A New Geometrical Approach to Void Statistics

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Theory of void distributions

From structure formation theory [e.g., Press-Schechter (1974); Bardeen, Bond, Kaiser & Szalay (1986)], applying the excursion set formalism to voids:

> Poissonian empty sets [e.g., Politzer & Preskill (1986); Betancourt-Rijo (1990)];

> Void hierarchy [e.g., Sheth & van de Weygaert (2004)];

> Non-Gaussianity [e.g., D'Amico, Musso, Noreña & Paranjape (2011)];

> Modified theories of gravity [e.g., Clampitt, Cai & Li (2013)].

From descriptive models, including:

> Voronoi tessellations [e.g., Icke & van de Weygaert (1987)];

> Self-similarity: fractals and Zipf's law [e.g., Mandelbrot (1977); Einasto, Einasto & Gramann (1989); Gaite & Manrubia (2002)], multifractals and non-lacunarity [e.g., Gaite (2007, 2009)].
A new geometrical approach

Sphericity of underdense regions increases due to gravitational dynamics [e.g. Icke (1984)]. Voids can be approximated by spheres in mutual contact at filaments.

Idea: model voids as unoriented spheres in Euclidean 3-space with radius \( r > 0 \) and position of the centre \( a \in E^3 \). Each sphere is represented by a point in the 4-dimensional configuration space of spheres, with coordinate \( (r, a) \).

Question: which geometry of the configuration space has isometries that preserves the conformal structure of sphere contact (e.g., \( \Delta \))?
**De-Sitter configuration space of spheres**

**Answer:** the configuration space of spheres has *de-Sitter geometry*. This emerges from classical Lie sphere geometry and is unrelated to the cosmological de-Sitter spacetime of general relativity.

**Application:** study the distribution of voids in terms of their distribution function on this configuration space of spheres $dS^4$, with its 4-dimensional de-Sitter structure.

**Result:** the power-law distribution of small voids can be described by a *uniform* distribution over $dS^4$. Interesting physical processes (e.g., mergers of smallest voids in hierarchy; exponential cut-off of largest voids) show up as deviations from uniform distribution.

Joint work with Gary Gibbons (DAMTP, University of Cambridge), Chon Sunmyon and Naoki Yoshida (University of Tokyo/ Kavli IPMU). Preprint coming out very soon!

**Lie geometry of spheres**

To each sphere $c = (r, a)$, a unique homogeneous Lie cycle coordinate can be assigned,

$$l(c) = \left[ -\frac{1}{2} \left( \frac{a^2 + 1}{r} - r \right), -\frac{a}{r}, -\frac{1}{2} \left( \frac{a^2 - 1}{r} - r \right), 1 \right]$$

$$= [X^0, X, X^4, 1]$$

Choosing $X^i = (X^0, X, X^4) \in E^{1,4}$ as configuration coordinates of the sphere, we see that it satisfies

$$-(X^0)^2 + X^2 + (X^4)^2 = 1$$

so it is a point on the de-Sitter quadric in a 5-dimensional Minkowski space. Hence on the de-Sitter space $dS^4$. 
**Metric of the de-Sitter configuration space**

The induced metric on the de-Sitter quadric becomes

\[ ds^2 = \left[ -\left( dX^0 \right)^2 + dX^2 + \left( dX^{n+2} \right)^2 \right]_{dS^4} = \frac{1}{r^2} \left( -dr^2 + da^2 \right) = d\Delta^2 \]

so that, indeed, isometries preserve the intersection angle. Writing \( x^i = (r, a) \in dS^4 \) for the coordinates of a sphere in configuration space, the metric is

\[ ds^2 = g_{ij} dx^i dx^j \quad \text{with} \quad g_{ij} = \text{diag} \left( -\frac{1}{r^2}, \frac{1}{r^2}, \frac{1}{r^2}, \frac{1}{r^2} \right) \]

and so the configuration space volume element is

\[ dV^4 = \sqrt{-\det g} \, dx^1 \ldots dx^4 = \frac{1}{r^4} \, dr da^1 \, da^2 \, da^3 = \frac{1}{r^4} \, dr dv \]

where \( dv = da^1 \, da^2 \, da^3 \) is the volume element in \( E^3 \) of void centres.
Application to void distributions

Modelling voids as spheres, consider a uniform distribution over their configuration space so that

\[ dN = f(r) dr dv \propto dV^4 = \frac{1}{r^4} dr dv \]

Hence, the cumulative number density of voids with radius \( r' > r \) is found to be

\[ n(> r) = \int_r^\infty f(r') dr' \propto \frac{1}{r'^3} \]

Then in terms of void volume, \( V \propto r^3 \), this is of course

\[ n(> V) \propto \frac{1}{V} \]

as from the Press-Schechter/excursion set formalism.
Comparison with a simulation

A cosmological simulation with the GADGET2 code [Springel (2005)] with $256^3$ dark matter particles in a $240 \ (\text{Mpc}/h)^3$ box, with WMAP7-parameters, and an adaptive Gaussian window spherical void finder applying void merging conditions [Colberg et al. (2005)] shows:

> Cumulative void number density: green;

> Good agreement with the fitting formula derived by v. Benda-Beckmann & Müller (2008): blue;

> Good agreement also with uniform de-Sitter distribution for small voids: pink;

> Very small voids are systematically overcounted by uniform de-Sitter distribution, since mergers in void hierarchy are ignored;

> Currently studying the effect of survey geometry restrictions.
Conclusions

> Classical Lie sphere geometry gives rise to a de-Sitter geometry for the configuration space of unoriented spheres.

> A uniform distribution of such spheres over this configuration spaces implies an inverse power-law for the cumulative number density as function of volume.

> Such a distribution is observed to hold approximately for voids in cosmology, in absence of the exponential cut-off and void mergers, as expected from the excursion set formalism.

> Hence, it may be useful to investigate the physical implications of deviations from the uniform de-Sitter configuration space distribution.

> It is an application of de-Sitter geometry in a new cosmological context, and also provides a new geometrical perspective on self-similarity.