INFLATION

Lecture I

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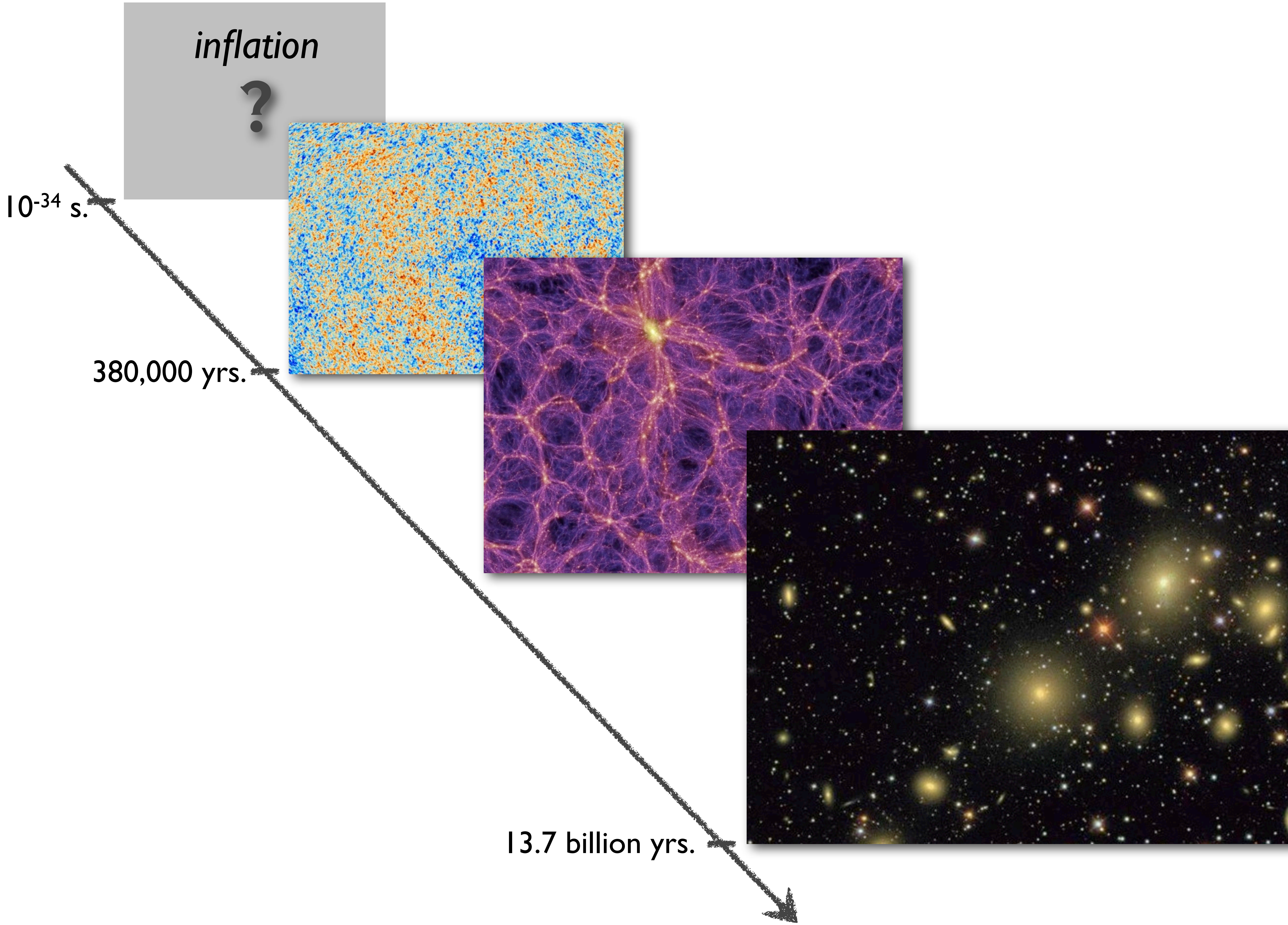
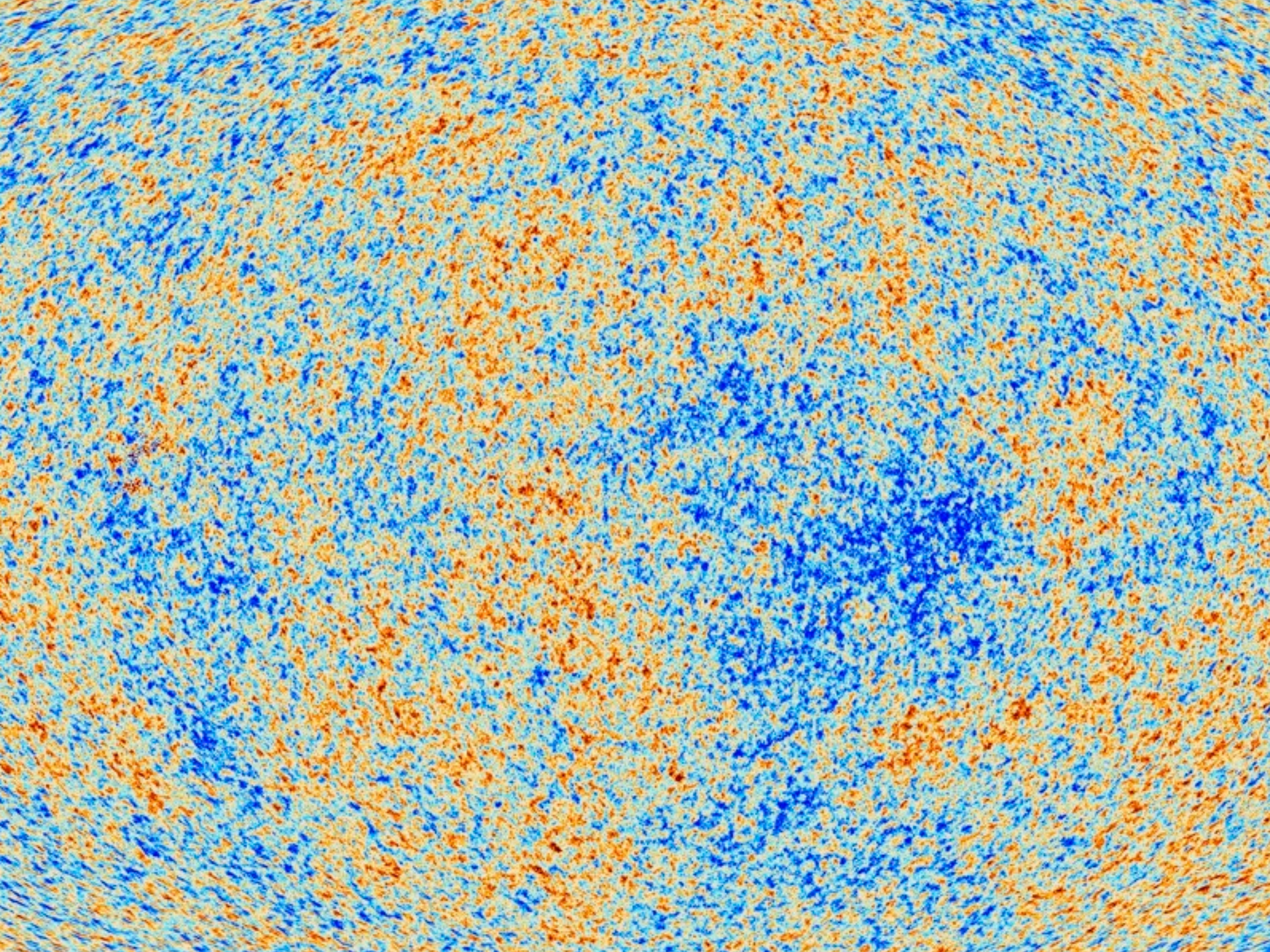
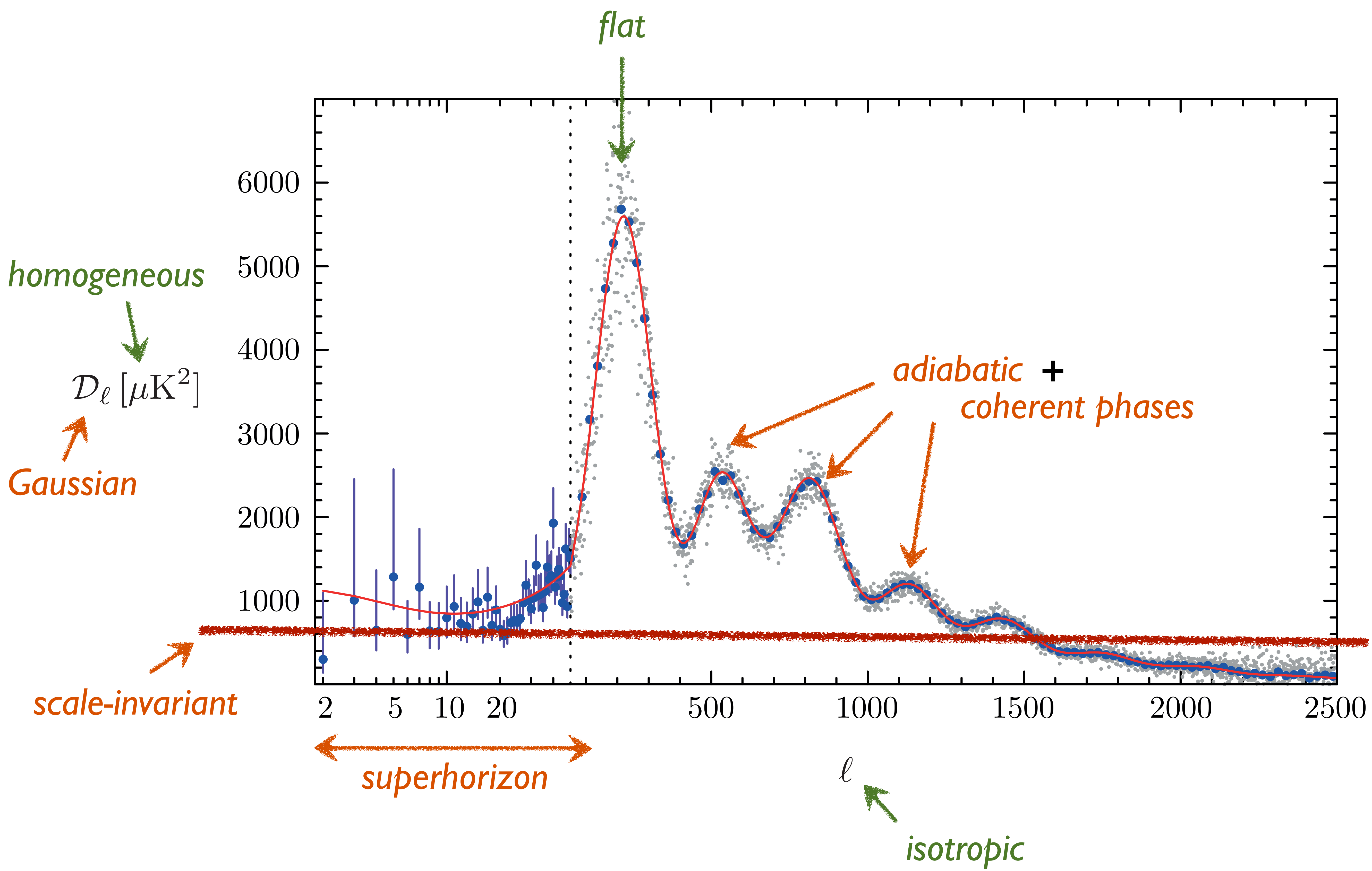

inflation
?
10-34 s.
380,000 yrs.
13.7 billion yrs.




flat
homogeneous
$\mathcal{D}_\ell\ [\mu\mathrm{K}^2]$
Gaussian
6000
5000
4000
3000
2000
1000
adiabatic +
coherent phases
scale-invariant
2
5
10
20
500
1000
1500
2000
2500
superhorizon
ℓ
isotropic

Why?

Outline

- Classical Dynamics of Inflation
    - The Horizon Problem
    - Slow-Roll Inflation
- Primordial Perturbations
    - Quantum Fluctuations in de Sitter
    - Curvature Perturbations
    - Gravitational Waves
- Advanced Topics
    - Non-Gaussianity
    - CMB Polarization
    - Inflation in Effective Field Theory
    - Inflation in String Theory

References

Notes: www.damtp.cam.ac.uk/user/db275/Cosmology.pdf

... /Inflation.pdf

Questions: dbaumann@damtp.cam.ac.uk

Please ask questions

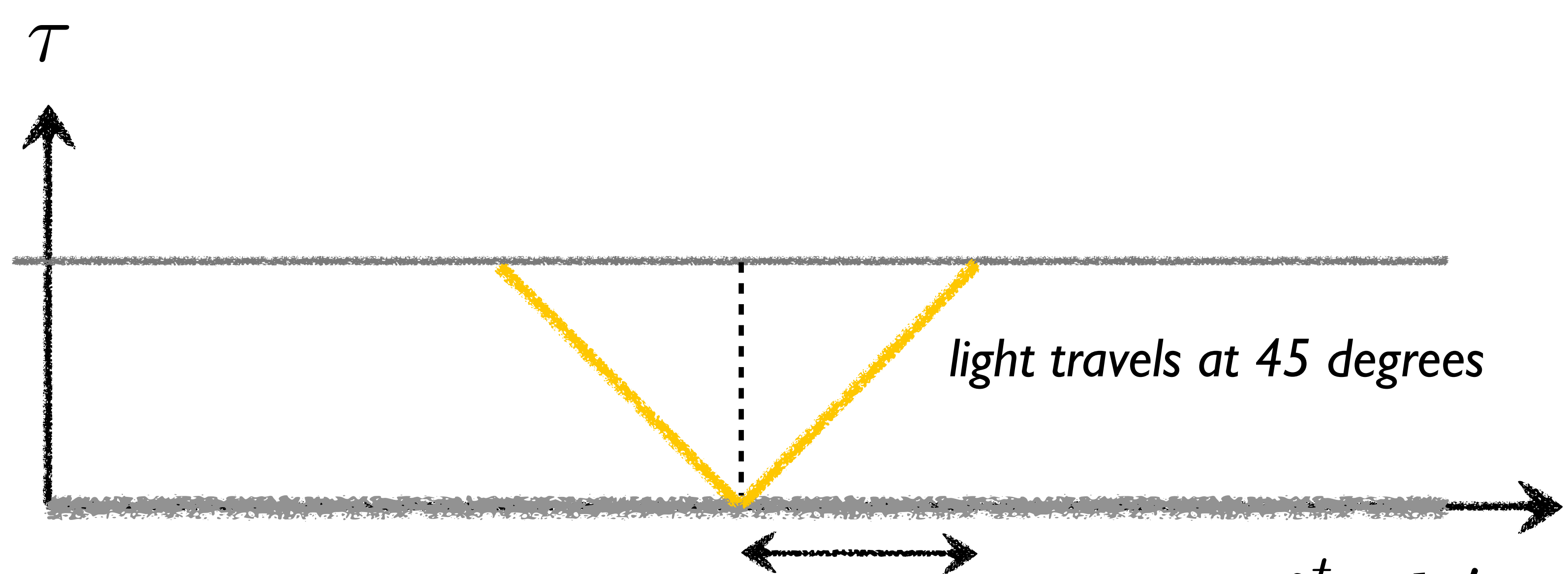
The Horizon Problem

Why is the CMB so uniform?

Let's talk about the history of the universe ...

... we will do it in conformal time $\quad \mathrm{d}\tau = \dfrac{\mathrm{d}t}{a(t)}$ *physical time* / *scale factor*

- *Light has travelled a finite distance since the Big Bang:*



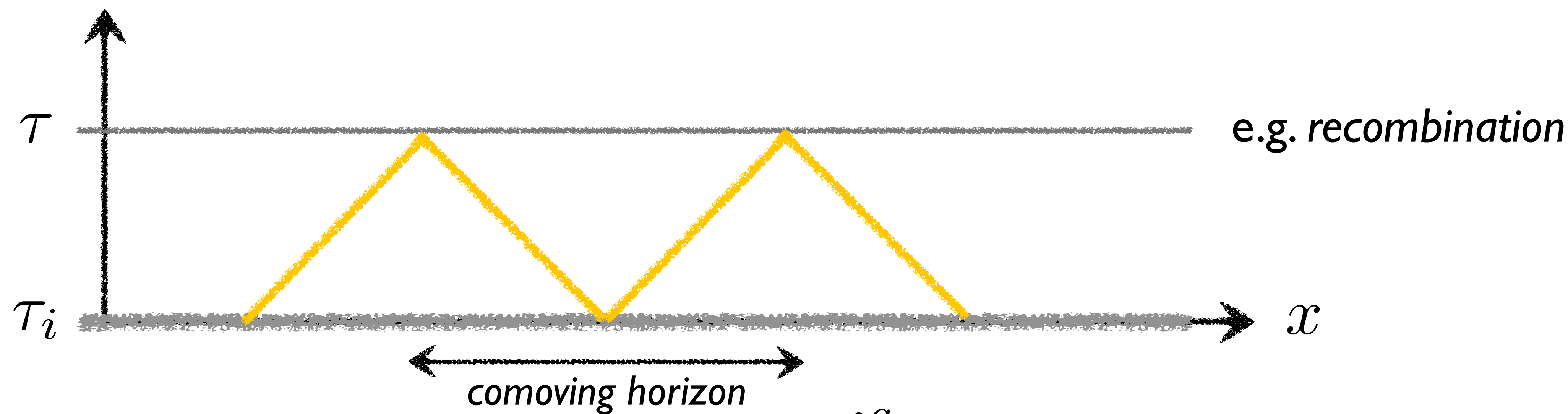
$$\Delta x = \Delta \tau = \int_0^t \frac{\mathrm{d}t'}{a(t')}$$

$$= \int_{a_i}^{a} (aH)^{-1} \, \mathrm{d} \ln a \qquad aH \equiv \dot{a}$$

comoving Hubble radius

$$(aH)^{-1} \propto a^{\frac{1}{2}(1+3w)} \qquad \textit{for a fluid}$$

- *Two points have never been in causal contact if their past light cones don't intersect:*



$$2\,\Delta\tau \;=\; 2\int_{a_i}^{a} (aH)^{-1}\,\mathrm{d}\ln a$$

- *Ordinary matter satisfies the SEC:* $1 + 3w > 0$

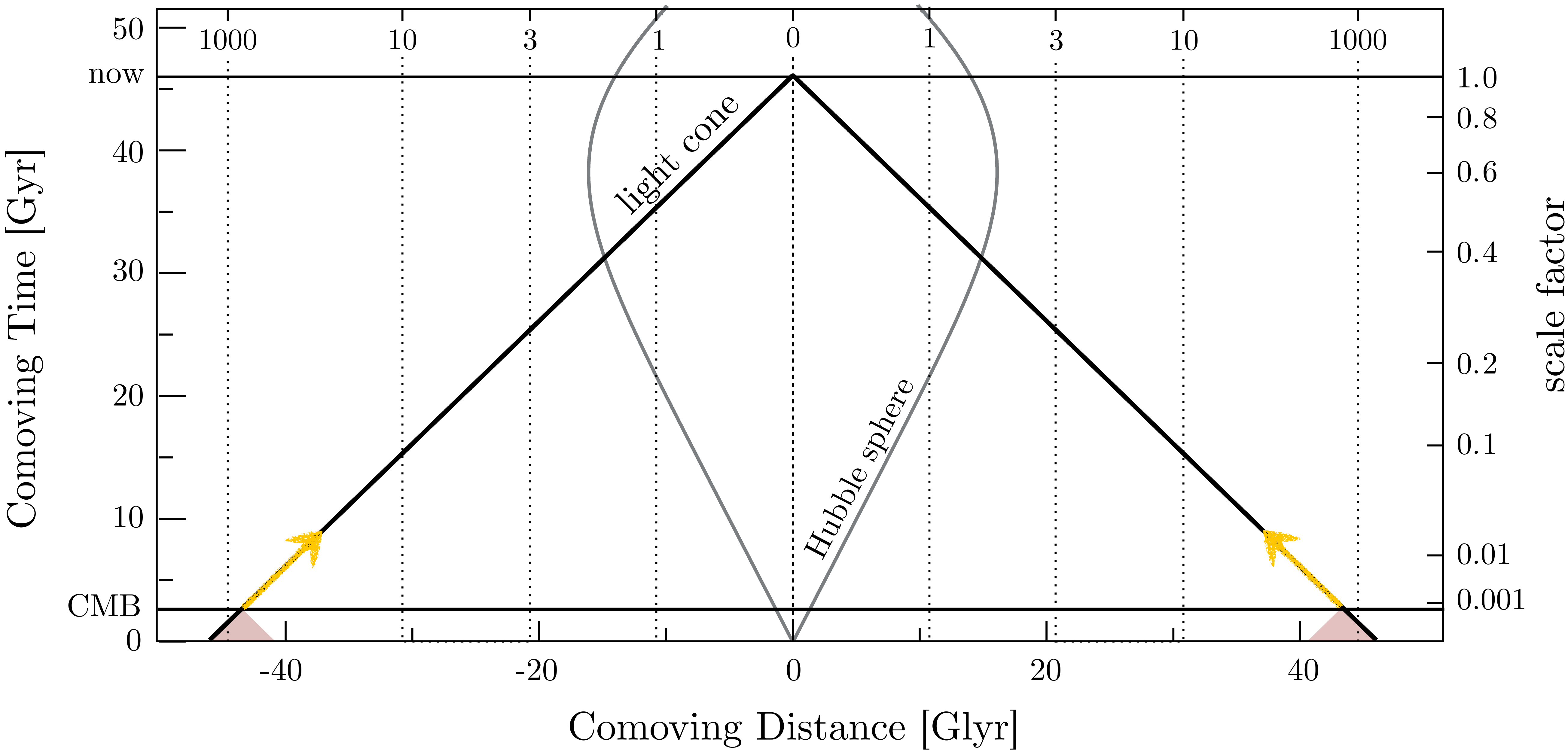
The comoving Hubble radius grows and the comoving horizon gets its largest contribution from late times :

$$\Delta\tau \propto \left[a^{\frac{1}{2}(1+3w)} - a_i^{\frac{1}{2}(1+3w)}\right] \qquad \textit{for a fluid}$$

$$0 \equiv \tau_i$$

In the standard Big Bang cosmology we therefore have:

$$\tau_0 - \tau_{\rm CMB} \gg \tau_{\rm CMB} - \tau_i$$



Exercise: Show that two points in the CMB have never been in causal contact if $\theta \gtrsim 2^\circ$

Inflation

A Shrinking Hubble Sphere

Maybe the early universe was not filled with ordinary matter ?

We need something that leads to a
shrinking Hubble sphere

$$\boxed{\frac{d}{dt}(aH)^{-1} < 0}$$

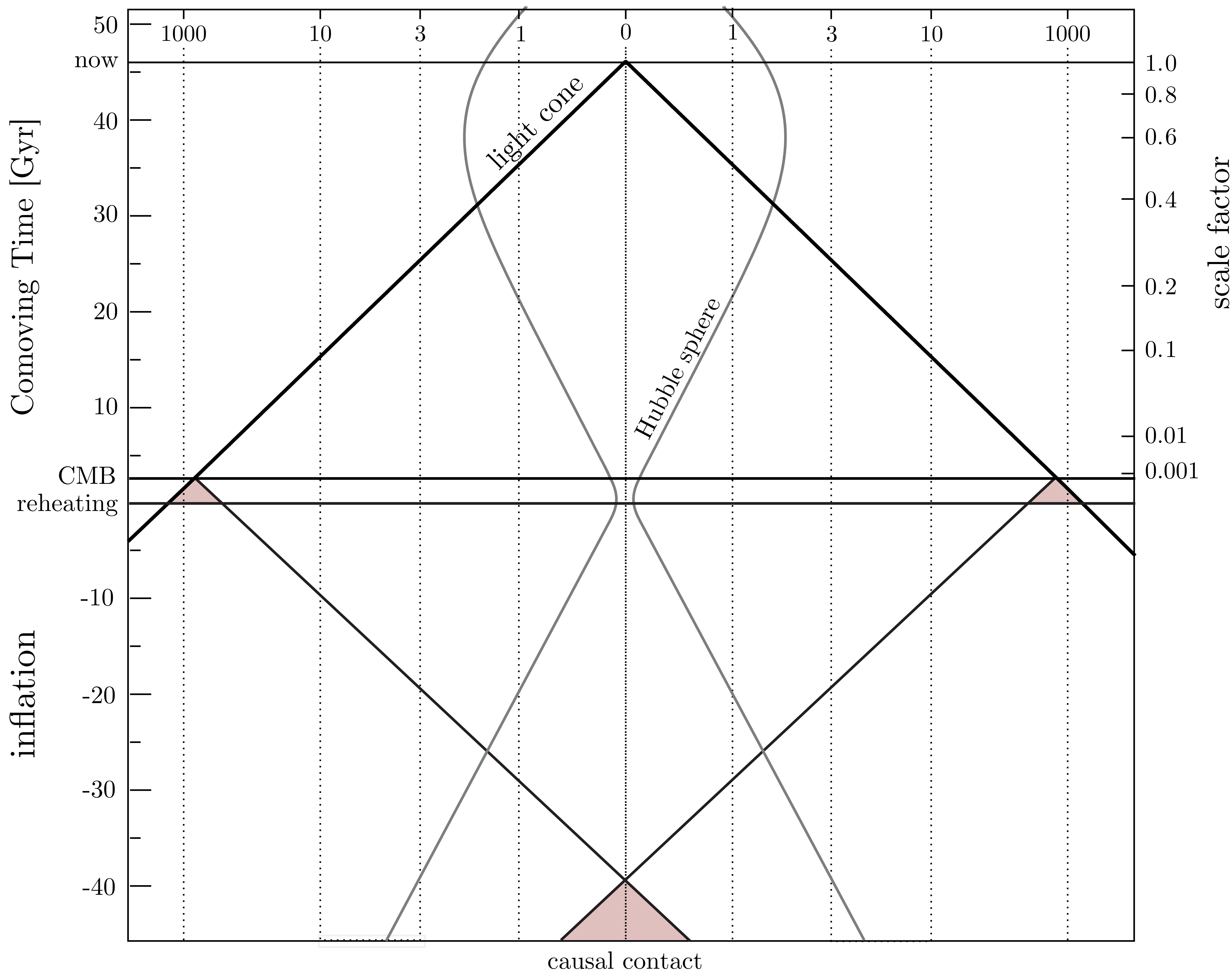
The comoving horizon then gets its largest contribution from early times :

$$\Delta\tau = \int_{a_i}^{a} (aH)^{-1}\, \mathrm{d}\ln a$$

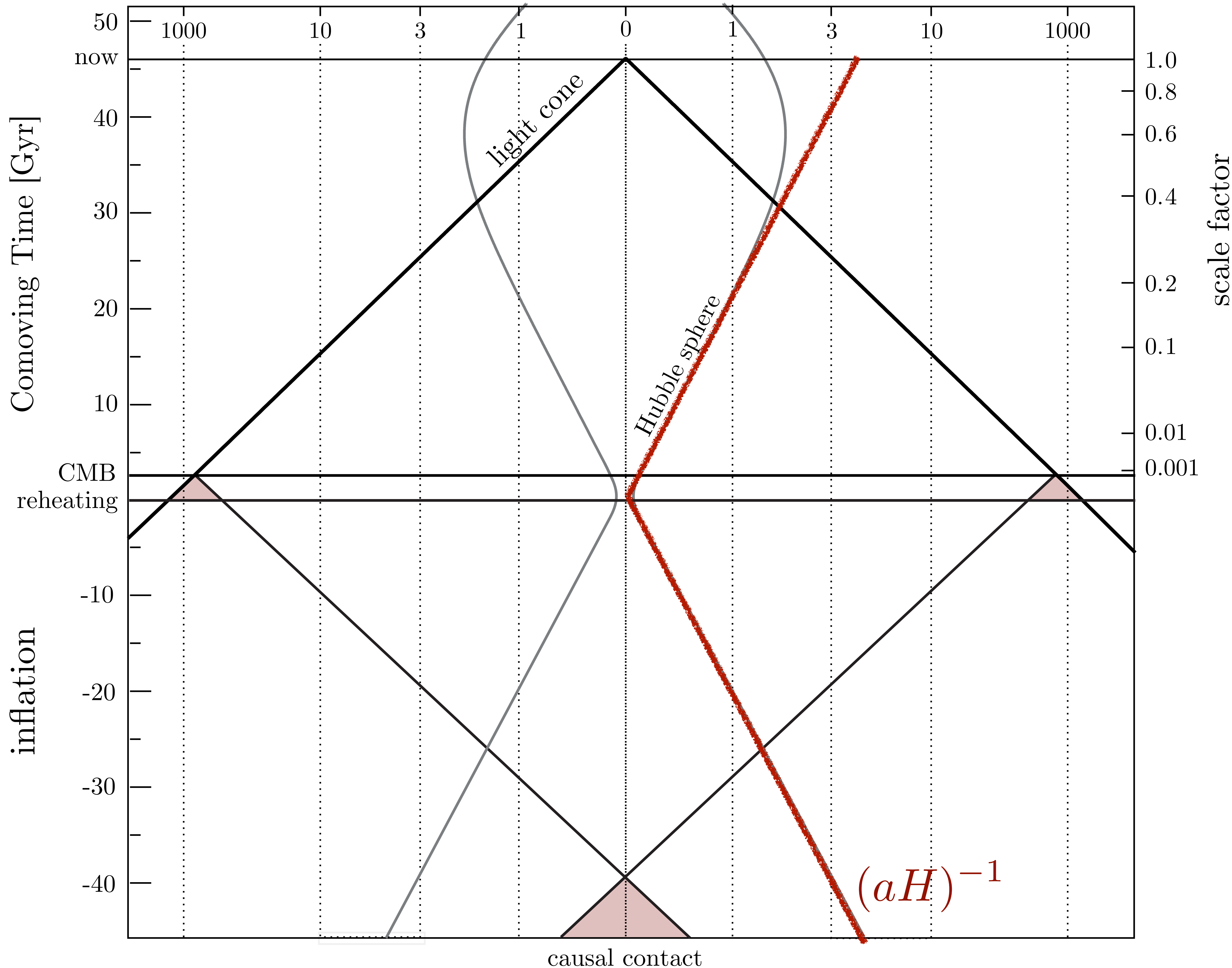
$$\propto -\frac{2}{1+3w}\left[a^{\frac{1}{2}(1+3w)} - a_i^{\frac{1}{2}(1+3w)}\right] \qquad \textit{for a fluid}$$

$$-\infty \equiv \tau_i$$

There was more time between the singularity and recombination than we had thought!

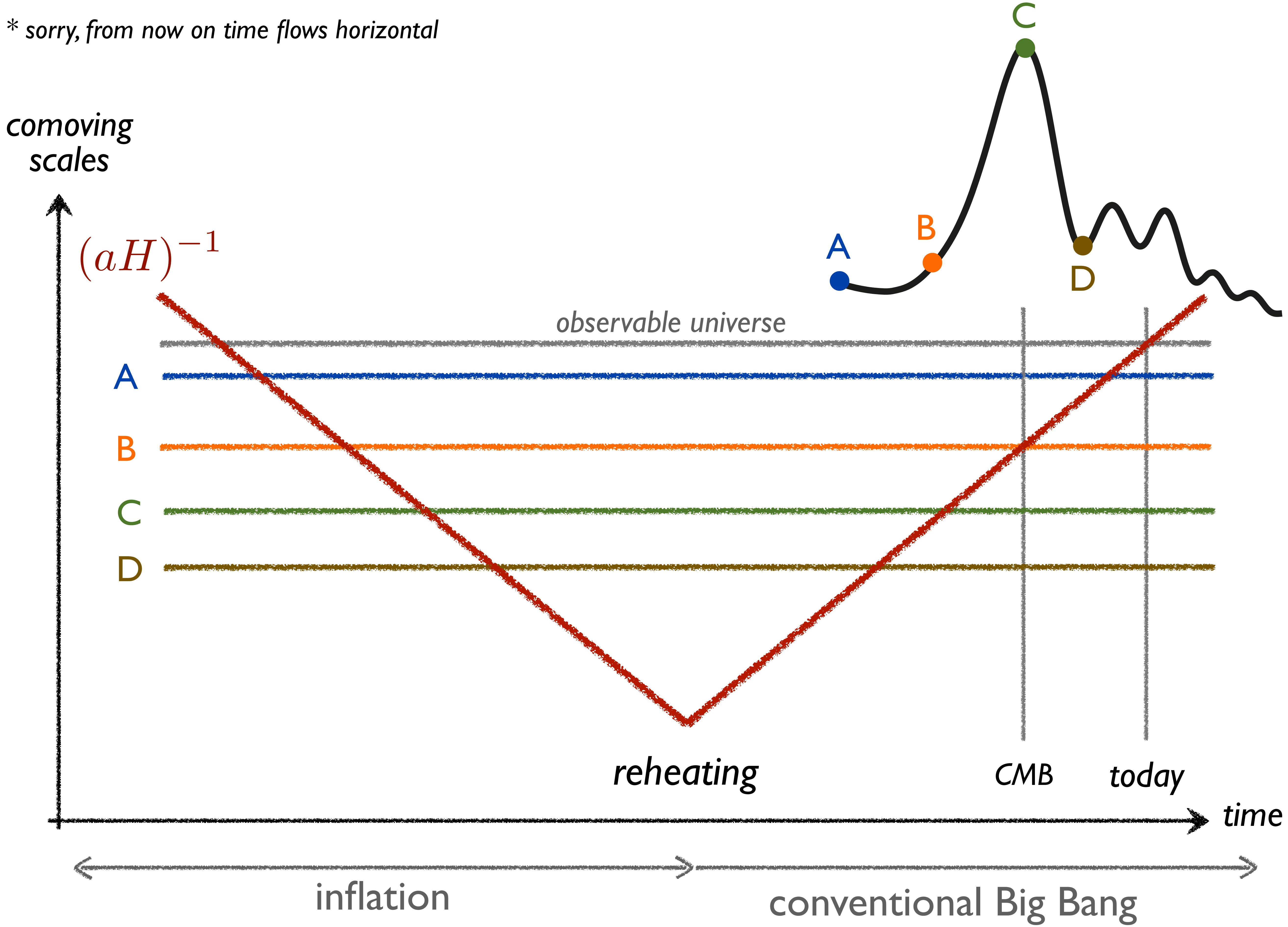


Comoving Time [Gyr]
inflation
scale factor
50
now
40
30
20
10
CMB
reheating
-10
-20
-30
-40
1000
10
3
1
0
1
3
10
1000
1.0
0.8
0.6
0.4
0.2
0.1
0.01
0.001
light cone
Hubble sphere
causal contact



50
now
40
30
20
10
CMB
reheating
-10
-20
-30
-40
Comoving Time [Gyr]
inflation
1000
10
3
1
0
1
3
10
1000
1.0
0.8
0.6
0.4
0.2
0.1
0.01
0.001
scale factor
light cone
Hubble sphere
$(aH)^{-1}$
causal contact

* sorry, from now on time flows horizontal



Exercise: Show that the following are equivalent definitions of inflation:

$1 + 3w < 0$

~~SEC~~

$\ddot{a} > 0$

acceleration

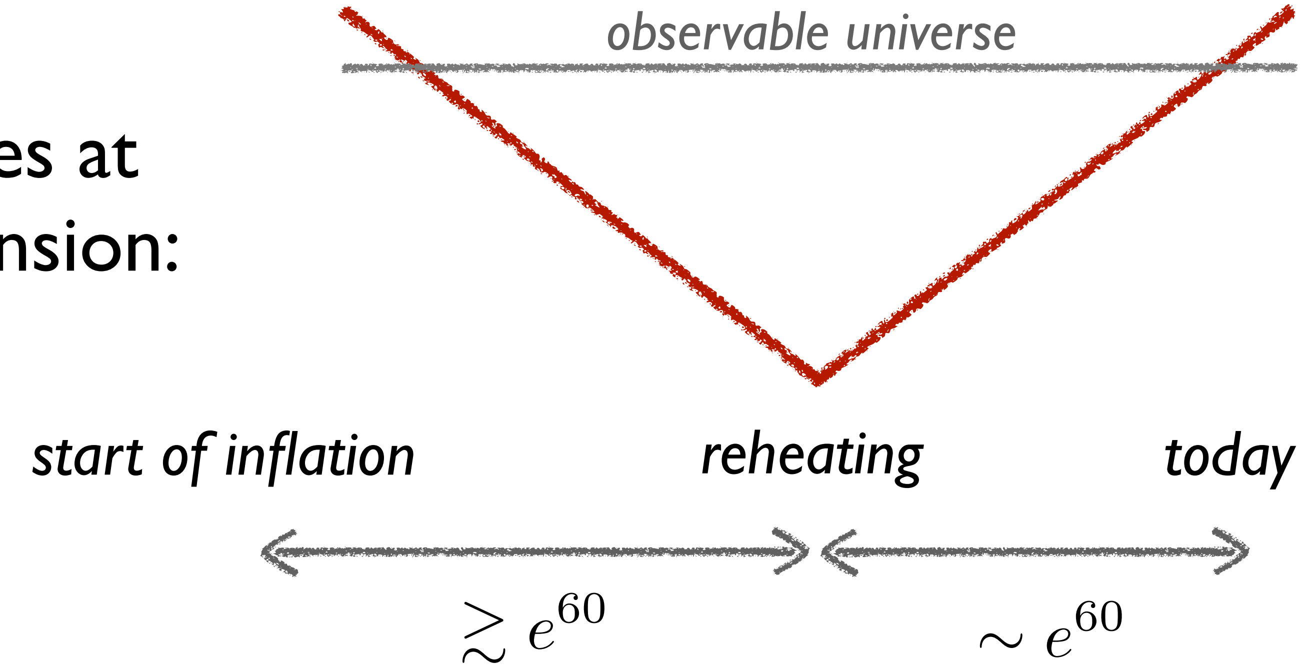
$$\boxed{\frac{d}{dt}(aH)^{-1} < 0}$$

$$\boxed{\varepsilon \equiv -\frac{\dot{H}}{H^2} < 1}$$

$$= -\frac{d \ln H}{dN}$$

$$\mathrm{d}N \equiv H \mathrm{d}t = \mathrm{d} \ln a$$

To solve the horizon problem requires at least 60 e-foldings of inflationary expansion:



We don't just require that inflation occurs ...

1) $$\boxed{\varepsilon = -\frac{d \ln H}{dN} < 1}$$

... but also that inflation lasts

2) $$\boxed{\delta = -\frac{d \ln \dot{H}}{dN} < 1}$$

$$= -\frac{\ddot{H}}{H \dot{H}}$$

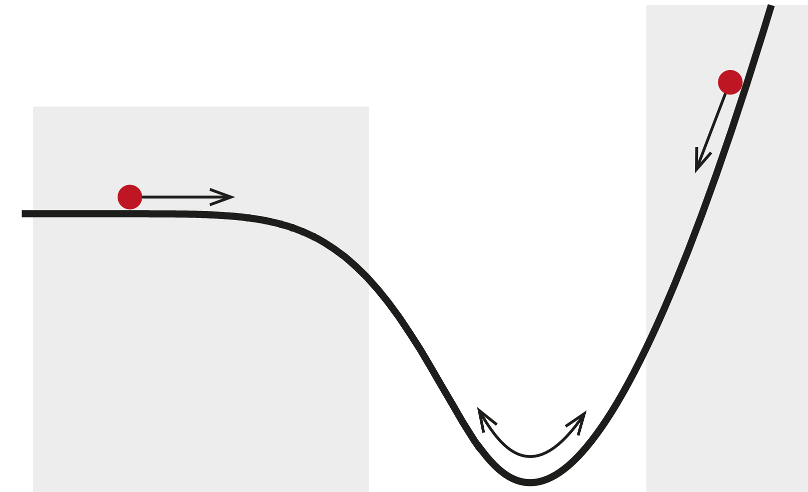
Slow-Roll Inflation

What microphysics leads to $\{\varepsilon\,,\,|\delta|\} \ll 1$ *?*

Scalar Field Dynamics

Consider a scalar field minimally coupled to gravity

$$S = \int \mathrm{d}^4x\sqrt{-g}\left[\frac{M_{\rm pl}^2}{2}R - \frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - V(\phi)\right]$$



In a flat FRW background, we have:

Friedmann

$$H^2 = \frac{1}{3M_{\rm pl}^2}\left[\frac{1}{2}\dot\phi^2 + V(\phi)\right]$$

Klein-Gordon

$$\ddot\phi + 3H\dot\phi = -V'$$

Continuity

$$\dot H = -\frac{1}{2}\frac{\dot\phi^2}{M_{\rm pl}^2}$$

$$\varepsilon = -\frac{\dot H}{H^2} = \frac{\frac{1}{2}\dot\phi^2}{M_{\rm pl}^2 H^2} < 1$$

slow-roll

$$\delta = -\frac{\ddot H}{H\dot H} = -\frac{2\ddot\phi}{H\dot\phi} < 1$$

Slow-Roll Approximation

Friedmann

$$H^2 = \frac{1}{3M_{\rm pl}^2}\left[\frac{1}{2}\dot\phi^2 + V(\phi)\right] \xrightarrow{\varepsilon \ll 1} H^2 \approx \frac{V}{3M_{\rm pl}^2}$$

Klein-Gordon

$$\ddot\phi + 3H\dot\phi = -V' \xrightarrow{|\delta| \ll 1} 3H\dot\phi \approx -V'$$

Exercise: Show that

$$\varepsilon \approx \frac{M_{\rm pl}^2}{2}\left(\frac{V'}{V}\right)^2 \equiv \epsilon$$

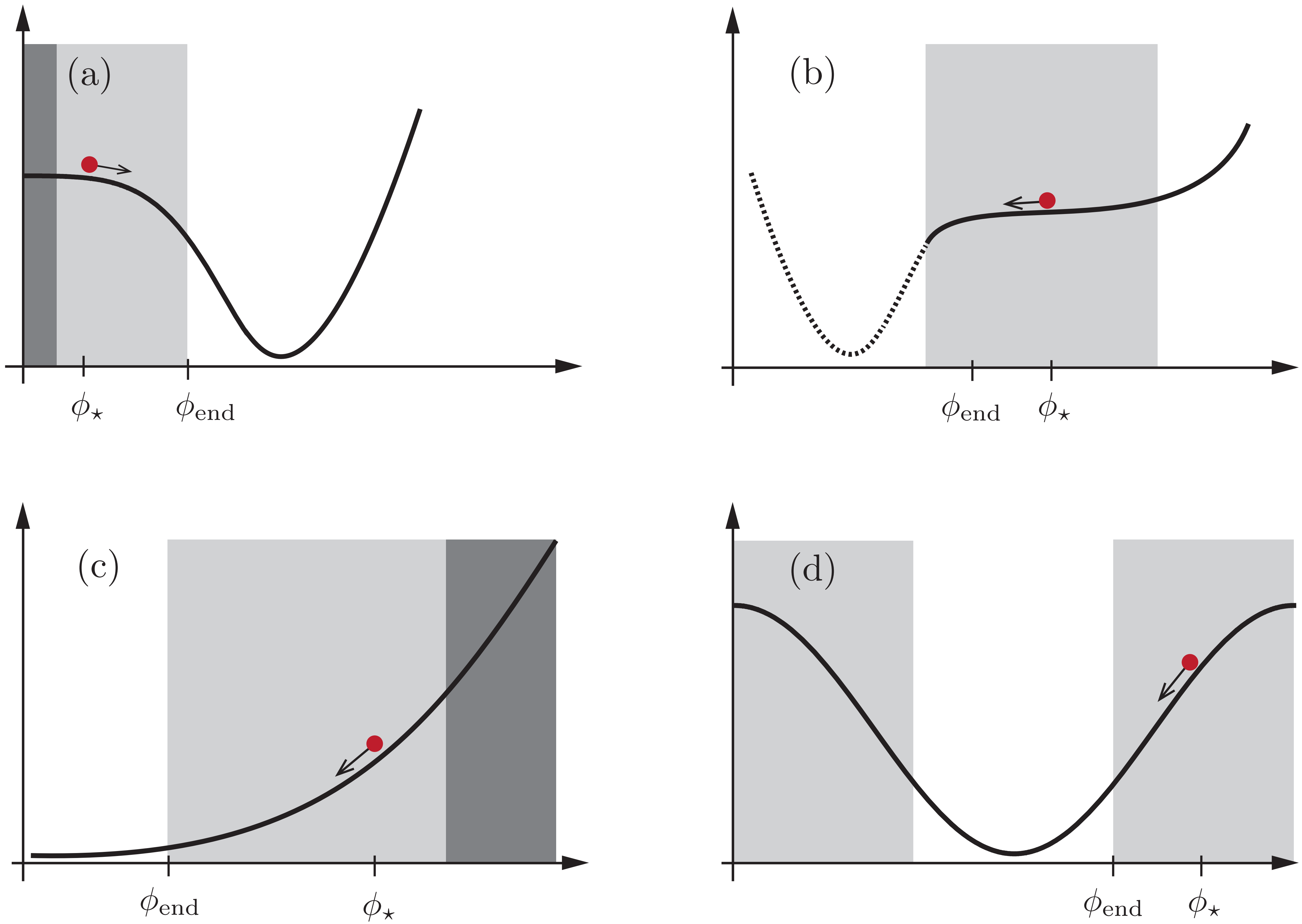
$$\frac{1}{2}\delta + \varepsilon \approx M_{\rm pl}^2 \frac{V''}{V} \equiv \eta$$

potential slow-roll parameters

Hubble slow-roll parameters

allow us to evaluate the prospect of inflation purely from the shape of the potential

Slow-Roll Examples



The Eta Problem

The eta parameter can be written as a ratio of the inflaton mass and the Hubble scale:

$$\eta = M_{\rm pl}^2 \frac{V''}{V} = \frac{1}{3}\frac{m_\phi^2}{H^2} < 1$$

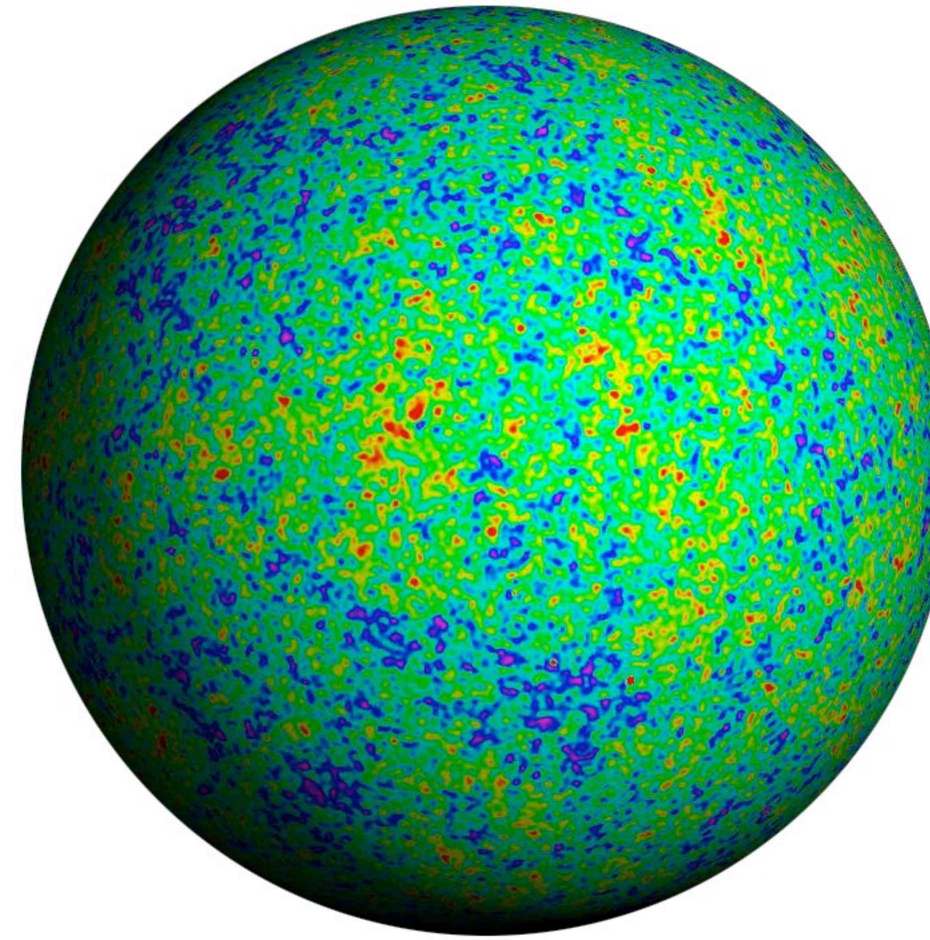
Achieving and stabilizing this mass hierarchy is one of the main challenges for models of inflation.

We will come back to this ...

Primordial Perturbations

So far, we have explained why the universe is homogeneous, isotropic and flat.

If this were the end of the story it would be a disaster ...



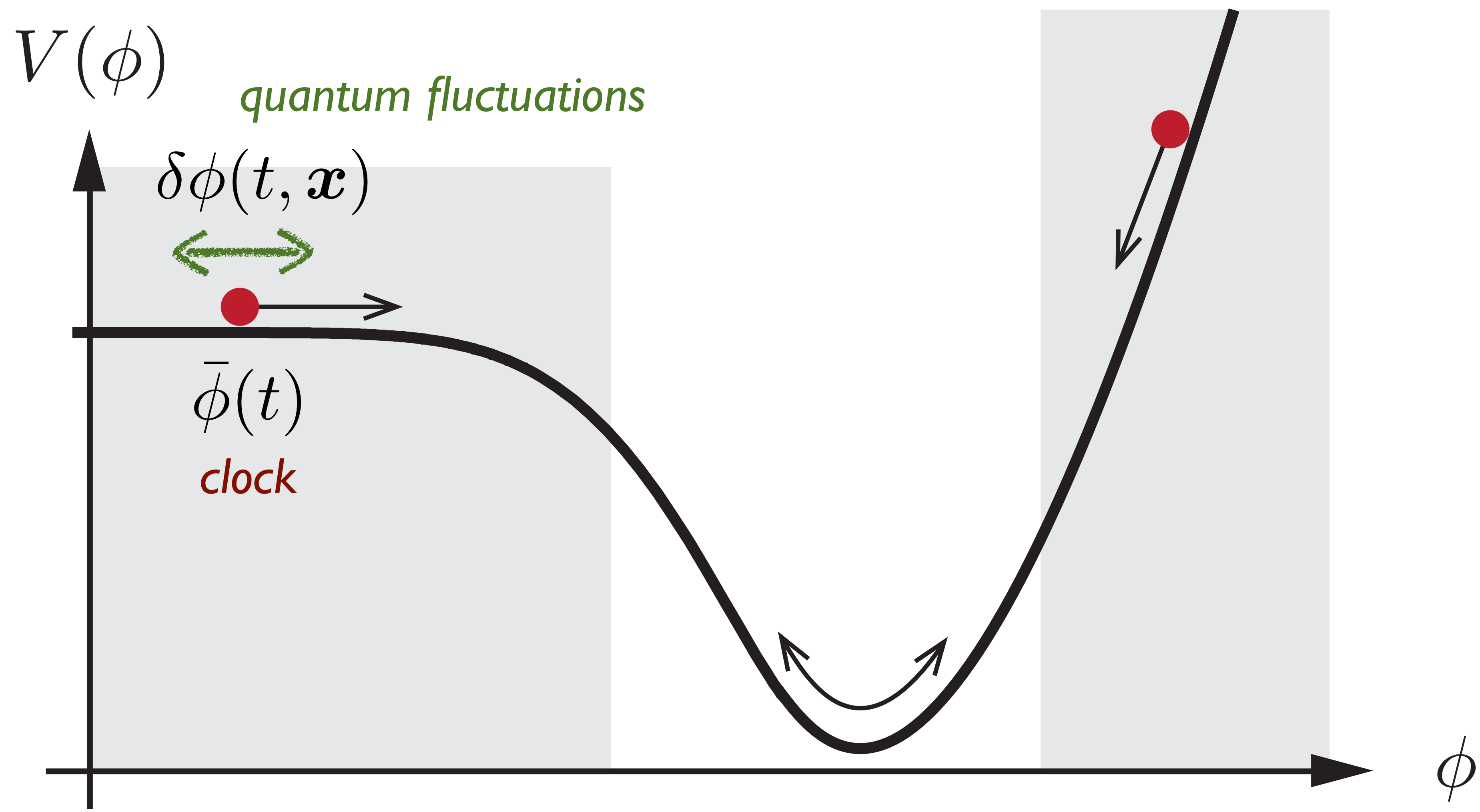
... we need a source for the anisotropies of the CMB:

Remarkably, inflation automatically contains a mechanism to produce primordial fluctuations:

Quantum Mechanics

Quantum Fluctuations

The quantum origin of density perturbations is quite intuitive:



vacuum fluctuations spread the inflaton vev ...

... which translates into density fluctuations after inflation

$$\delta\phi(\boldsymbol{x}) \longrightarrow \delta t(\boldsymbol{x}) \longrightarrow \delta\rho(\boldsymbol{x}) \longrightarrow \delta T(\boldsymbol{x})$$

... which induces a local time delay for the end of inflation

... which become the CMB anisotropies.

On the Back of an Envelope

The vacuum fluctuations can be estimated on the back of an envelope.

Hollands and Wald

- Linearized inflaton fluctuations satisfy an harmonic oscillator equation:

$$\underbrace{\delta\ddot{\phi}_k}_{\text{mass}=1} + \underbrace{3H}_{\text{friction}}\,\delta\dot{\phi}_k + \underbrace{\frac{k^2}{a^2}}_{\text{spring 'constant'}}\delta\phi_k \approx 0 \; {}^{*}$$

• inside the horizon: friction negligible

• outside the horizon: friction dominates

$\longrightarrow$ freeze-out at $k/a_\star = H_\star$

- This comes from the following action:

$$S = \frac{1}{2}\int \mathrm{d}t\,\mathrm{d}^3x\,a^3\Big[(\delta\dot{\phi})^2 - a^{-2}(\partial_i\delta\phi)^2\Big]$$

* *This ignores metric fluctuations. I will fix this later.*

On the Back of an Envelope

The vacuum fluctuations can be estimated on the back of an envelope.

Hollands and Wald

- The zero-point fluctuations of the quantum oscillator are

$$\langle (\delta\phi_k)^2 \rangle = \frac{1}{a^3} \frac{1}{2(k/a)}$$

cf. $\langle x^2 \rangle = \frac{\hbar}{2\omega}$ Wikipedia

This holds as long as the mode evolves adiabatically (inside the horizon).

- Fluctuations freeze in at horizon crossing:

$$\langle (\delta\phi_k)^2_\star \rangle = \frac{1}{a^3_\star} \frac{1}{2(k/a_\star)}$$

$$= \frac{1}{2} \frac{H^2_\star}{k^3}$$

de Sitter fluctuations

scale-invariance

Of course, we don't do precision cosmology on the back of an envelope.

Let's get the same answer a bit more formally.

(*Warning:* This will be a bit technical, but everybody should have seen this once!)

Canonical Quantization

We first write the action in terms of conformal time ...

$$S = \frac{1}{2}\int \mathrm{d}\tau\, \mathrm{d}^3x\, a^2\left[(\delta\phi')^2 - (\partial_i\delta\phi)^2\right]$$

... and canonically normalize the field $v \equiv a\,\delta\phi$

We arrive at the action for a harmonic oscillator in Minkowski space with time-dependent mass:

$$\boxed{S = \frac{1}{2}\int \mathrm{d}\tau\, \mathrm{d}^3x \left[(v')^2 - (\partial_i v)^2 + \frac{a''}{a}v^2\right]}$$

$$\eta^{\mu\nu}\partial_\mu v\,\partial_\nu v$$

$$m^2(\tau) \sim (aH)^2$$

(captures the expansion of the universe)

Canonical Quantization

The associated equation of motion is the Mukhanov-Sasaki equation:

$$\boxed{v''_{\boldsymbol{k}} + \left(k^2 - \frac{a''}{a}\right) v_{\boldsymbol{k}} = 0}$$

- In the subhorizon limit, the mass is negligible and the mode oscillates:

$$v''_{\boldsymbol{k}} + k^2 v_{\boldsymbol{k}} \approx 0 \longrightarrow v_{\boldsymbol{k}} \propto e^{\pm ik\cdot\tau}$$

- In the superhorizon limit, gradients are negligible and the mode freezes: *

$$v''_{\boldsymbol{k}} - \frac{a''}{a} v_{\boldsymbol{k}} \approx 0 \longrightarrow v_{\boldsymbol{k}} \propto \left\{ \begin{array}{l} a \\ a^{-2} \end{array} \right. \longrightarrow \delta\phi_{\boldsymbol{k}} \equiv \frac{v_{\boldsymbol{k}}}{a} \propto \left\{ \begin{array}{l} 1 \\ a^{-3} \end{array} \right.$$

- The general solution can be written as:

initial condition

$$\boxed{v_{\boldsymbol{k}}(\tau) \equiv v_k(\tau) a_{\boldsymbol{k}} + v_k^*(\tau) a^*_{-\boldsymbol{k}}}$$

mode function: $v''_k + \left(k^2 - \frac{a''}{a}\right) v_k = 0$

Canonical Quantization

We have arrived at the following mode expansion:

$$v(\tau, \boldsymbol{x}) \equiv \int \frac{\mathrm{d}^3 k}{(2\pi)^{3/2}} \left[v_k(\tau)\, a_{\boldsymbol{k}}\, e^{i\boldsymbol{k}\cdot\boldsymbol{x}} + v_k^*(\tau)\, a_{\boldsymbol{k}}^*\, e^{-i\boldsymbol{k}\cdot\boldsymbol{x}} \right]$$

So far, this is still classical.

To make it quantum, we promote fields to operators ...

$$\begin{matrix} v \\[2em] \pi \equiv \dfrac{\partial \mathcal{L}}{\partial v'} = v' \end{matrix} \quad \longrightarrow \quad \boxed{\begin{matrix} \hat{v} \\[2em] \hat{\pi} \end{matrix}} \quad \text{and} \quad \boxed{\begin{matrix} \hat{a}_{\boldsymbol{k}} \\[2em] \hat{a}_{\boldsymbol{k}}^\dagger \end{matrix}}$$

... and impose the canonical commutation relation:

$$\left[\hat{v}(\tau, \boldsymbol{x}), \hat{\pi}(\tau, \boldsymbol{y})\right] = i\delta(\boldsymbol{x} - \boldsymbol{y})$$

Canonical Quantization

Exercise: *By substituting*

$$\hat{v}(\tau, \boldsymbol{x}) = \int \frac{\mathrm{d}^3 k}{(2\pi)^{3/2}} \left[v_k(\tau)\, \hat{a}_{\boldsymbol{k}}\, e^{i\boldsymbol{k}\cdot\boldsymbol{x}} + v_k^*(\tau)\, \hat{a}_{\boldsymbol{k}}^\dagger\, e^{-i\boldsymbol{k}\cdot\boldsymbol{x}} \right]$$

Show that the commutation relation becomes

(Wronskian)

$$W(v_k, v_k^*) \times \left[\hat{a}_{\boldsymbol{k}}, \hat{a}_{\boldsymbol{k}'}^\dagger\right] = \delta(\boldsymbol{k} - \boldsymbol{k}') \qquad \text{where} \qquad W(v_k, v_k^*) \equiv -i \left(v_k {v'_k}^* - v'_k v_k^*\right)$$

We use our freedom to normalize the mode functions to set $W(v_k, v_k^*) \equiv 1$

The commutation relation then becomes that of the creation and annihilation operators of a harmonic oscillator

$$\left[\hat{a}_{\boldsymbol{k}}, \hat{a}_{\boldsymbol{k}'}^\dagger\right] = \delta(\boldsymbol{k} - \boldsymbol{k}')$$

annihilation *creation*

The vacuum state is defined in the standard way: $\hat{a}_{\boldsymbol{k}}|0\rangle = 0$

However, at this point neither $\hat{a}_{\boldsymbol{k}}$ nor $|0\rangle$ are uniquely defined, since they depend on the form of $v_k(\tau)$ which hasn't been fixed.

Canonical Quantization

$$a(\tau) = -\frac{1}{H\tau}$$

Let's be concrete and solve the theory in de Sitter space:

The Mukhanov-Sasaki equation becomes

$$v_k'' + \left(k^2 - \frac{2}{\tau^2}\right) v_k = 0$$

Exercise: *Show that this is solved by*

$$v_k(\tau) = \alpha\, e^{-ik\tau}\left(1 - \frac{i}{k\tau}\right) + \beta\, e^{+ik\tau}\left(1 + \frac{i}{k\tau}\right)$$

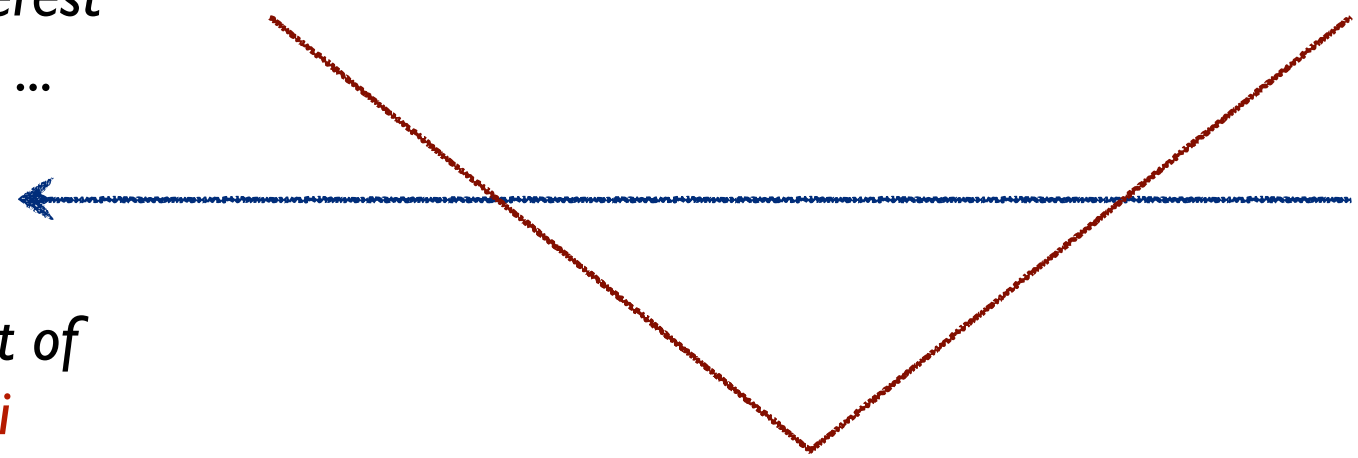
Different choices of $\{\alpha, \beta\}$ correspond to different vacuum states.

How do we choose the "correct vacuum"?

Canonical Quantization

Key insight: *At early times, all modes of interest were deep inside the horizon ...*

$$v_k'' + k^2 v_k \approx 0$$



... and the mode equation is that of a massless field in Minkowski

The minimum energy mode function in Minkowski is:

$$v_k(\tau) = \frac{1}{\sqrt{2k}}\, e^{-ik\tau}$$

Exercise: *Prove this.*

Therefore, we choose the de Sitter mode function such that at early times it matches Minkowski:

$$v_k(\tau) = \frac{e^{-ik\tau}}{\sqrt{2k}} \left(1 - \frac{i}{k\tau}\right)$$

Bunch-Davies mode function

Vacuum Fluctuations

We can now compute the vacuum expectation values of fluctuations:

- Mean: $\langle 0|\hat{v}_{\boldsymbol{k}}|0\rangle = \langle 0|\, v_k\, \underset{=0}{\hat{a}_{\boldsymbol{k}}} + v_k^*\, \overset{=0}{\hat{a}^\dagger_{\boldsymbol{k}}}\, |0\rangle \;= 0$

- Variance: $\langle 0|\hat{v}^\dagger_{\boldsymbol{k}}\hat{v}_{\boldsymbol{k}'}|0\rangle \;= |v_k|^2\langle 0|[\hat{a}_{\boldsymbol{k}}, \hat{a}^\dagger_{\boldsymbol{k}'}]|0\rangle = |v_k|^2\,\delta(\boldsymbol{k}-\boldsymbol{k}')$

The power spectrum is the square of the mode function:

$$P_v(k) \equiv |v_k|^2 = \frac{1}{2}\frac{(aH)^2}{k^3}\left[1 + \underbrace{\cancel{\left(\frac{k}{aH}\right)^2}}_{=0}\right]$$

superhorizon limit

The power spectrum of inflaton fluctuations is:

$$\boxed{P_{\delta\phi}(k) = \frac{P_v(k)}{a^2} = \frac{1}{2}\frac{H^2}{k^3}}$$

Curvature Perturbations

Deficiencies of the treatment so far:

- It is inconsistent to ignore metric fluctuations:

$$\delta\phi(\boldsymbol{x}) \xleftrightarrow[\text{Einstein}]{} \delta g_{\mu\nu}(\boldsymbol{x})$$

- Inflaton fluctuations are not observables.

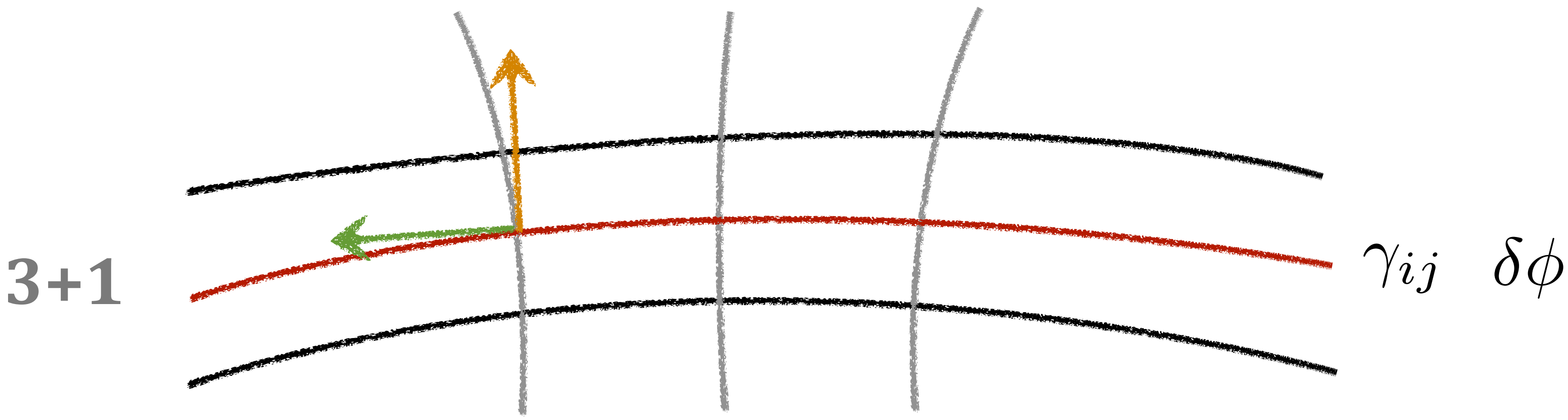
We will fix this now.

(*Warning:* This will be a bit tedious, but everybody should have seen this once!)

Metric Perturbations

We will treat metric perturbations in the ADM formalism:

Arnowitt, Deser and Misner [gr-qc/0405109]



$$\mathrm{d}s^2 = -\underbrace{N^2}_{\text{lapse}}\mathrm{d}t^2 + \gamma_{ij}\left(\mathrm{d}x^i + \underbrace{N^i}_{\text{shift}}\mathrm{d}t\right)\left(\mathrm{d}x^j + N^j\mathrm{d}t\right)$$

$$N \equiv 1 + \alpha(t, \boldsymbol{x}) \qquad N_i \equiv \partial_i \beta(t, \boldsymbol{x})$$

$$\gamma_{ij} \equiv a^2(t)\left[\underbrace{(1 - 2\Phi(t, \boldsymbol{x}))\delta_{ij}}_{\text{trace}} + \underbrace{\partial_{\langle i}\partial_{j\rangle}\gamma(t, \boldsymbol{x})}_{\text{traceless}}\right]$$

scalar modes		*gauge*		*constraints*		*physical d.o.f.*
5	-	2	-	2	=	1

4 metric:

$\alpha \quad \beta \quad \gamma \quad \Phi$

1 matter:

$\delta\phi$

1 time:

$t \mapsto t + \epsilon_0$

1 space:

$x_i \mapsto x_i + \partial_i \epsilon$

Gauge Fixing

- Using a time shift, we can remove any fluctuations in the scalar field:

  1. $\delta\phi \equiv 0$ *(comoving gauge)*

- A spatial shift, sets the traceless part of the metric to zero:

  2. $\gamma \equiv 0$

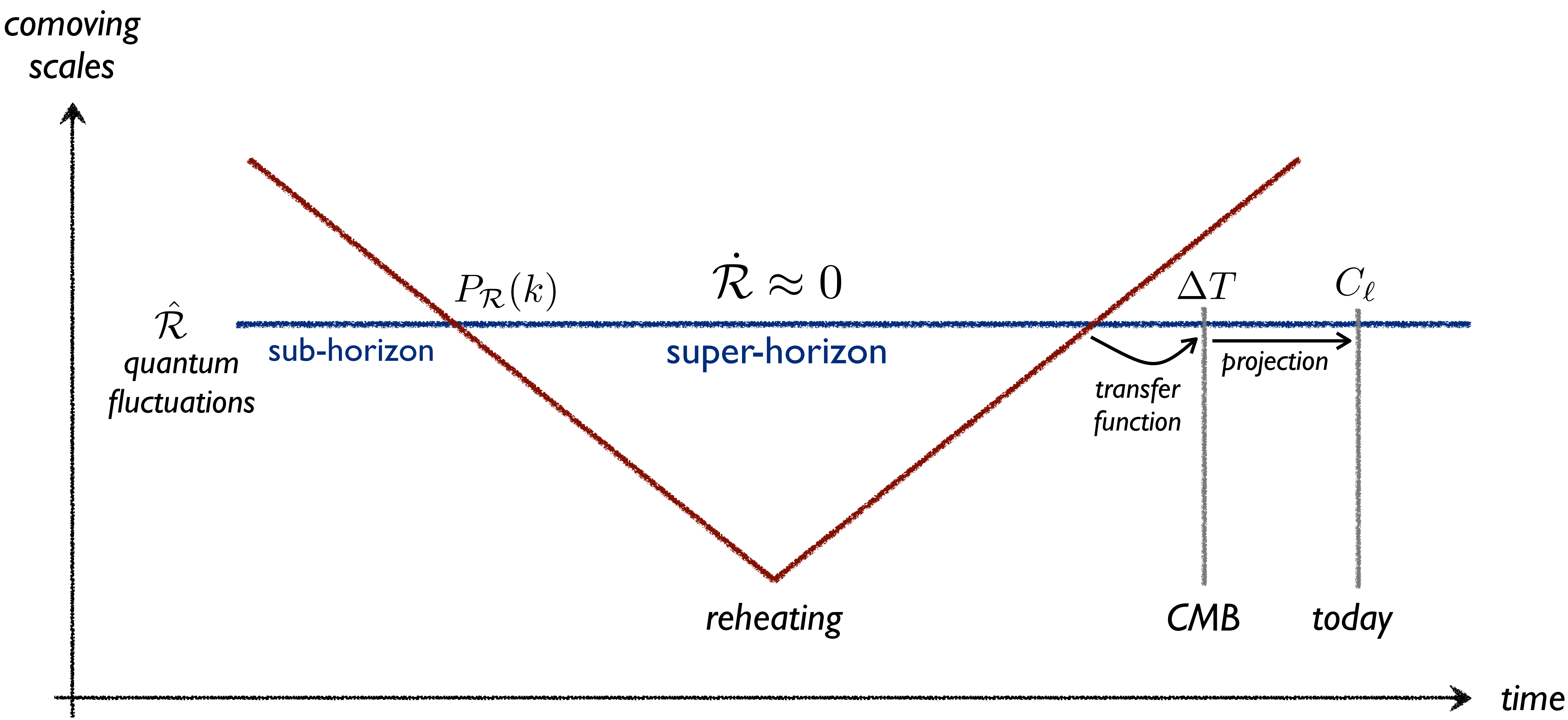
The trace of the metric contains the *comoving curvature perturbation*:

$$\gamma_{ij} = a^2(1 - 2\mathcal{R})\delta_{ij}$$

It measures the intrinsic curvature of the spatial slice. $R_{(3)} = \frac{4}{a^2}\nabla^2\mathcal{R}$

The comoving curvature perturbation is conserved on superhorizon scales.

Weinberg [astro-ph/0302326]



This allows us to be ignorant of the uncertain details of reheating.

Constraint Equations

We still have the lapse and the shift to take care of.
The Einstein equations relate them to the curvature perturbation.

We start with the action:

$$S = \int \mathrm{d}^4x \sqrt{-g} \left[\frac{1}{2} R - \frac{1}{2} g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi - V(\phi) \right]$$

with $M_{\rm pl} \equiv 1$

Constraint Equations

Exercise: *Show that in ADM:*

$$S = \frac{1}{2}\int \mathrm{d}^4x\sqrt{\gamma}\left[NR_{(3)} + N^{-1}(E_{ij}E^{ij} - E^2) + N^{-1}\dot{\phi}^2 - 2NV\right]$$

where $E_{ij} \equiv \frac{1}{2}\left(\dot{\gamma}_{ij} - \nabla_i N_j - \nabla_j N_i\right)$ *(extrinsic curvature)*

The lapse and the shift are non-dynamical, i.e. they satisfy constraint equations:

$$\nabla_i[N^{-1}(E^i_j - \delta^i_j E)] = 0 \longleftarrow \frac{\partial \mathcal{L}}{\partial N} = 0$$

$$R_{(3)} - 2V - N^{-2}(E_{ij}E^{ij} - E^2) - N^{-2}\dot{\phi}^2 = 0 \longleftarrow \frac{\partial \mathcal{L}}{\partial N_i} = 0$$

- substitute: $N \equiv 1 + \alpha \quad N_i \equiv \partial_i \beta \quad \gamma_{ij} \equiv a^2(1 - 2\mathcal{R})\delta_{ij}$
- linearize and solve: $\alpha = \frac{\dot{\mathcal{R}}}{H} \quad \beta = -\frac{\mathcal{R}}{H} + \frac{a^2\varepsilon}{H}\partial^{-2}\dot{\mathcal{R}}$

Quadratic Action

- plug back into the action:
  [integrate by parts and use background e.o.m.]

$$\boxed{S_{(2)} = \int \mathrm{d}t\,\mathrm{d}^3x \; a^3\varepsilon \left[\dot{\mathcal{R}}^2 - a^{-2}(\partial_i\mathcal{R})^2\right]}$$

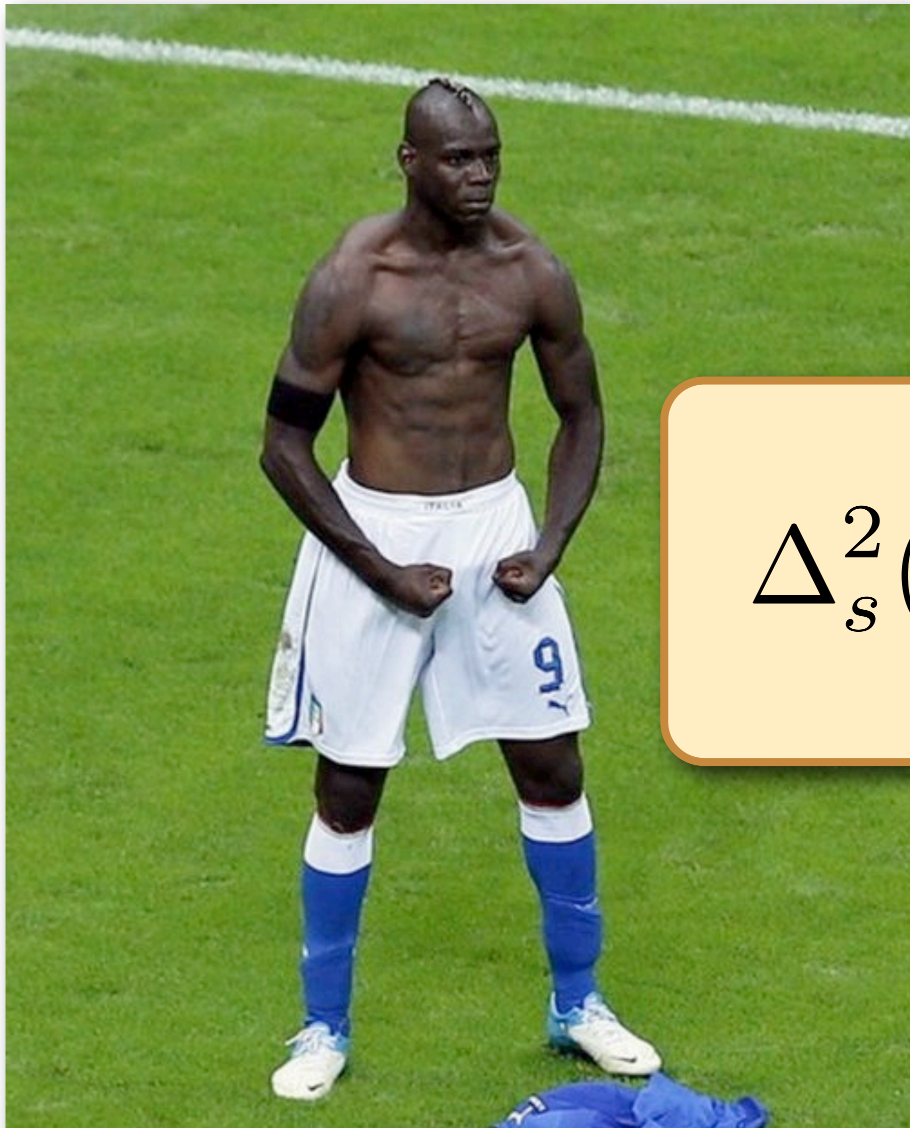
This is the same action as before, except for the factor of 2ε *.*

Hence,

$$P_{\mathcal{R}}(k) = \frac{P_{\delta\phi}(k)}{2\varepsilon}$$ this is infinite in de Sitter: $\varepsilon = 0$

Trick: use de Sitter mode functions even for $\varepsilon \neq 0$
but evaluate the power spectrum at horizon crossing:

$$\boxed{P_{\mathcal{R}}(k) = \frac{1}{4k^3}\frac{H^2}{\varepsilon}\bigg|_{k=aH}}$$



$$\Delta_s^2(k) \equiv \frac{k^3}{2\pi^2} P_{\mathcal{R}}(k) = \frac{1}{8\pi^2} \frac{H^2}{\varepsilon}\bigg|_{k=aH}$$

Scale Dependence

Time dependence becomes scale dependence:

$$\frac{H^2(t)}{\varepsilon(t)} \sim \frac{H^4(t)}{\dot{H}(t)} \longrightarrow \quad n_s - 1 \equiv \frac{d\ln \Delta_s^2}{d\ln k}$$

Exercise: *Show that*

$$\begin{aligned} n_s - 1 &\approx -4\varepsilon + \delta \\ &\approx -3M_{\rm pl}^2 \left(\frac{V'}{V}\right)^2 + 2M_{\rm pl}^2 \frac{V''}{V} \end{aligned}$$