Introduction to the physics of artificial gauge fields

Lecture 2: Magnetism in a periodic lattice

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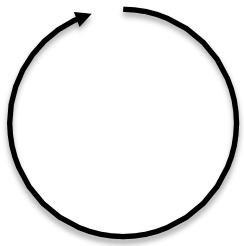
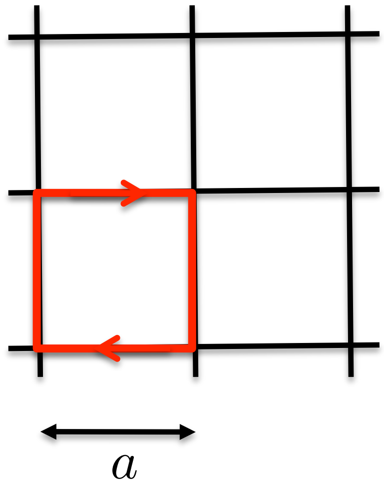
Magnetism and periodic potential

cyclotron motion



$$\ell_{\mathrm{m}} = \sqrt{\hbar/qB}$$

motion on a square lattice



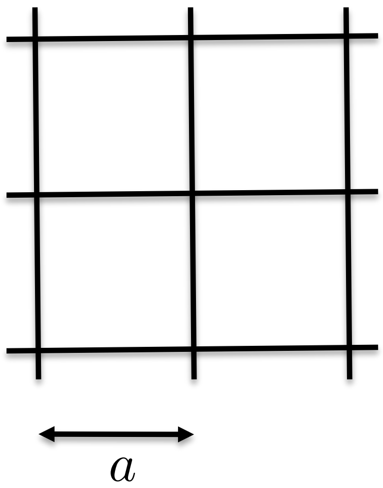
Competition between two phenomena (frustration) :

$$\frac{a^2}{\ell_{\mathrm{m}}^2} = \frac{qBa^2}{\hbar} = 2\pi\frac{\Phi}{\Phi_0}$$

$$\begin{cases} \Phi = Ba^2 & \text{: flux across a cell} \\ \Phi_0 = h/q & \text{: flux quantum} \end{cases}$$

= Aharonov-Bohm phase along the contour of a cell

Relevant values for Aharonov-Bohm phase $2\pi\Phi/\Phi_0$



In a regular solid:

$$B \sim 100\,\mathrm{T} \qquad\qquad a \sim 1\,\text{Å} = 10^{-10}\,\mathrm{m}$$

$$\Phi = Ba^2 \sim 10^{-18}\,\mathrm{Wb}$$

$$\Phi_0 = h/q \approx 4\;10^{-15}\,\mathrm{Wb}$$

$$\longrightarrow \quad 2\pi\,\Phi/\Phi_0 \sim 10^{-3}$$

No drastically new effect expected in this "weak field" regime

If the flux Ba^2 becomes much larger thanks to synthetic materials or artificial gauge fields, frustration may play a dominant role.

Outline

1. Gauge fields on a lattice

    Tight-binding model, gauge choice

2. Hofstadter butterfly

    Sub-bands, Chern number

3. Shaken lattices

    How to obtain non real tunnel matrix elements

4. Lattices combining several internal states

    Laser assisted tunnelling, flux lattices

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   Lewenstein et al., Advances in Physics, 56:2, 243-379 (2007)

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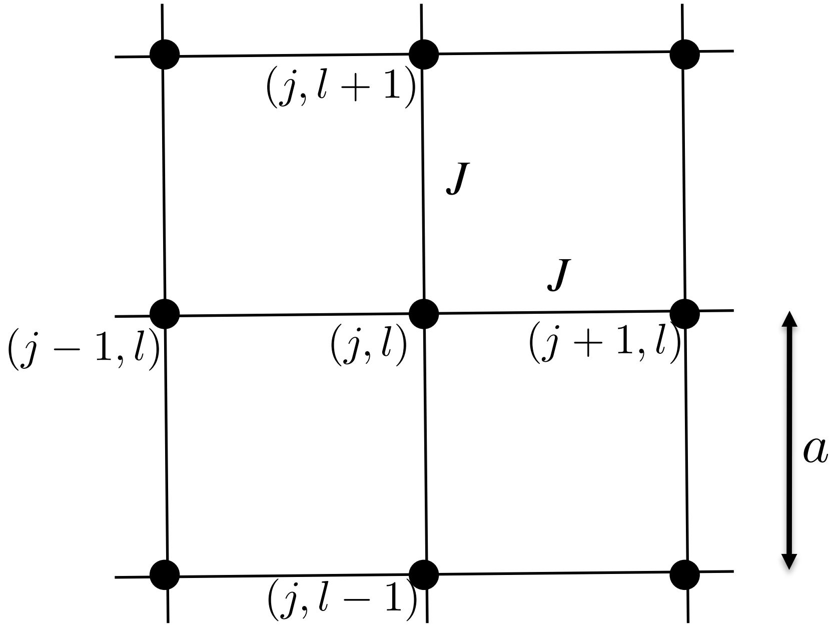
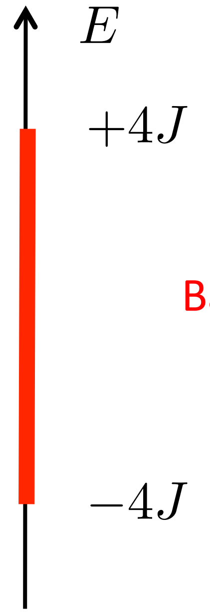
2D tight-binding model



$\hat{H} = \hat{H}_x + \hat{H}_y$: separable

$$\hat{H} = -J \sum_{j,l} \left(|w_{j+1,l}\rangle\langle w_{j,l}| + |w_{j,l+1}\rangle\langle w_{j,l}| \right) + \text{h.c.}$$



Band width $8J$

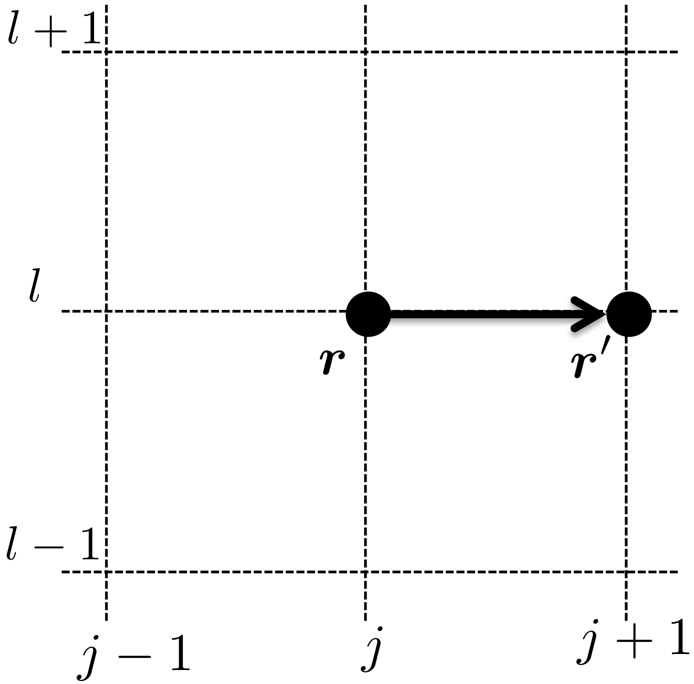
2D Bloch waves

$$\boldsymbol{q} = (q_x, q_y) \qquad |\psi_{\boldsymbol{q}}\rangle = \sum_{j,l} \mathrm{e}^{\mathrm{i}a(jq_x + lq_y)} \, |w_{j,l}\rangle$$

$$E(\boldsymbol{q}) = -2J \left(\cos(aq_x) + \cos(aq_y) \right)$$

Brillouin zone: $q_j \in]-\pi/a, \pi/a]$

Lattice in the presence of a magnetic field



Aharonov-Bohm phase associated to the link $\boldsymbol{r} \to \boldsymbol{r}'$

$$\phi(\boldsymbol{r} \to \boldsymbol{r}') = \frac{q}{\hbar} \int_{\boldsymbol{r}}^{\boldsymbol{r}'} \boldsymbol{A} \cdot \mathrm{d}\boldsymbol{r}$$

Peierls substitution in the tight-binding regime:
Each tunnel term of the Hamiltonian gets the corresponding phase

$$-J\,|w_{j+1,l}\rangle\langle w_{j,l}| \quad \longrightarrow \quad -J\,\mathrm{e}^{\mathrm{i}\phi(j,l\to j+1,l)}\;|w_{j+1,l}\rangle\langle w_{j,l}|$$

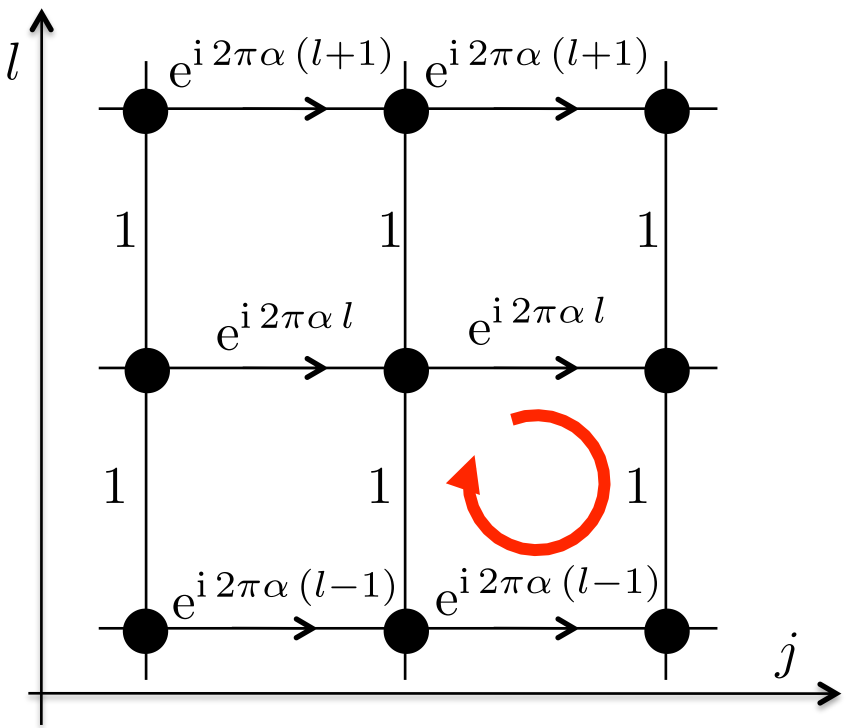
Uniform field and Landau gauge for a square lattice

On a given lattice cell, the only relevant physical quantity (gauge invariant) is $\mathrm{e}^{\mathrm{i}\Theta}$, where Θ is the sum of the phase of the tunnel coefficients around the cell.

Choose a zero-phase for all vertical links : discrete version of $\boldsymbol{A} = (-By, 0, 0)$

Phase along a cell: $2\pi\alpha$ with $\alpha = \Phi/\Phi_0$

$$\begin{cases} \Phi = Ba^2 & \text{: flux across a cell} \\ \Phi_0 = h/q & \text{: flux quantum} \end{cases}$$



Here:

$$\Theta = 2\pi\alpha l + 0 - 2\pi\alpha(l-1) + 0$$

$$= 2\pi\alpha$$

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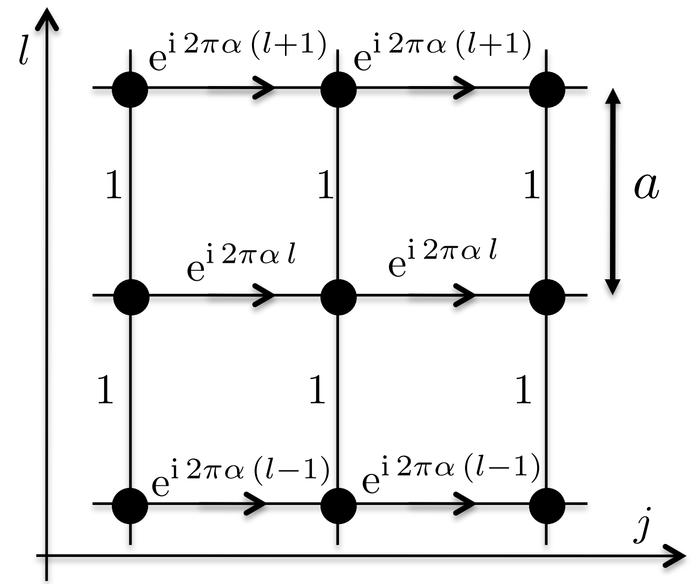
   Di Xiao, Ming-Che Chang & Qian Niu, Rev. Mod. Phys. 82, p. 1959 (2010)

3. Shaken lattices

4. Lattices combining several internal states

General formulation of the problem



We look for the spectrum of the Hamiltonian

$$\hat{H} = -J \sum_{j,l} \mathrm{e}^{\mathrm{i}\,2\pi\alpha\,l} |w_{j+1,l}\rangle\langle w_{j,l}| + |w_{j,l+1}\rangle\langle w_{j,l}| \; + \; \mathrm{h.c.}$$

We keep the periodicity along x, but loose the periodicity along y

We look for the eigenstates as Bloch functions for x

$$|\Psi\rangle = \sum_{j,l} C_l \; \mathrm{e}^{\mathrm{i}ja q_x} |w_{j,l}\rangle$$

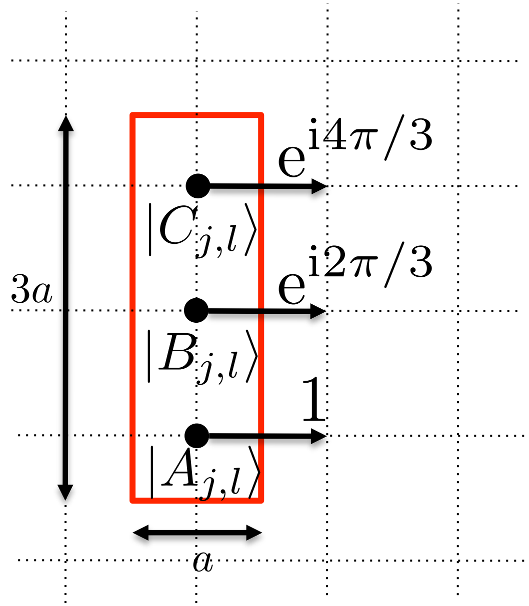
Recursion equation (Harper) for the coefficients C_l

Important particular case when the coefficient α is rational, $\alpha = p'/p$

One then recovers periodicity along y, but with the period pa instead of a.

$$\mathrm{e}^{\mathrm{i}2\pi\alpha l} = \mathrm{e}^{\mathrm{i}2\pi\alpha(l+p)}$$

Example for the flux $\alpha = 1/3$

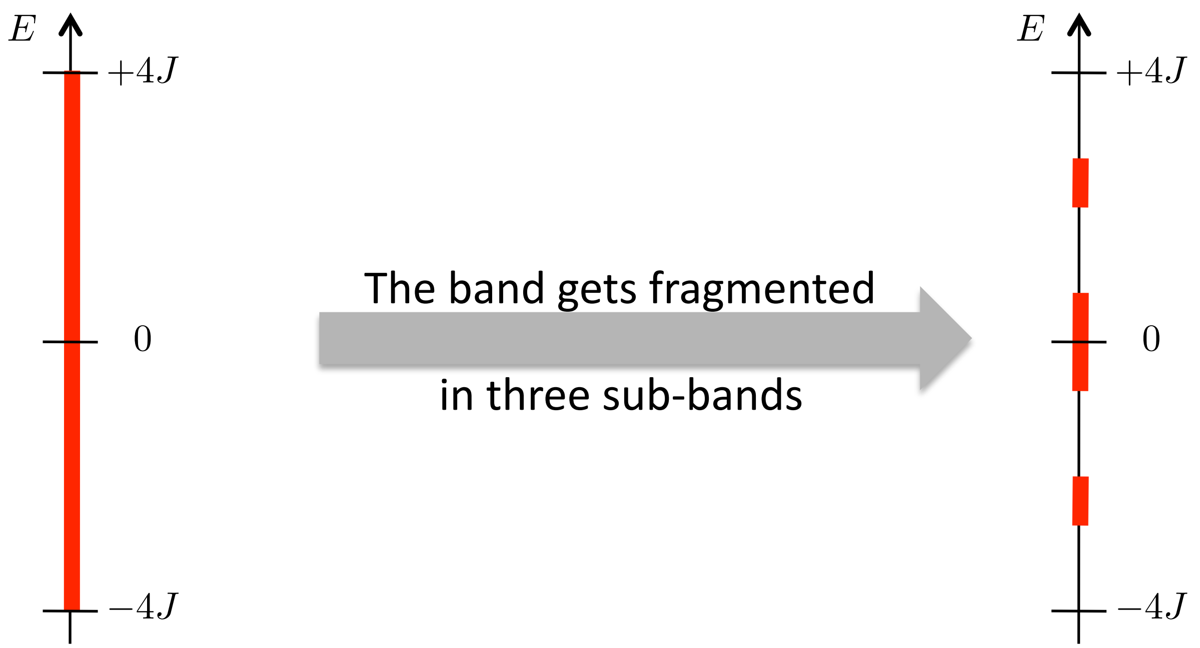


Unit cell with 3 sites $|A\rangle, |B\rangle, |C\rangle$ of size $a \times 3a$

Look for eigenstates as Bloch functions:

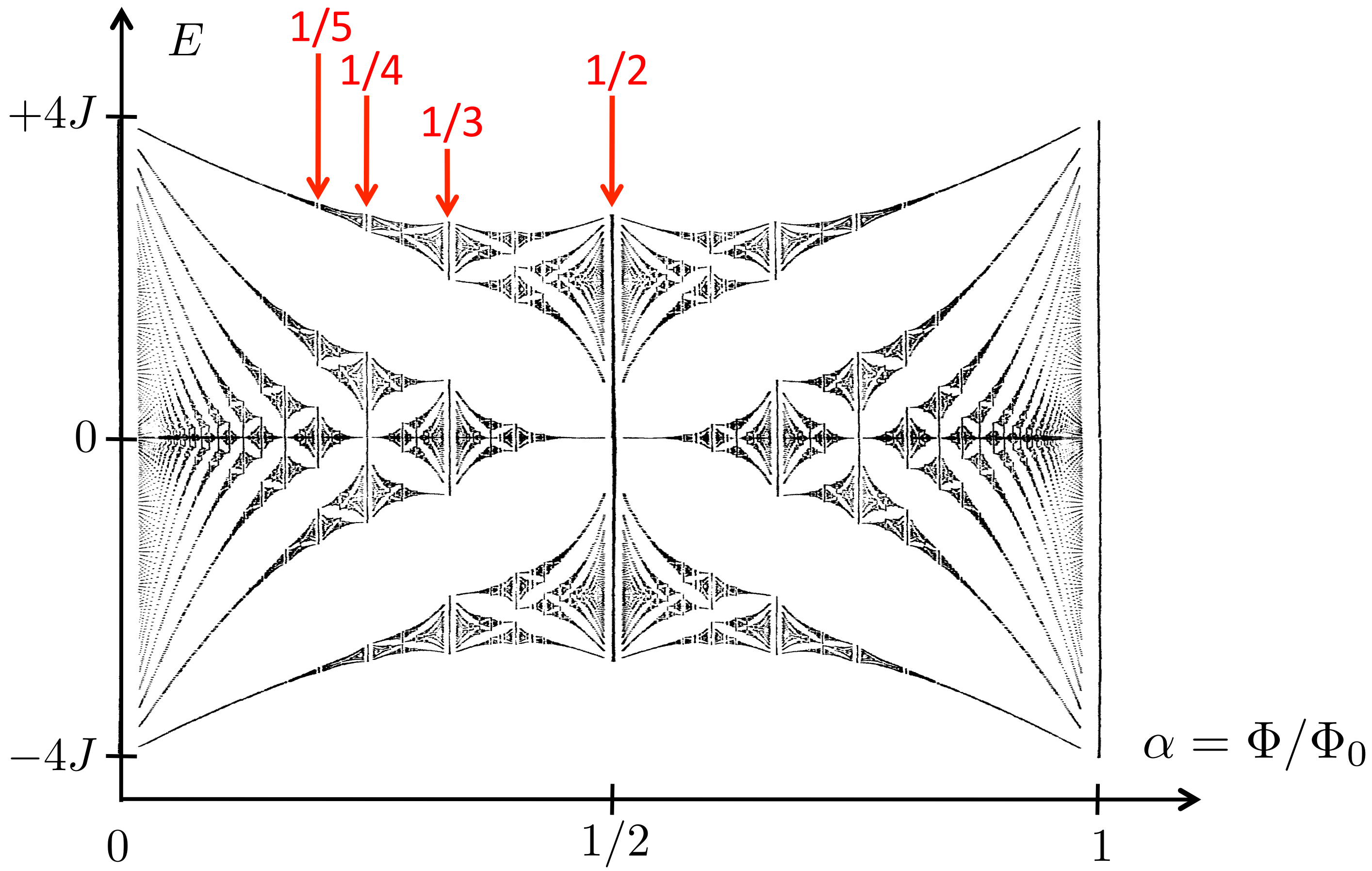
$$|\Psi_{\boldsymbol{q}}\rangle = \sum_{j,l} \mathrm{e}^{\mathrm{i}a(jq_x + 3lq_y)} \left(\alpha|A_{j,l}\rangle + \beta|B_{j,l}\rangle + \gamma|C_{j,l}\rangle\right)$$

$$\hat{H}|\Psi_{\boldsymbol{q}}\rangle = E(\boldsymbol{q})\ |\Psi_{\boldsymbol{q}}\rangle \longrightarrow \text{3 x 3 matrix to diagonalize for each } \boldsymbol{q}$$



General shape of the spectrum

Hofstadter, 1976

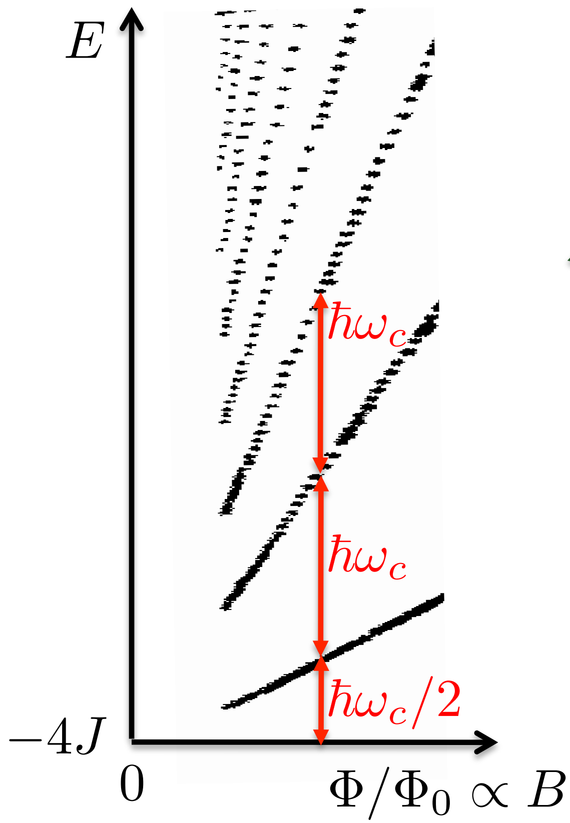


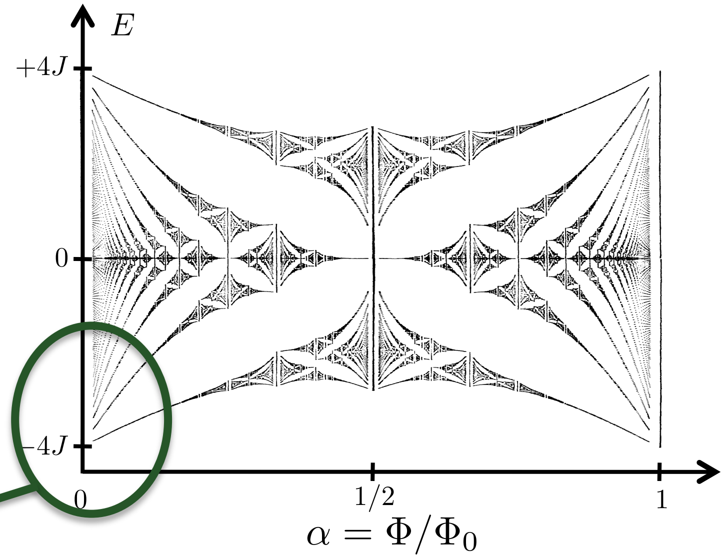
Fractal structure, self -similarity

Dirac points for α = 1/2, 1/4, ...

Landau levels recovered

Hofstadter, 1976





Very narrow equidistant levels

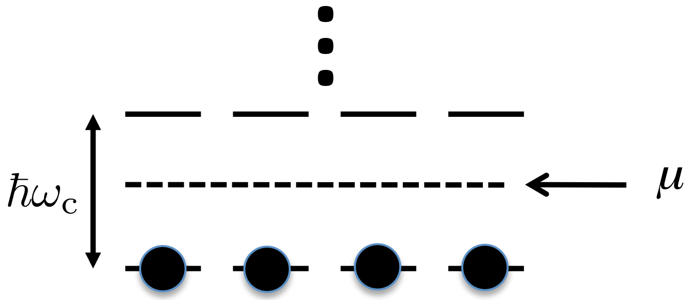
$$E_n = (n + 1/2)\hbar\omega_{\rm c}$$

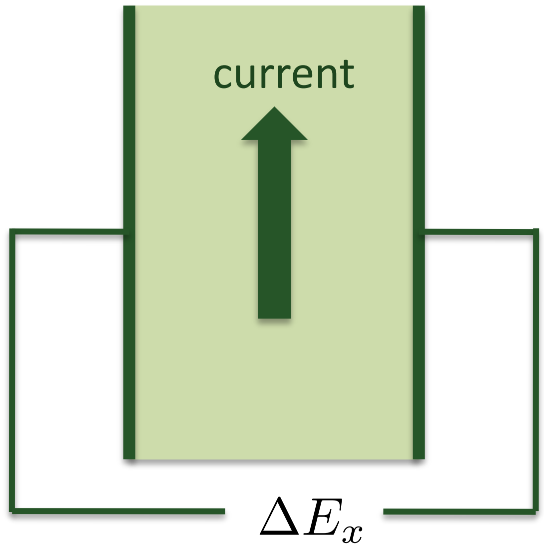
with $\omega_{\rm c} = qB/M_{\rm eff}$

For the square lattice: $M_{\rm eff} = \dfrac{\hbar^2}{2Ja^2}$

Hall conductance of a filled band (fermions)

Consider for example an electron gas with a chemical potential $\frac{\hbar\omega_c}{2} < \mu < \frac{3\hbar\omega_c}{2}$





Transverse current: $\dot{N}_y = \frac{1}{h}\Delta E_x$ corresponding to a single channel conduction

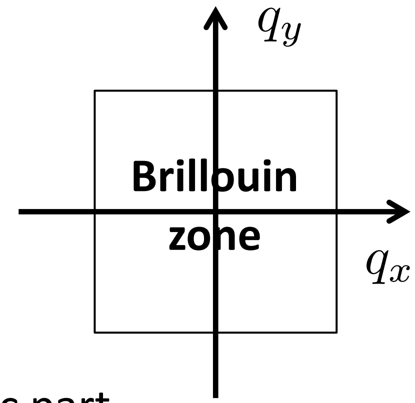
Generalization to insulator-type filling of any band structure: $\dot{N}_y = \frac{C}{h}\Delta E_x$

where C is an integer (Chern number) [Thouless et al, 1982]

What does one get for the lowest band of the Hofstadter butterfly?

$\alpha = 1/p \quad \Rightarrow \quad C = 1$ "topologically equivalent to LLL"

How to calculate the Chern number?



Suppose that only the lowest band is filled

Bloch functions $\psi_{\boldsymbol{q}}(\boldsymbol{r}) = \mathrm{e}^{\mathrm{i}\boldsymbol{q}\cdot\boldsymbol{r}}\, u_{\boldsymbol{q}}(\boldsymbol{r})$ $u_{\boldsymbol{q}}(\boldsymbol{r})$ or $|u_{\boldsymbol{q}}\rangle$: periodic part

Thouless et al (1982): linear response theory (Kubo formula) to calculate the current J_y for a given ΔE_x

Berry connection in quasi-momentum space: $\mathcal{A}(\boldsymbol{q}) = \mathrm{i}\,\langle u_{\boldsymbol{q}}|\boldsymbol{\nabla}_{\boldsymbol{q}} u_{\boldsymbol{q}}\rangle$

Associated Berry curvature: $\mathcal{B}(\boldsymbol{q}) = \boldsymbol{\nabla}_{\boldsymbol{q}} \times \mathcal{A}(\boldsymbol{q})$

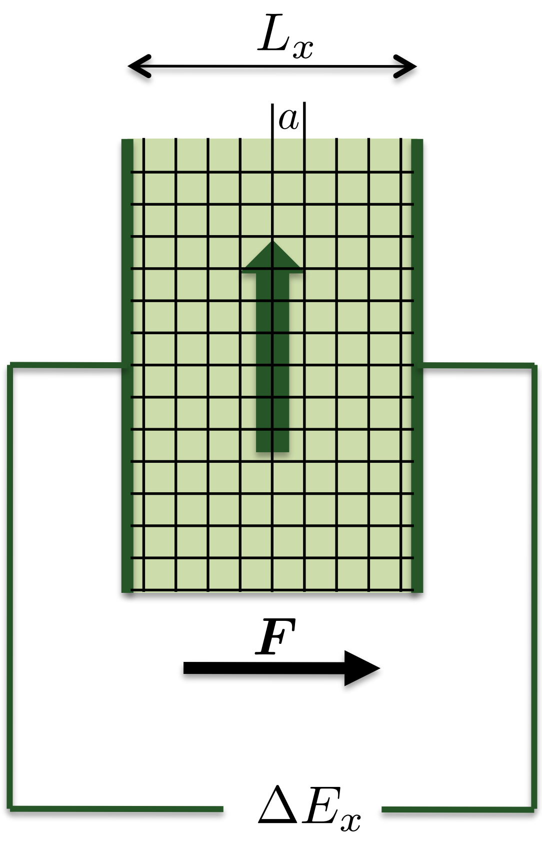
$$C = \frac{1}{2\pi}\int_{\mathrm{BZ}} \mathcal{B}_z(\boldsymbol{q})\ \mathrm{d}^2 q = \frac{1}{2\pi}\oint \boldsymbol{\mathcal{A}}(\boldsymbol{q})\cdot \mathrm{d}\boldsymbol{q}$$

Integer number: robust property that can be changed only with a contact between two bands

In cold atom physics, the Berry curvature $\boldsymbol{\mathcal{B}}(\boldsymbol{q})$ can be measured locally thanks to the Bloch oscillation technique

Physical interpretation of the Chern number

$$\dot{N}_y = C \Delta E_x / h$$



Apply a force F associated with the energy difference $\Delta E_x = F L_x$

Bloch oscillations along x with the period

$$\tau_\mathrm{B} = \frac{h}{Fa}$$

During one Bloch period, how many particles do cross a segment of length a ?

$$\begin{aligned} \delta N &= a J_y \tau_\mathrm{B} \\ &= a \, \frac{1}{L_x} \left(\frac{C}{h} \Delta E_x \right) \left(\frac{h}{Fa} \right) \\ &= C \qquad \text{Chern number} \end{aligned}$$

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How to obtain non real tunnel matrix elements

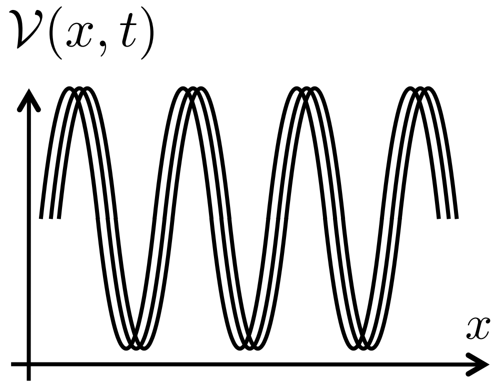
Non exhaustive presentation. There are also approaches based on "rotating" the lattice: Sorensen et al (2005), Tung et al (2006), Hemmerich et al (2007), Kitagawa et al (2010)

Goldman, Juzeliunas, Ohberg, Spielman, arXiv:1308.6533

4. Lattices combining several internal states

Goal of this approach

Lattice modulated in time



One wants to use the different parameters of the modulation

- amplitude
- frequency
- phase

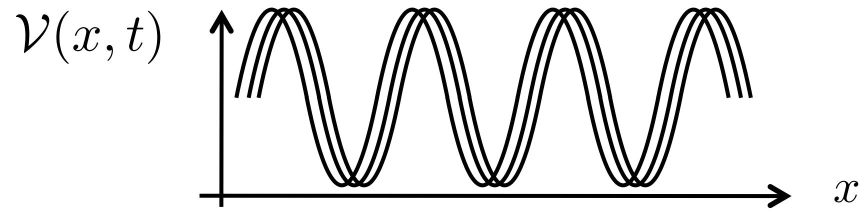
to engineer the tunnel matrix elements $J \longrightarrow J\,\mathrm{e}^{\mathrm{i}\theta}$

Systematic approaches to this type of problem:

- Floquet formalism
- Effective Hamiltonian: Rahav et al (2003); Goldman & Dalibard, arXiv 1404.4373

Here, we will use a simple treatment based on
a (non-rigourous) time-average of the Schrödinger equation

Modulated 1D lattice



$$\mathcal{V}(x,t) = V[x - x_0(t)]$$

Unitary transform $\hat{U}(t) = \exp\left(\mathrm{i}x_0(t)\hat{p}/\hbar\right)$ that allows one to go from

$\hat{H}(t) = \dfrac{\hat{p}^2}{2M} + V[x - x_0(t)]$ to $\hat{H}(t) = \dfrac{[\hat{p} - A(t)]^2}{2M} + V(x)$ with $A(t) = M\dot{x}_0(t)$

Tight-binding approach: $\hat{H}(t) = -J\, \mathrm{e}^{\mathrm{i}Ma\dot{x}_0(t)/\hbar} \sum_j |w_{j+1}\rangle\langle w_j| + \mathrm{h.c.}$

The Schrödinger equation for the state vector $|\Psi(t)\rangle = \sum_j \alpha_j(t)\, |w_j\rangle$ leads to

$$\mathrm{i}\,\hbar\,\dot{\alpha}_j = -J\left(\alpha_{j+1}\,\mathrm{e}^{-\mathrm{i}\,Ma\dot{x}_0(t)/\hbar} + \alpha_{j-1}\,\mathrm{e}^{+\mathrm{i}\,Ma\dot{x}_0(t)/\hbar}\right)$$

Two time scales: fast motion $\dot{x}_0(t)$, slow motion $J/\hbar$. A temporal average then gives:

$\mathrm{i}\,\hbar\,\dot{\alpha}_j = -\bar{J}^*\alpha_{j+1} - \bar{J}\alpha_{j-1}$ average matrix element: $\bar{J} = J\,\langle \mathrm{e}^{\mathrm{i}\,Ma\dot{x}_0(t)/\hbar}\rangle$

Changing the amplitude of the tunnel coefficient

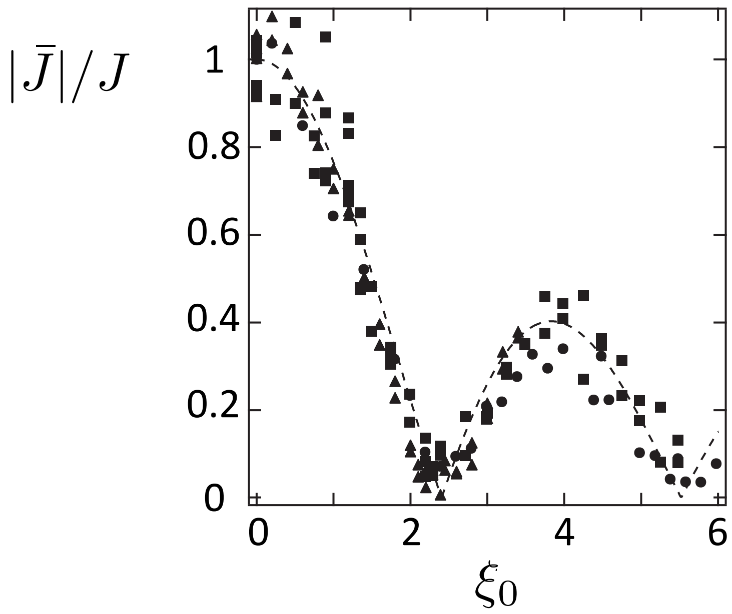
$$\bar{J} = J\ \langle \mathrm{e}^{\mathrm{i}\, Ma\dot{x}_0(t)/\hbar} \rangle$$

Eckardt, Weiss, Holthaus (2005)

Experiment in Arimondo's group (Pisa, 2007) : sine modulation

$$\frac{Ma}{\hbar}\dot{x}_0(t) = \xi_0\ \sin(\Omega t + \phi) \longrightarrow \bar{J} = J\langle \mathrm{e}^{\mathrm{i}\,\xi_0\ \sin(\Omega t+\phi)} \rangle = J\ \mathcal{J}_0(\xi_0)$$

Bessel function

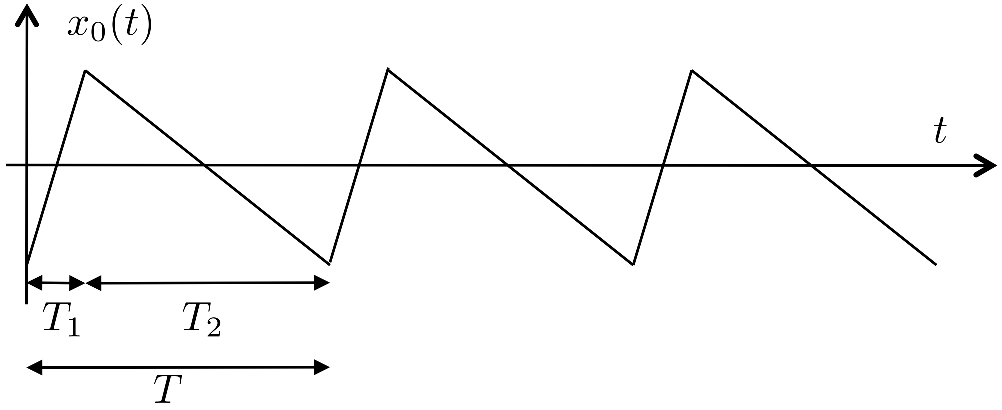


Allows one to change the sign of the tunnel coefficient, but not its phase

Changing the phase of the coefficient tunnel (version 1)

$$\bar{J} = J \, \langle \mathrm{e}^{\mathrm{i}\, M a \dot{x}_0(t)/\hbar} \rangle$$

Take a non-symmetric temporal modulation

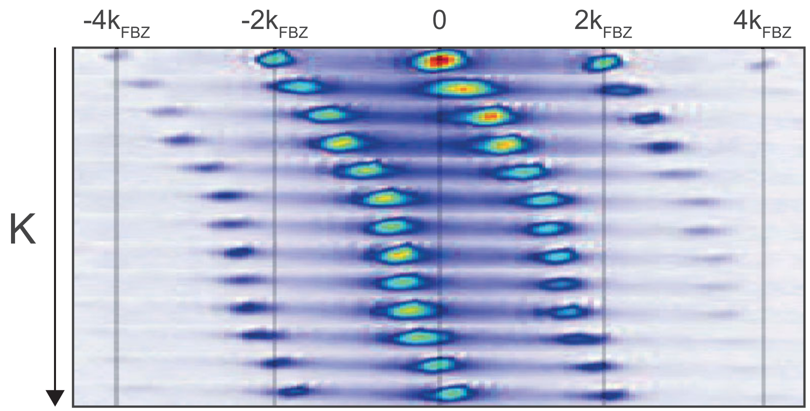


$$\frac{\bar{J}}{J} = \frac{T_1}{T} \mathrm{e}^{\mathrm{i} M a v_1/\hbar} + \frac{T_2}{T} \mathrm{e}^{\mathrm{i} M a v_2/\hbar}$$

generally non real

$$v_1 T_1 + v_2 T_2 = 0$$

Equivalent of a constant vector potential: shift of the minimum of the dispersion relation that can be measured in a time of flight experiment



Struck et al, Hamburg 2012

Changing the phase of the tunnel coefficient (version 2)

Back to a sine modulation $\frac{Ma}{\hbar}\dot{x}_0(t) = \xi_0 \sin(\Omega t + \phi)$

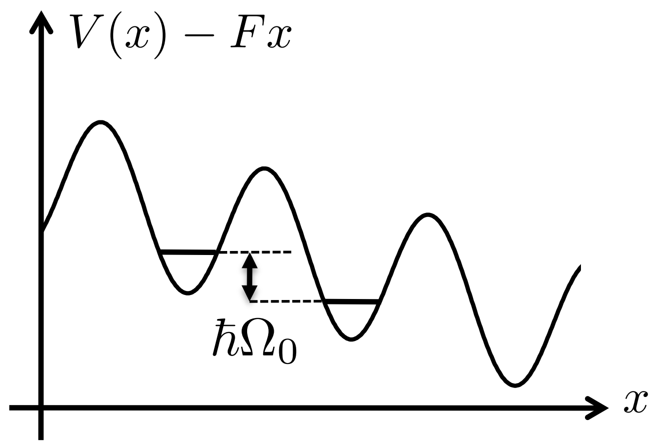
Try to print the phase ϕ of the modulation on the tunnel coefficient

- Until now we saw $\bar{J} = J\ \mathcal{J}_0(\xi_0)$, which does not work.
- But one can also produce $\bar{J} = J\ \mathcal{J}_1(\xi_0)\ \mathrm{e}^{\mathrm{i}\phi}$ using a resonance !

Step one: superimpose a uniform force F to the lattice.

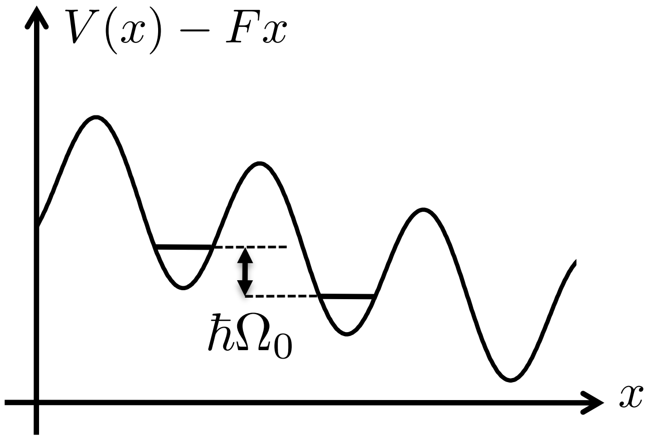
Energy offset of two adjacent sites $\hbar\Omega_0$

Step two: modulate at $\Omega \sim \Omega_0$



Kolovsky (2011)

Use a resonance (continued)



Look for the state vector as

$$|\Psi(t)\rangle = \sum_j \alpha_j(t)\, \mathrm{e}^{\mathrm{i}j\Omega_0 t}\, |w_j\rangle$$

Schrödinger equation

$$\mathrm{i}\,\hbar\,\dot{\alpha}_j = -J\left(\alpha_{j+1}\,\mathrm{e}^{-\mathrm{i}\,(Ma\dot{x}_0(t)/\hbar-\Omega_0 t)} + \alpha_{j-1}\,\mathrm{e}^{+\mathrm{i}\,(Ma\dot{x}_0(t)/\hbar-\Omega_0 t)}\right)$$

For $\frac{Ma}{\hbar}\dot{x}_0(t) = \xi_0\,\sin(\Omega t + \phi)$ and $\Omega = \Omega_0$, the average tunnel coefficient is :

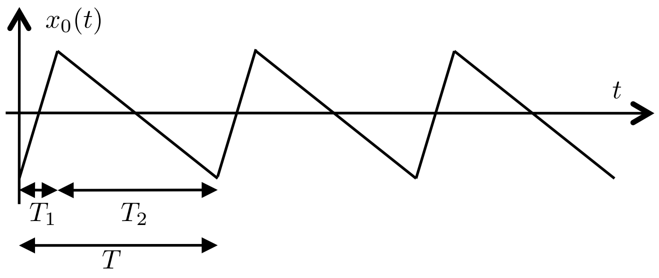
$$\bar{J} = J\langle \mathrm{e}^{\mathrm{i}[\xi_0 \sin(\Omega t+\phi)-\Omega_0 t]}\rangle \quad = J\,\langle \sum_n \mathcal{J}_n(\xi_0)\mathrm{e}^{\mathrm{i}n(\Omega t+\phi)}\;\mathrm{e}^{-\mathrm{i}\Omega_0 t}\rangle$$

$$\bar{J} = J\,\mathcal{J}_1(\xi_0)\;\mathrm{e}^{\mathrm{i}\phi}$$

OK for this 1D model (mere gauge transform)

Going to two dimensions

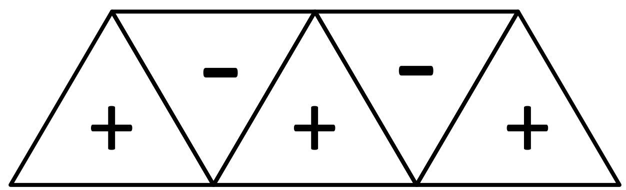
Non-symmetric modulation



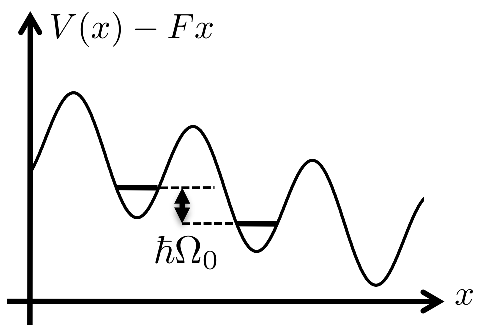
Can be used for a triangular lattice
(does not work if the sides of the unit cell are parallel)

Staggered flux

Hamburg 2013



Resonant modulation



Can be adapted to a square lattice to produce a uniform flux, with some subtleties ...

Munich 2013, MIT 2013