics Communications 282 (2009) 727–734
Gains for different polarizations
Area of focal spot

\[
\frac{\text{area}}{\lambda^2}
\]

\[\alpha \ (\text{radians})\]

- rad annulus rad dipole
- Helmholtz
- paraboloid
- mixed annulus
- mixed aplanatic

NA = 0.89
NA = 0.91
Focal volume

\[ \text{volume} = \frac{\text{NA}^3}{\lambda^3} \]

- rad dipole
- Helmholtz
- paraboloid
- aplanatic
- mixed
- TE1 smallest
- NA = 0.98

\[ \alpha \text{ (radians)} \]
Rotationally symmetric beams

• TM0 = radial polarized input (longitudinal field in focus)
• TE0 = azimuthal polarization

• x polarized + \( i \) y polarized = circular polarized
• TE1_x + \( i \) TE1_y = azimuthal polarization with a phase singularity (bright centre)
• ED_x + \( i \) ED_y = elliptical polarization with a phase singularity (bright centre) (ellipticity increases with angle from axis)
• (TM1_x + \( i \) TM1_y = radial polarization with a phase singularity)
• Same \( G_T \) as for average over \( \phi \)
Normalized width for rotationally symmetric

$\text{TE}_1 = \text{azimuthal polarization with phase singularity (vortex)}$
Bessel beams:
Transverse behaviour for rotationally symmetric (also average over φ)
Bessel beams for rotationally symmetric

Transverse gain

TE1 is narrowest

ED has weakest sidelobes

Rad has weakest sidelobes

NA = 0.83

Eccentricity

ED has weakest sidelobes

Rad has weakest sidelobes

$\alpha$, radians

$G_T$

$\varepsilon$

$\alpha$, radians
Points to note

• Focusing plane polarized light results in a large focal spot
• Focusing is improved using radially polarized illumination
  - Strong longitudinal field on axis
• Electric dipole polarization gives higher electric energy density at focus
• Transverse electric (TE1) polarization gives smallest central lobe
  (smaller than radially polarized for Bessel beam)
• TE1 is asymmetric: symmetric version is azimuthal polarization with a phase singularity (vortex)
4 Pi microscope (Hell)

Fig. 3 The cut-off of the three-dimensional fluorescence transfer function: (a) conventional, (b) confocal, (c) illuminated coherently from both sides.
Resolution in 4 Pi

• Axial resolution is improved
• Longitudinal field components from counter-propagating beams cancel out, so transverse resolution is also improved
Performance parameters for 4 Pi

\[ G_p = 1 \]

As \( G_T \) increases, \( G_A \) decreases

*Micron*, to be published
spherical spot
axial
transverse

4 Pi

Micron, to be published
4 Pi:
Need to match electric dipole polarization
Electric field in input plane
Table 1. Values of the parameters $F$, $G_T$ and $M$ for NA = 1.46.  
“% increase in resolution” is for 4Pi compared with the single lens case.

<table>
<thead>
<tr>
<th></th>
<th>$F$</th>
<th>$G_T$ (4Pi)</th>
<th>$M = F$ $G_T$ (4Pi)</th>
<th>$G_T$ (single lens)</th>
<th>% increase in resoln.</th>
</tr>
</thead>
<tbody>
<tr>
<td>aplanatic</td>
<td>0.746</td>
<td>0.713</td>
<td>0.532</td>
<td>0.573</td>
<td>10.4</td>
</tr>
<tr>
<td>mixed dipole</td>
<td>0.747</td>
<td>0.729</td>
<td>0.545</td>
<td>0.581</td>
<td>10.7</td>
</tr>
<tr>
<td>electric dipole</td>
<td>0.797</td>
<td>0.756</td>
<td>0.603</td>
<td>0.694</td>
<td>4.2</td>
</tr>
<tr>
<td>Helmholtz</td>
<td>0.463</td>
<td>0.995</td>
<td>0.461</td>
<td>0.680</td>
<td>17.3</td>
</tr>
<tr>
<td>parabolic mirror</td>
<td>0.697</td>
<td>0.833</td>
<td>0.581</td>
<td>0.630</td>
<td>13.0</td>
</tr>
<tr>
<td>$TE_1$</td>
<td>0.552</td>
<td>0.833</td>
<td>0.460</td>
<td>0.833</td>
<td>0</td>
</tr>
<tr>
<td>radial</td>
<td>0.612</td>
<td>1.032</td>
<td>0.632</td>
<td>0.821</td>
<td>10.8</td>
</tr>
</tbody>
</table>
Localization of a highly convergent wave can be specified by a variety of different parameters that can be expressed in terms of moments of the pupil. The simplest of these is the Strehl ratio. The transverse, axial and polar gains for arbitrary, real (but possibly negative) pupils can also be simply expressed in terms of the moments, even for the high-aperture case. They can also be explained in terms of the moments of the three-dimensional generalized pupil. For beams in systems without hard-edged apertures, the moments of the amplitude or intensity of the focused beam are alternative measures of localization. Uncertainty relationships exist between these moments in the near-field and the far-field.

Localization measures for high-aperture wavefields based on pupil moments

Colin J R Sheppard, Miguel A Alonso and Nicole J Moore

Localization in terms of pupil moments, $\mu_n$ (scalar)

Table 1. Summary of localization measures, both spatial and directional, including their definition in terms of moments of the pupil, any conditions upon the pupil function required for validity, their value for a Gaussian complex focus beam and their variation with respect to the pupil function.

<table>
<thead>
<tr>
<th>Measure</th>
<th>Definition</th>
<th>Conditions</th>
<th>Complex focus</th>
<th>Variation</th>
</tr>
</thead>
<tbody>
<tr>
<td>(F)</td>
<td>$\frac{</td>
<td>\omega</td>
<td>^2}{2\mu_{2,0}^2}$</td>
<td>(\mu_0)</td>
</tr>
<tr>
<td>(F_1)</td>
<td>$\frac{\mu_0^2}{2\mu_{2,-1}^2}$</td>
<td>(\mu_0)</td>
<td>$\lim_{k \to \infty} Q = 0$</td>
<td>not well defined</td>
</tr>
</tbody>
</table>

Focal curvature gain

<table>
<thead>
<tr>
<th>Measure</th>
<th>Definition</th>
<th>Conditions</th>
<th>Complex focus</th>
<th>Variation</th>
</tr>
</thead>
<tbody>
<tr>
<td>(G_A)</td>
<td>$3 \left( \frac{\mu_0^2}{\mu_0} - \frac{\mu_0^2}{\mu_0^2} \right)$</td>
<td>(\mu_0)</td>
<td>$\frac{3 - \frac{6}{k_0^2} L(kz_0)}{3L^2(kz_0)}$</td>
<td>$\frac{3}{\mu_0} \left( \frac{c - \mu_0}{c} \right)^2 - \frac{Q}{\mu_0}$</td>
</tr>
<tr>
<td>(G_T)</td>
<td>$\frac{3}{2} \left( 1 - \frac{\mu_0^2}{\mu_0^2} \right)$</td>
<td>(\mu_0)</td>
<td>$\frac{1}{k_0^2} L(kz_0)$</td>
<td>$\frac{3}{2\mu_0} \left( \frac{c^2 - \mu_0^2}{c} \right)$</td>
</tr>
<tr>
<td>(G_P)</td>
<td>$3 \left( 1 - \frac{\mu_0^2}{\mu_0^2} \right)$</td>
<td>(\mu_0)</td>
<td></td>
<td>$1 - L^2(kz_0)$</td>
</tr>
</tbody>
</table>

Intensity second moment

<table>
<thead>
<tr>
<th>Measure</th>
<th>Definition</th>
<th>Conditions</th>
<th>Complex focus</th>
<th>Variation</th>
</tr>
</thead>
<tbody>
<tr>
<td>(k^2 \Delta^2)</td>
<td>$\frac{\mu_{2,0}^2}{\mu_{2,0}}$</td>
<td>(\mu_{2,0})</td>
<td>$\lim_{k \to \infty} Q = 0$</td>
<td>not well defined</td>
</tr>
<tr>
<td>(k^2 \langle \rho^2 \rangle)</td>
<td>$\frac{1}{\mu_{2,0}} \left( \mu_{2,0}^2 - \mu_{2,0}^2 \right) + \mu_{2,0}^2 - \mu_{2,0}^2 - 2\mu_{2,0}^2 + 2\mu_{2,0}^2$</td>
<td>(\mu_{2,0})</td>
<td>$\lim_{k \to \infty} Q = 0$</td>
<td>not well defined</td>
</tr>
<tr>
<td>(L^2)</td>
<td>$1 + \frac{\mu_{2,0}^4 - \mu_{2,0}^2}{\mu_{2,0}^2}$</td>
<td>(\mu_{2,0})</td>
<td></td>
<td>$1 + k^2 L(2kz_0)$</td>
</tr>
</tbody>
</table>

Directional Aperture size

<table>
<thead>
<tr>
<th>Measure</th>
<th>Definition</th>
<th>Conditions</th>
<th>Complex focus</th>
<th>Variation</th>
</tr>
</thead>
<tbody>
<tr>
<td>(\alpha^*)</td>
<td>(\mu_{3,0}^2 - \mu_{2,0}^2 )</td>
<td>(\mu_{3,0})</td>
<td></td>
<td>not well defined</td>
</tr>
</tbody>
</table>

Pupil intensity moments

<table>
<thead>
<tr>
<th>Measure</th>
<th>Definition</th>
<th>Conditions</th>
<th>Complex focus</th>
<th>Variation</th>
</tr>
</thead>
<tbody>
<tr>
<td>(R_A^2)</td>
<td>$\frac{\mu_{2,0}^2}{\mu_{3,0}^2} - \frac{\mu_{2,0}^2}{\mu_{3,0}^2}$</td>
<td>(\mu_{3,0})</td>
<td>$\frac{1}{\mu_{3,0}} \left( \frac{c - \mu_{2,0}^2}{\mu_{2,0}^2} \right)^2 - R_A^2$</td>
<td>$Q$</td>
</tr>
<tr>
<td>(R_T^2)</td>
<td>$\frac{1}{2} \left( 1 - \frac{\mu_{2,0}^2}{\mu_{2,0}^2} \right)$</td>
<td>(\mu_{2,0})</td>
<td>$\frac{1}{2k_0} L(2kz_0)$</td>
<td>$\frac{1}{\mu_{2,0}^2} \left( \frac{c^2 - \mu_{2,0}^2}{\mu_{2,0}^2} \right) Q$</td>
</tr>
<tr>
<td>(R_P^2)</td>
<td>$1 - \frac{\mu_{2,0}^2}{\mu_{2,0}^2}$</td>
<td>(\mu_{2,0})</td>
<td></td>
<td>$1 - L^2(2kz_0)$</td>
</tr>
</tbody>
</table>

Propagation of second moments (scalar)

(Transverse width)$^2$

\[
k^2 \langle \rho^2 \rangle = \frac{1}{\mu_{2,-1}} \int \left| \frac{d}{dc} \left( \frac{Q(c)}{c} \exp(ikzc) \right) \right|^2 \frac{(1-c^2)}{c} dc
\]

\[
= \frac{1}{\mu_{2,-1}} \int \left[ \left| \frac{dQ}{dc} \right|^2 + \left| \frac{Q^2}{c^2} \right| + k^2 z^2 |Q|^2 - \frac{1}{2c} \frac{d}{dc} |Q|^2 
- i k z \left( Q^* \frac{dQ}{dc} - Q \frac{dQ^*}{dc} \right) \right] \frac{(1-c^2)}{c^3} dc.
\]

(Axial width)$^2$

\[
k^2 \langle z^2 \rangle = \frac{1}{\mu_{2,0}} \int \left| \frac{dQ}{dc} \right|^2 dc.
\]

\[
k^2 \langle z^2 \rangle = \frac{1}{\mu_{2,0}} \int_{\cos \alpha}^{1} dQ^* \frac{dQ}{dc} dc = -\frac{1}{\mu_{2,0}} \int_{\cos \alpha}^{1} Q^* \frac{d^2Q}{dc^2} dc.
\]

Axial position of centre of gravity:

\[
k \bar{z} = k \langle z \rangle = \frac{i}{2\mu_{2,0}} \int \left( Q^* \frac{dQ}{dc} - Q \frac{dQ^*}{dc} \right) dc.
\]

(Central second moment width)$^2$

\[
k^2 \Delta_z^2 = k^2 \langle (z - \bar{z})^2 \rangle = k^2 \left( \langle z^2 \rangle - \bar{z}^2 \right)
\]

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