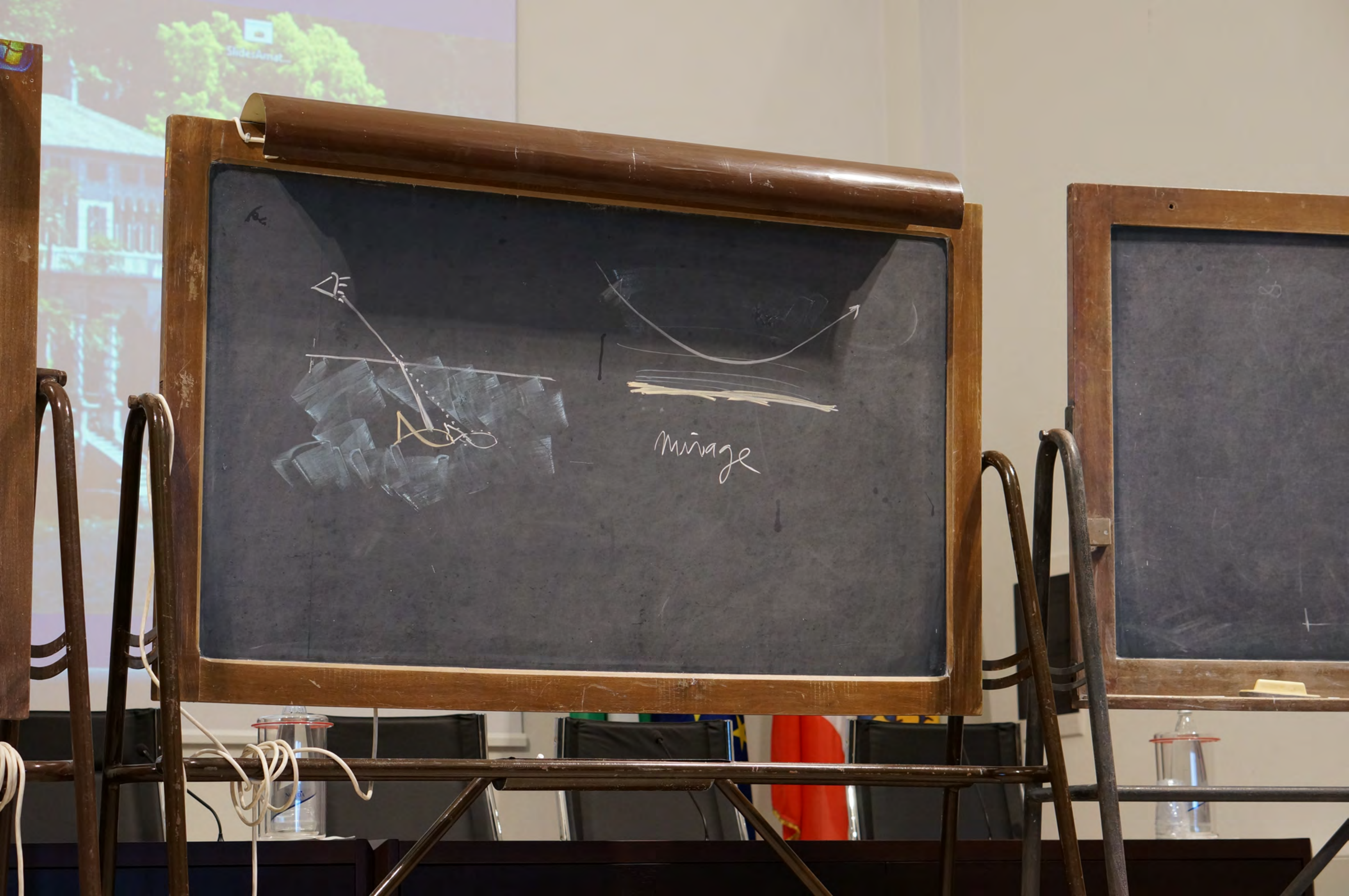
Prof. Ulf Leonhardt

“The Geometry of Light”

Slides of the Lecture

Wednesday 2 July 2014

International School of Physics Enrico Fermi
Varenna, Lake Como
Course 190 -Frontiers in Modern Optics


mirage


C
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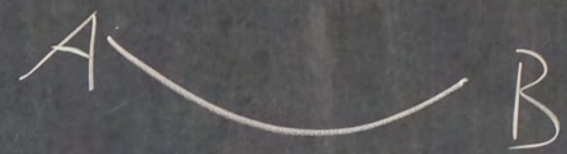
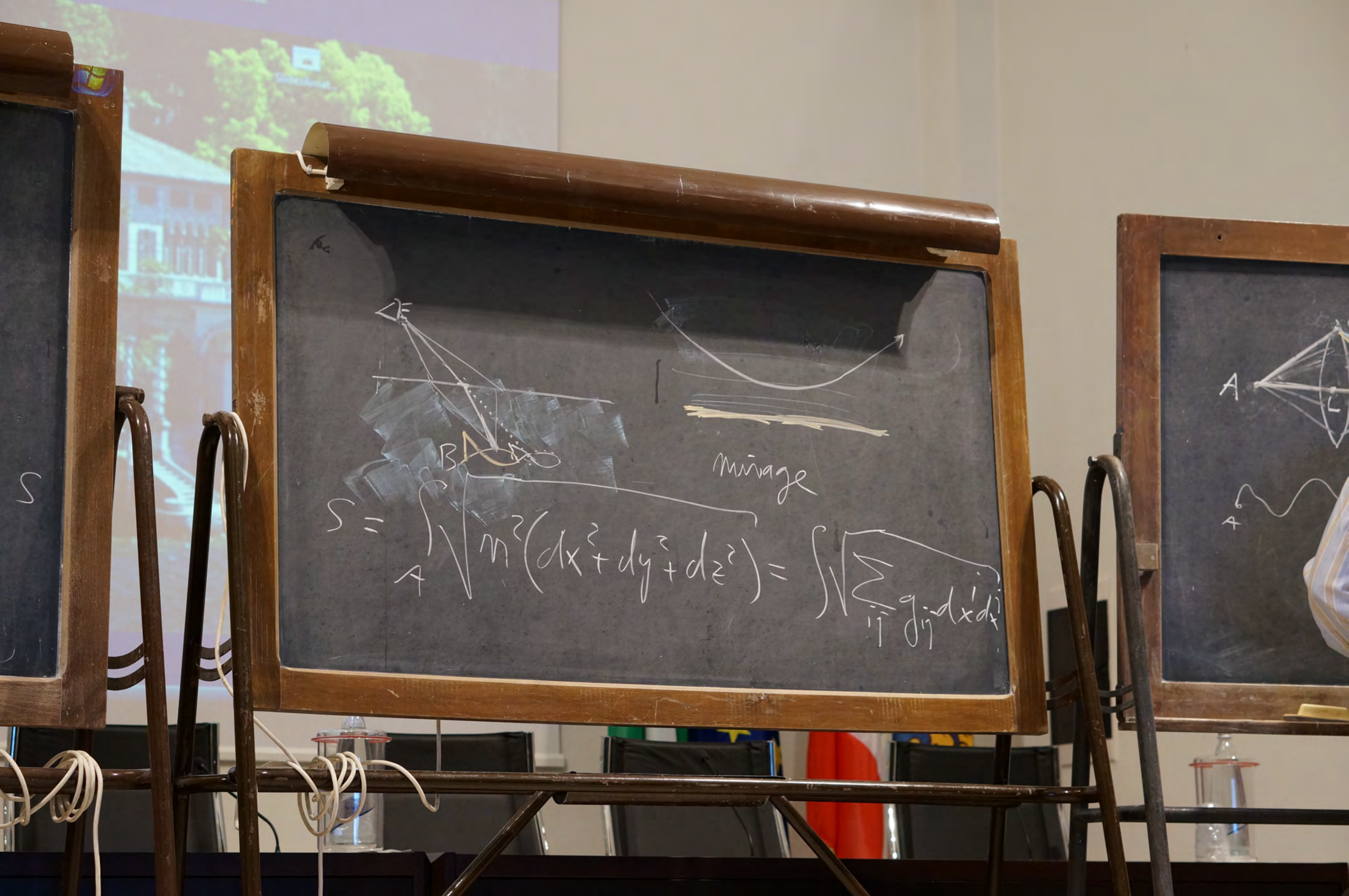
GEOMETRY & LIGHT

Dover, 2010

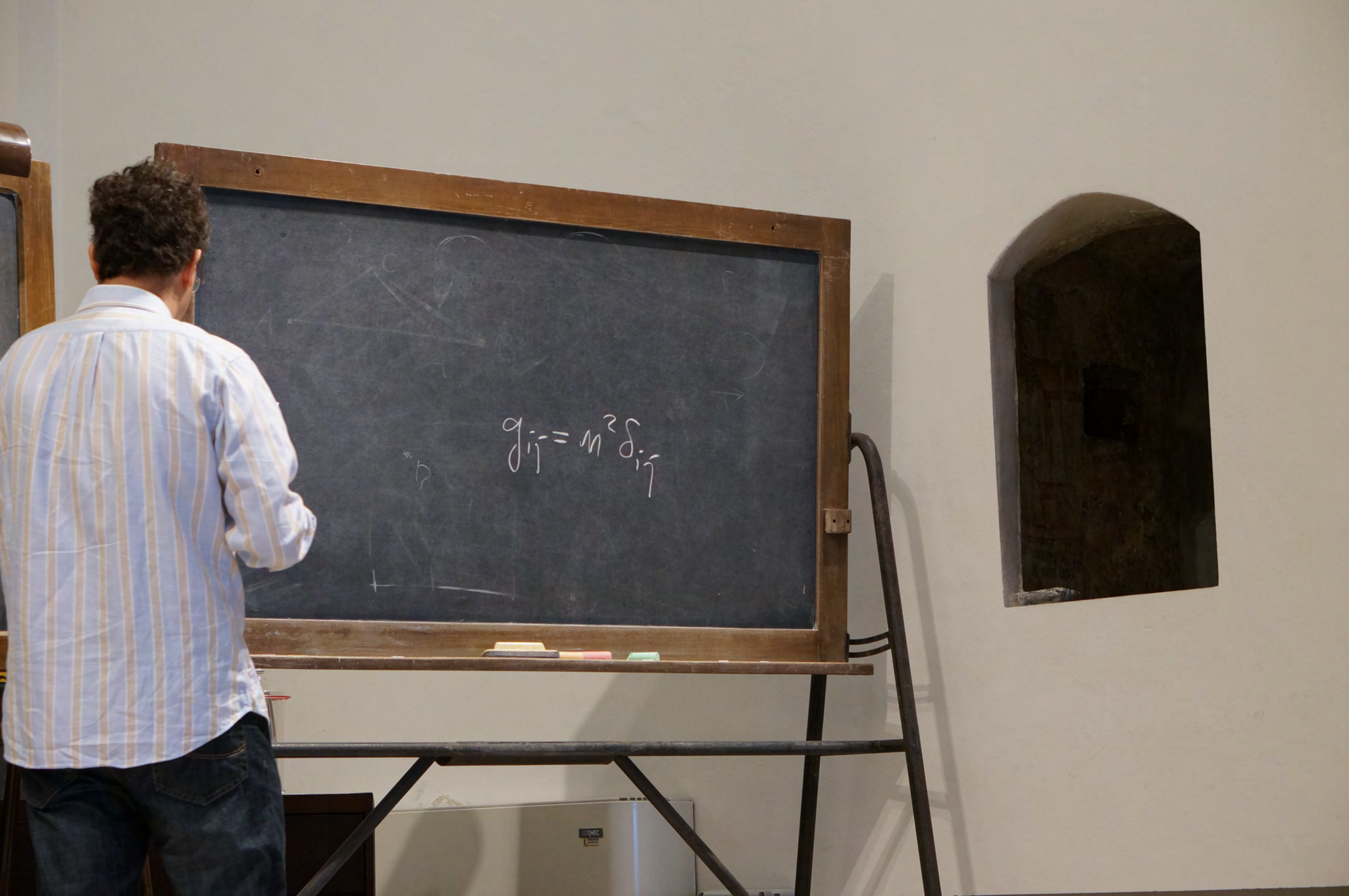
1662 Fermat's principle



$$\text{extremum} = \int_A^B n\,dl = \int_A^B n\sqrt{dx^2+dy^2+dz^2}$$



mirage
$S = \int_A \sqrt{n^2(dx^2+dy^2+dz^2)} = \int \sqrt{\sum_{ij} g_{ij} dx^i dx^j}$


$g_{ij} = n^2 \delta_{ij}$

2. Spatial geometries

$$ds^2 = \sum_{ij} g_{ij}\, dx^i dx^j \qquad s = \int ds$$

g_{ij} : metric tensor, x^i : coordinates

$x^1 = x,\ x^2 = y,\ x^3 = z$ Cartesian

$x^1 = r,\ x^2 = \vartheta,\ x^3 = \phi$

$$\sum_{ij} g_{ij}\, dx^i dx^j$$

$$ds^2 = dx^2 + dy^2 + dz^2$$

$$= dr^2 + r^2 d\vartheta^2 + r^2 \sin^2\vartheta \, d\phi^2$$

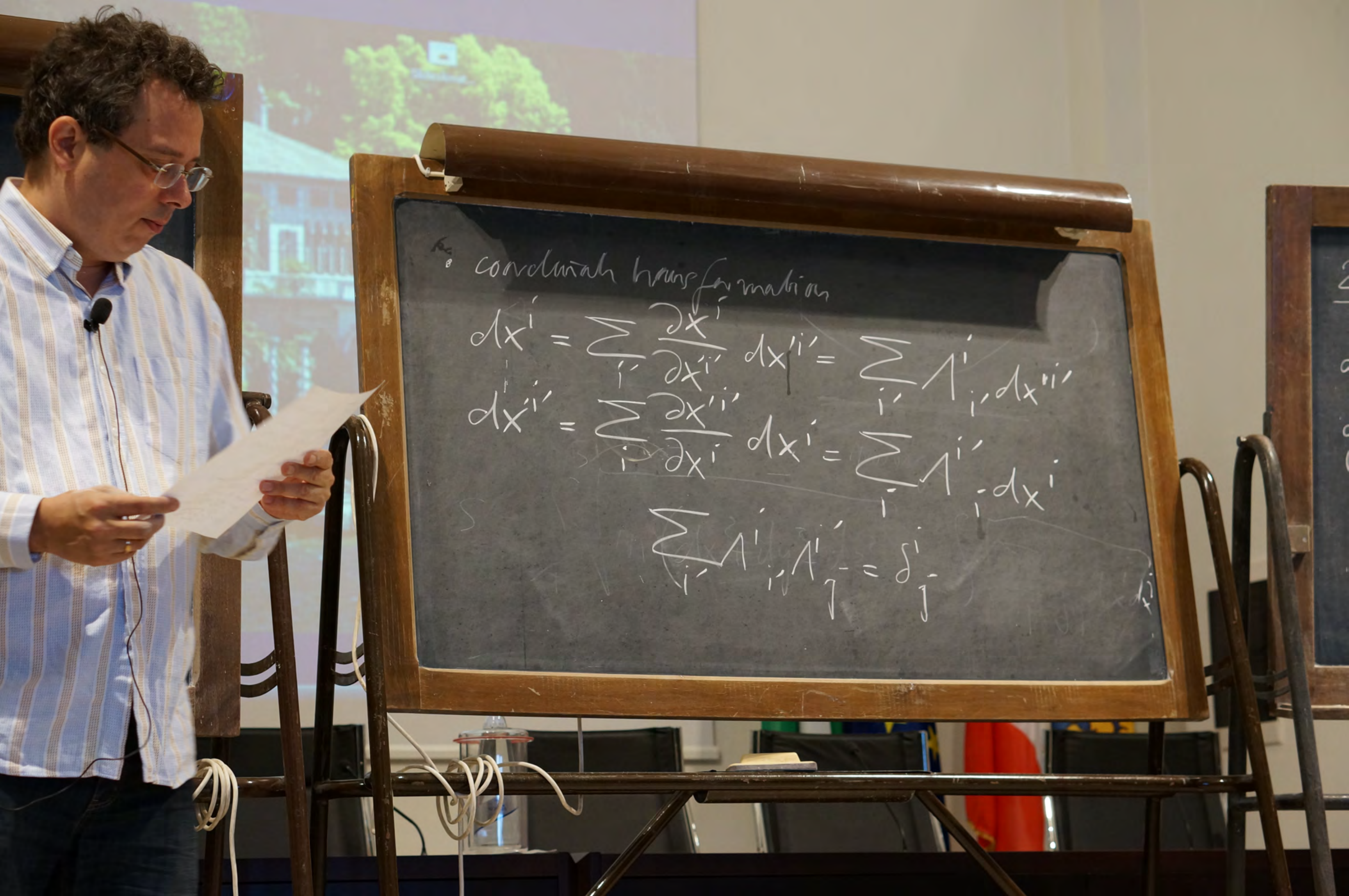
$$g_{ij} = \mathrm{diag}\left(1, r^2, r^2 \sin^2\vartheta\right)$$

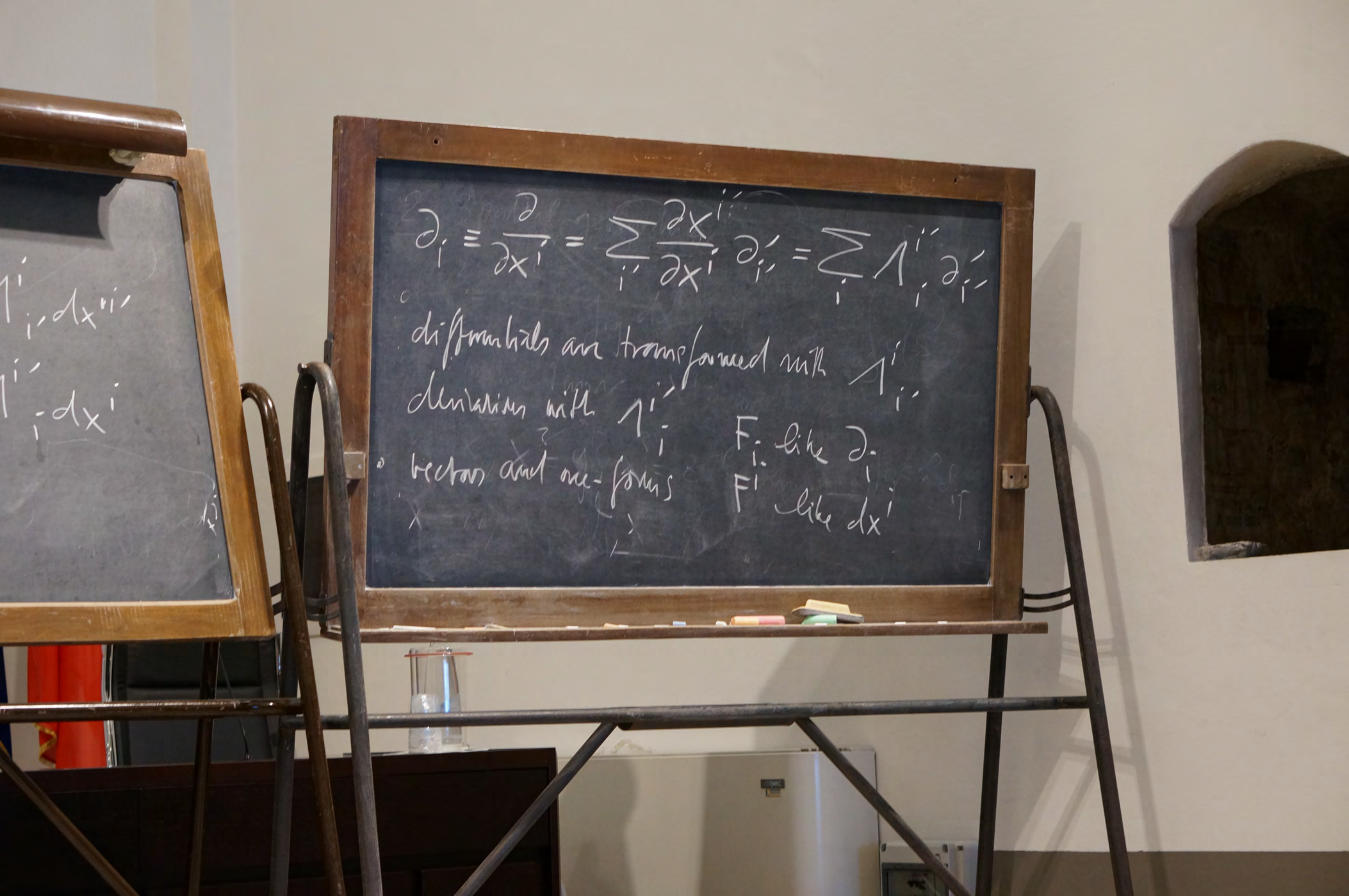
$$g^{ij} \equiv \left(g_{ij}\right)^{-1} \text{ matrix inverse}$$

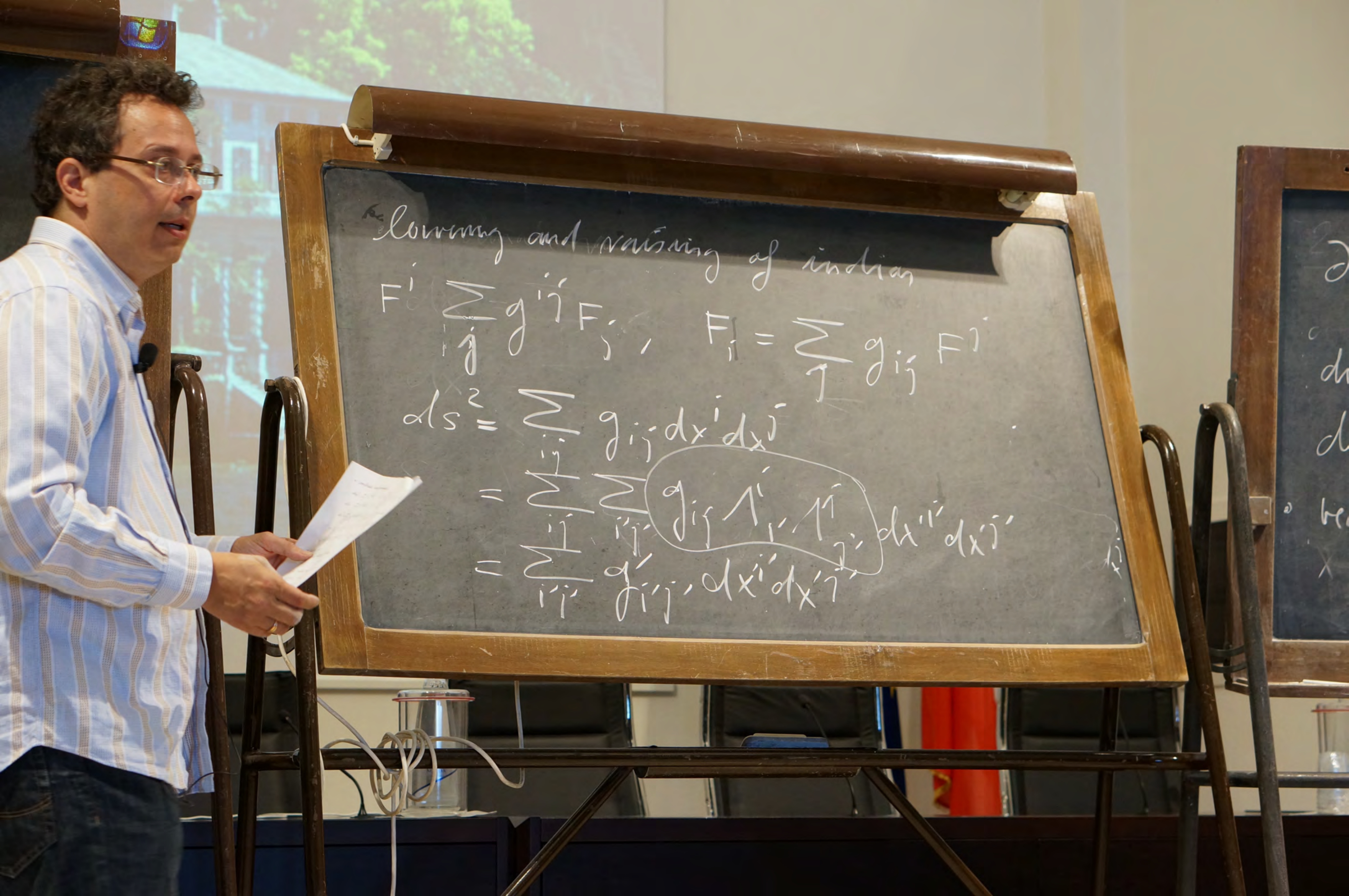
$$g^{ij} = \mathrm{diag}\left(1, \frac{1}{r^2}, \frac{1}{r^2 \sin^2\vartheta}\right)$$

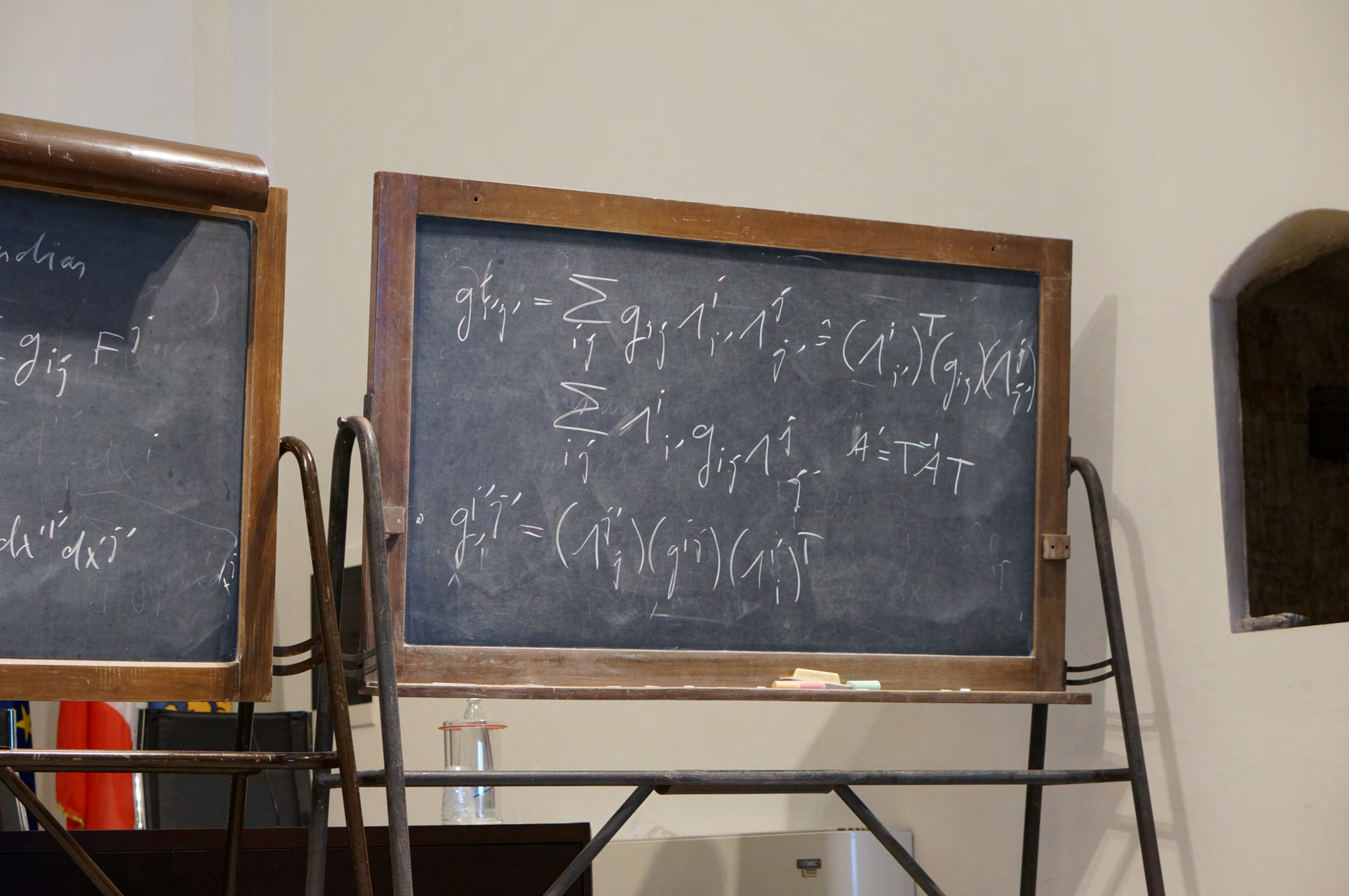
$$g = \det(g_{ij})$$

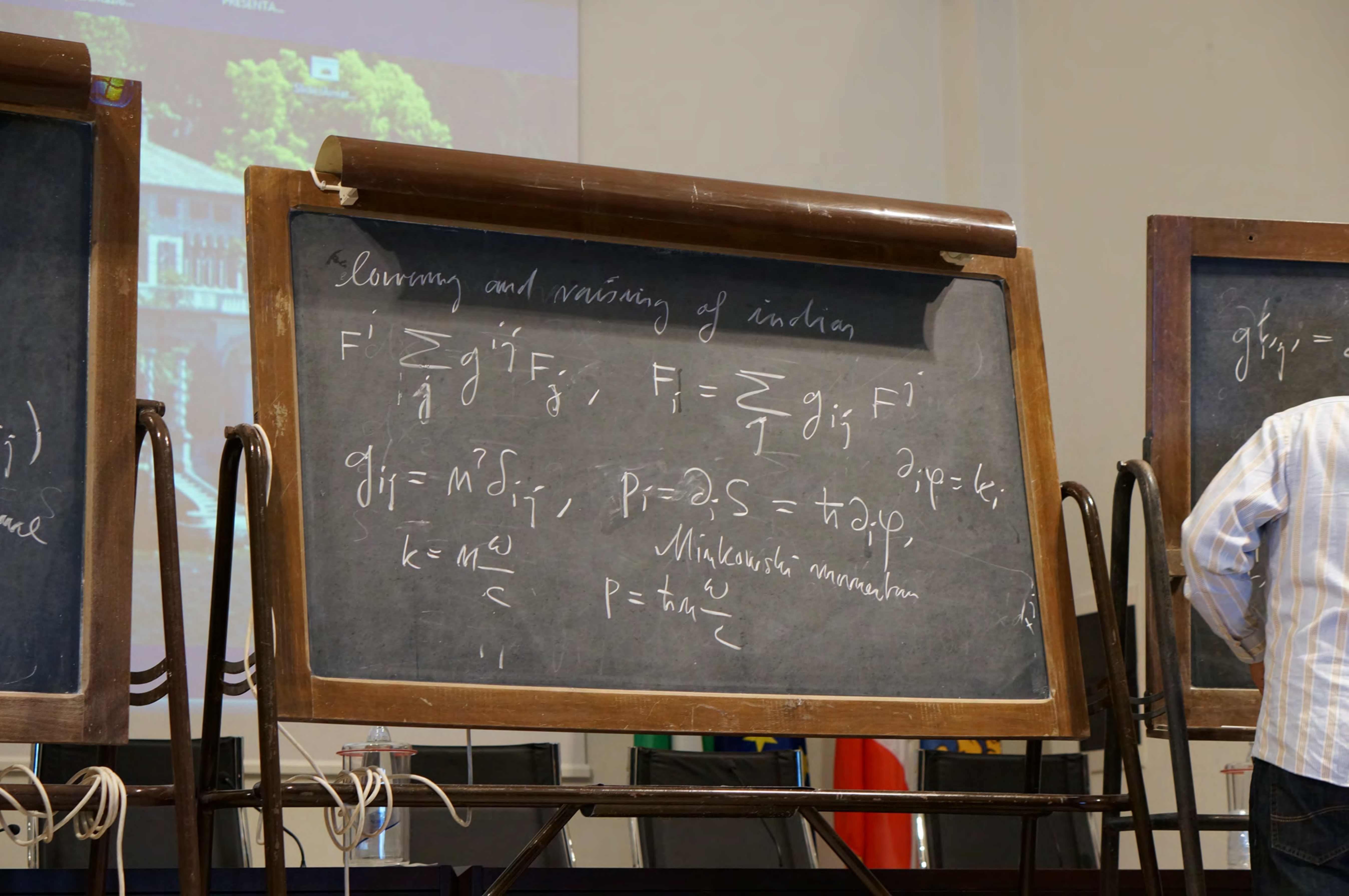
$\sqrt{g}$: 3D volume element

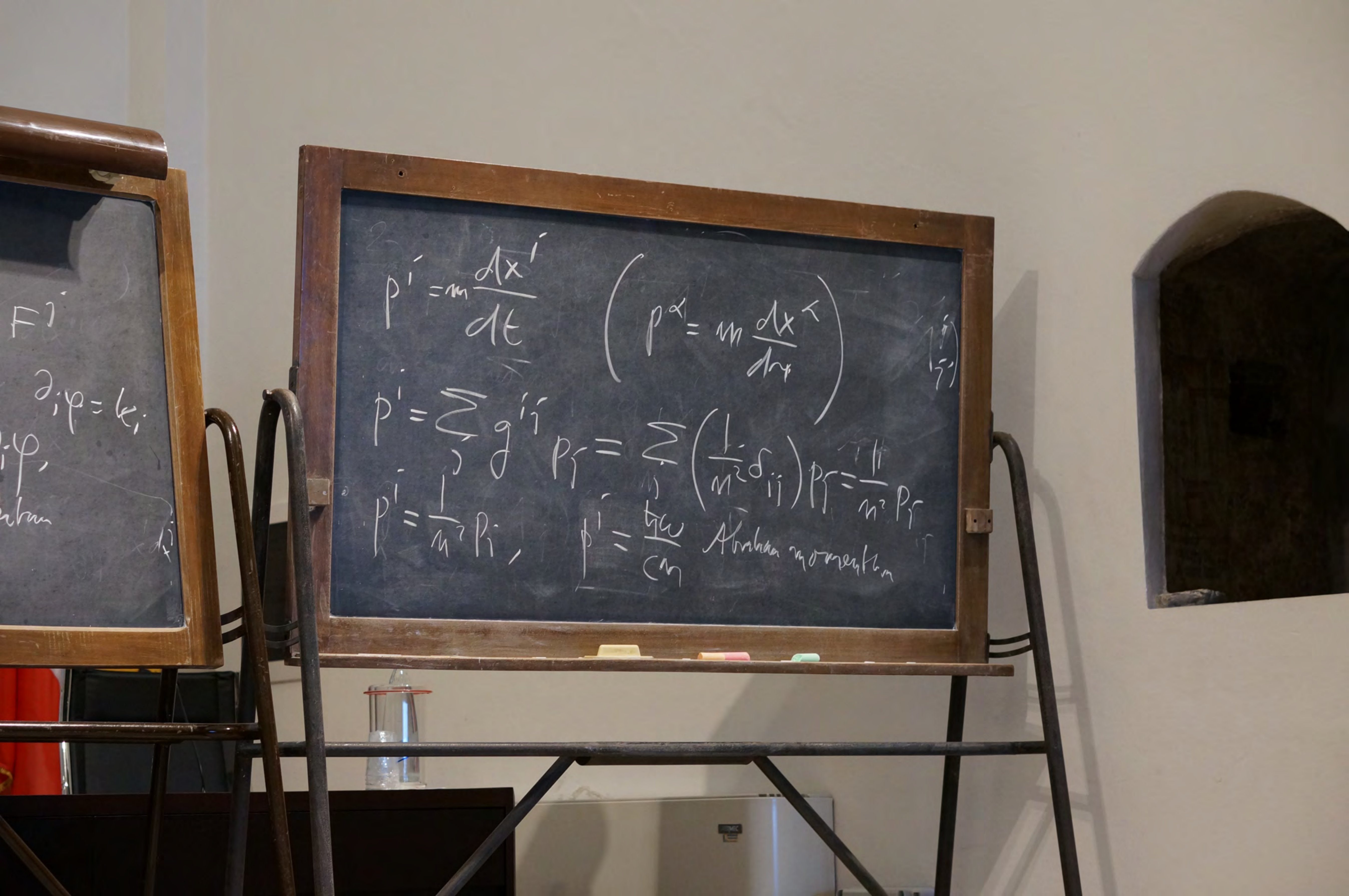

coordinate transformation
$dx^i = \sum_{i'} \frac{\partial x^i}{\partial x'^{i'}} dx'^{i'} = \sum_{i'} \Lambda^i{}_{i'} dx'^{i'}$
$dx'^{i'} = \sum_i \frac{\partial x'^{i'}}{\partial x^i} dx^i = \sum_i \Lambda^{i'}{}_i dx^i$
$\sum_{i'} \Lambda^i{}_{i'} \Lambda^{i'}{}_j = \delta^i_j$


$\partial_i \equiv \frac{\partial}{\partial x^i} = \sum_{i'} \frac{\partial x^{i'}}{\partial x^i} \partial_{i'} = \sum_{i'} \Lambda^{i'}_{\ i} \partial_{i'}$
differentials are transformed with $\Lambda^{i'}_{\ i}$
derivatives with $\Lambda^{i}_{\ i'}$
vectors and one-forms
F_i like ∂_i
F^i like dx^i


lowering and raising of indices
$F^i = \sum_j g^{ij} F_j$, $F_i = \sum_j g_{ij} F^j$
$ds^2 = \sum_{ij} g_{ij} dx^i dx^j$
$= \sum_{ij} \sum_{i'j'} g_{ij} \Lambda^i_{i'} \Lambda^j_{j'} dx^{i'} dx^{j'}$
$= \sum_{i'j'} g_{i'j'} dx^{i'} dx^{j'}$




lowering and raising of indices
$F^i = \sum_j g^{ij} F_j$, $F_i = \sum_j g_{ij} F^j$
$g_{ij} = m^2 \delta_{ij}$, $p_i = \partial_i S = \hbar \partial_i \varphi$, $\partial_i \varphi = k_i$
$k = m \frac{\omega}{c}$
Minkowski momentum
$p = \hbar m \frac{\omega}{c}$


$p^i = m \frac{dx^i}{dt}$
$\left(p^\alpha = m \frac{dx^\alpha}{d\tau} \right)$
$p^i = \sum_j g^{ij} p_j = \sum_j \left(\frac{1}{n^2} \delta_{ij} \right) p_j = \frac{1}{n^2} p_j$
$p^i = \frac{1}{n^2} p_i ,$
$p^i = \frac{\hbar\omega}{cn}$ Abraham momentum

- Maxwell's equations

$$\nabla \cdot \vec{E} = 0, \quad \nabla \times \vec{E} = -\partial_t \vec{B}$$

$$\nabla \cdot \vec{B} = 0, \quad \nabla \times \vec{B} = \frac{1}{c^2} \partial_t \vec{E}$$

from differential geometry

$$\nabla \cdot \vec{E} = \frac{1}{\sqrt{g}} \sum_{ij} \partial_i \sqrt{g}\, g^{ij} E_j$$

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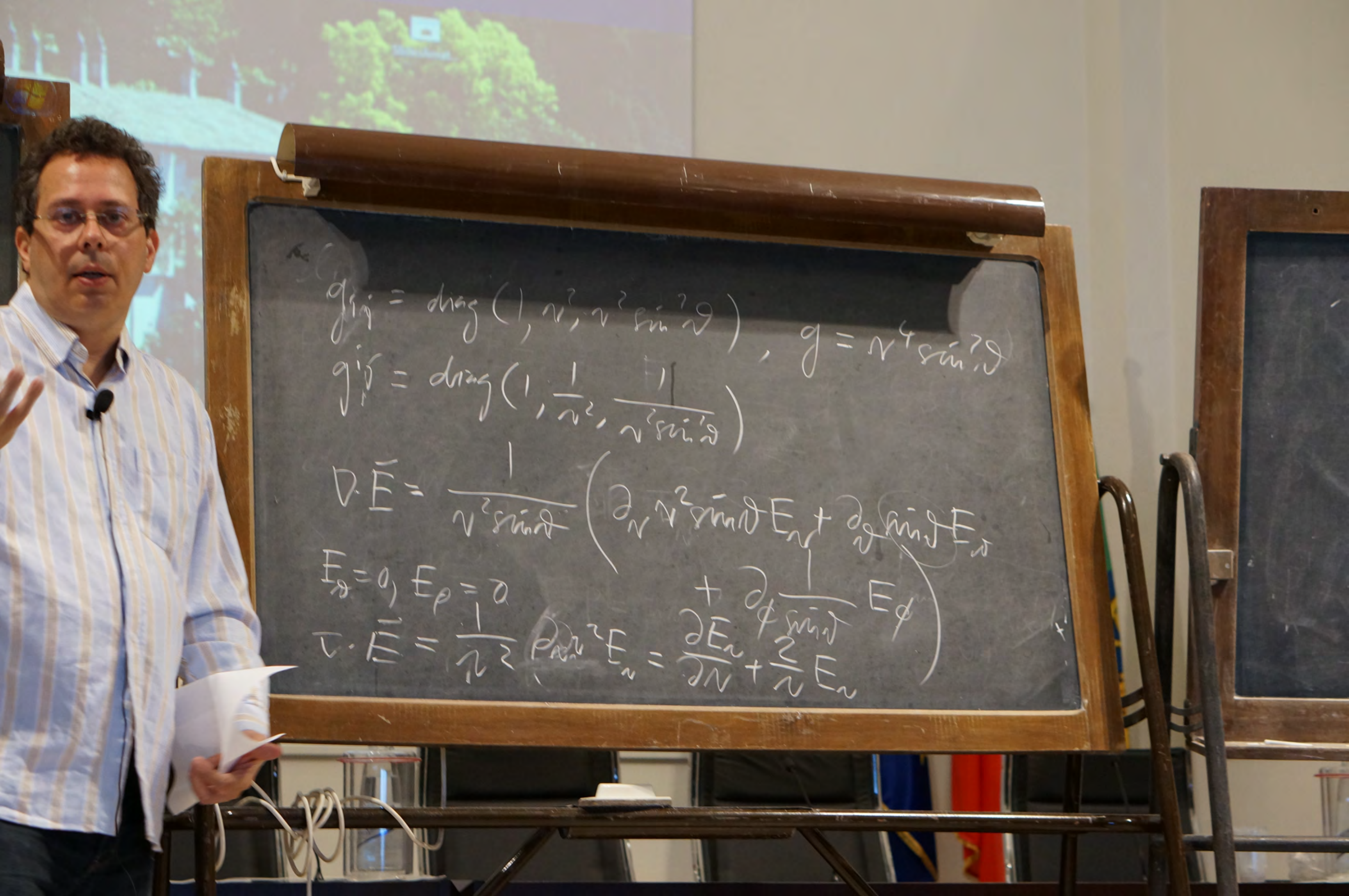
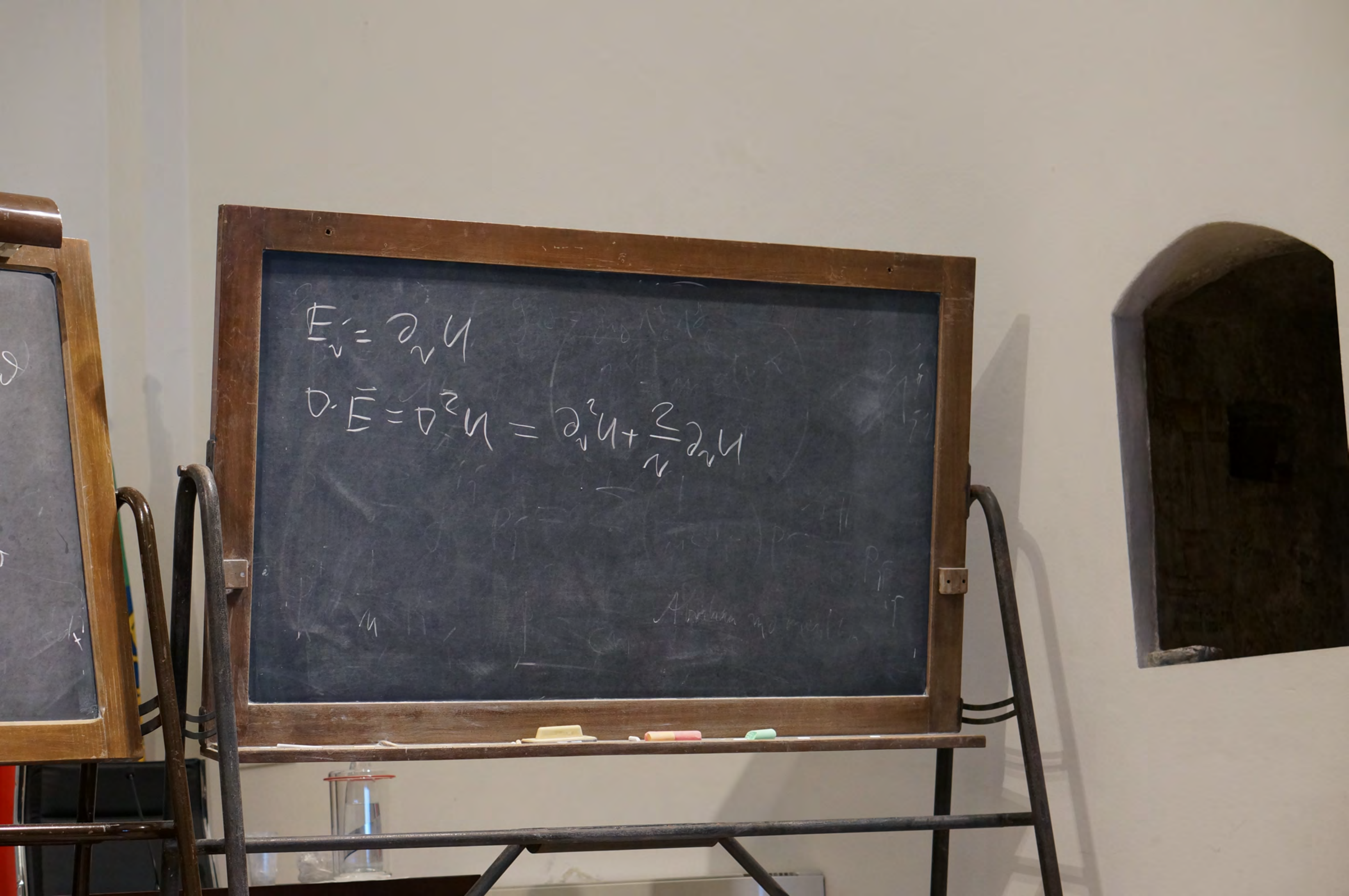
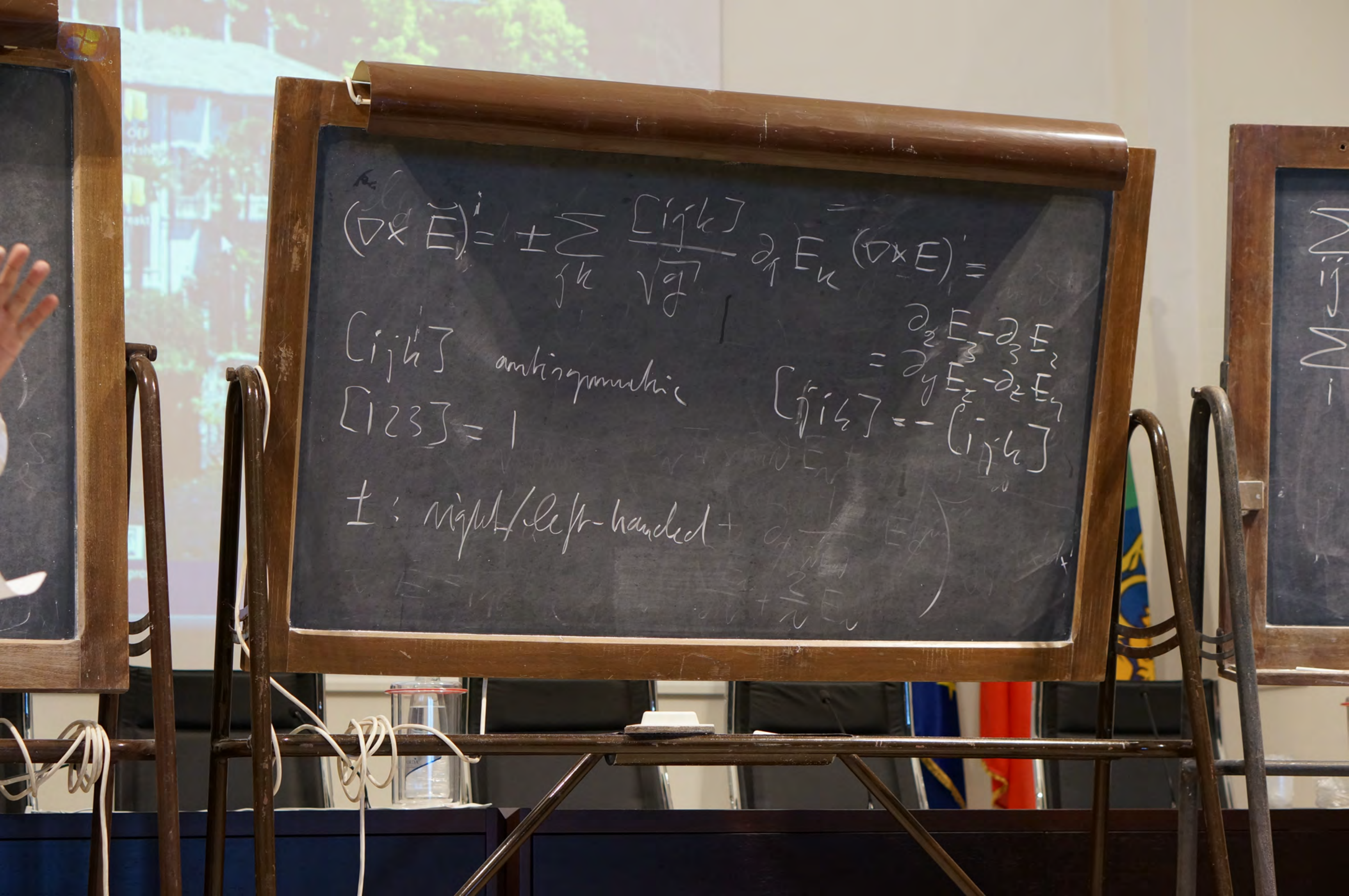


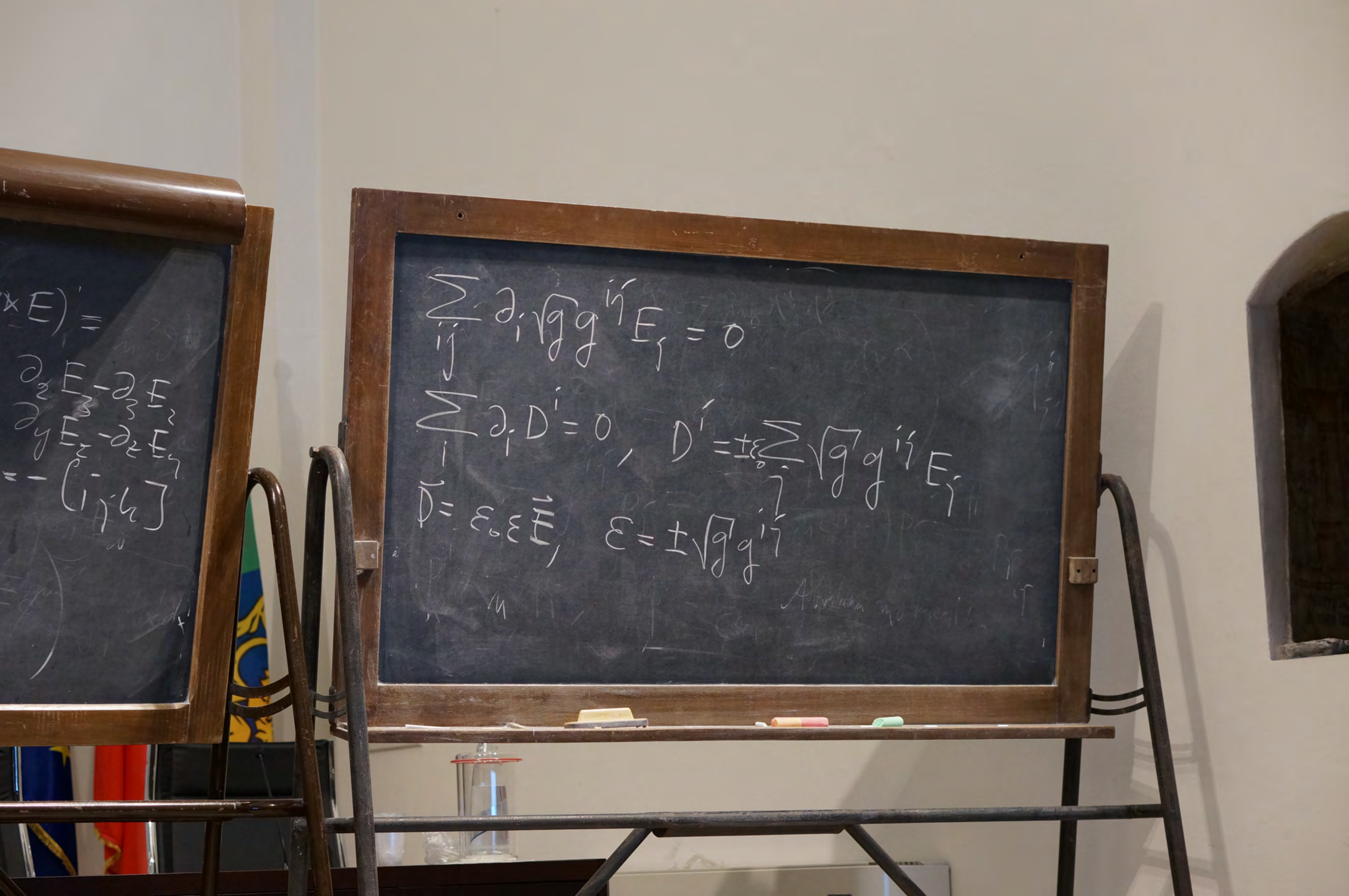
$g_{ij} = \mathrm{diag}(1, r^2, r^2 \sin^2\vartheta)$, $g = r^4 \sin^2\vartheta$
$g^{ij} = \mathrm{diag}\left(1, \frac{1}{r^2}, \frac{1}{r^2 \sin^2\vartheta}\right)$
$\nabla \cdot \vec{E} = \frac{1}{r^2 \sin\vartheta}\left(\partial_r r^2 \sin\vartheta E_r + \partial_\vartheta \sin\vartheta E_\vartheta + \partial_\phi \frac{1}{\sin\vartheta} E_\phi\right)$
$E_\vartheta = 0, E_\phi = 0$
$\nabla \cdot \vec{E} = \frac{1}{r^2} \partial_r r^2 E_r = \frac{\partial E_r}{\partial r} + \frac{2}{r} E_r$



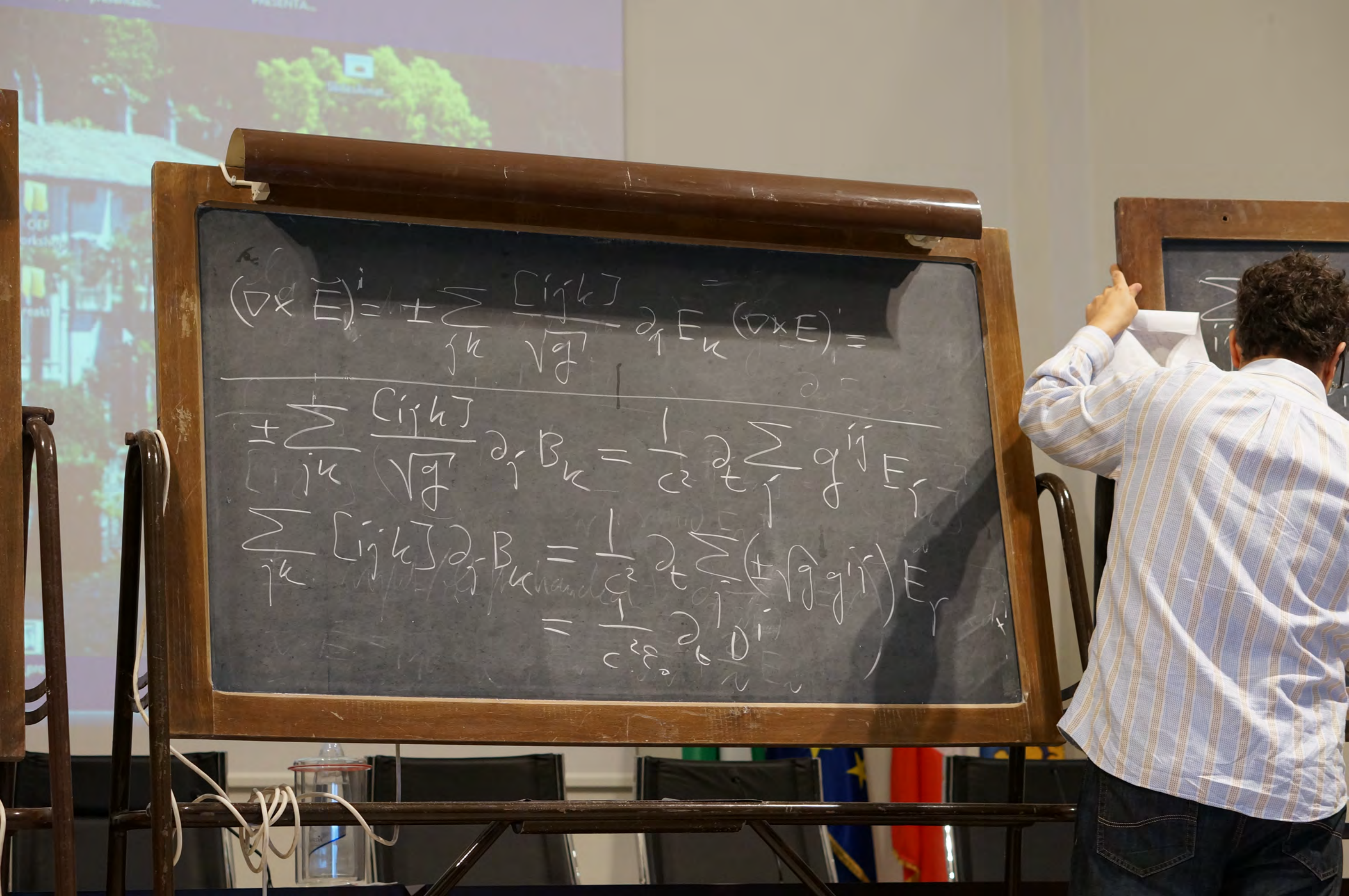
$E_r = \partial_r u$
$\nabla \cdot E = \nabla^2 u = \partial_r^2 u + \frac{2}{r}\partial_r u$

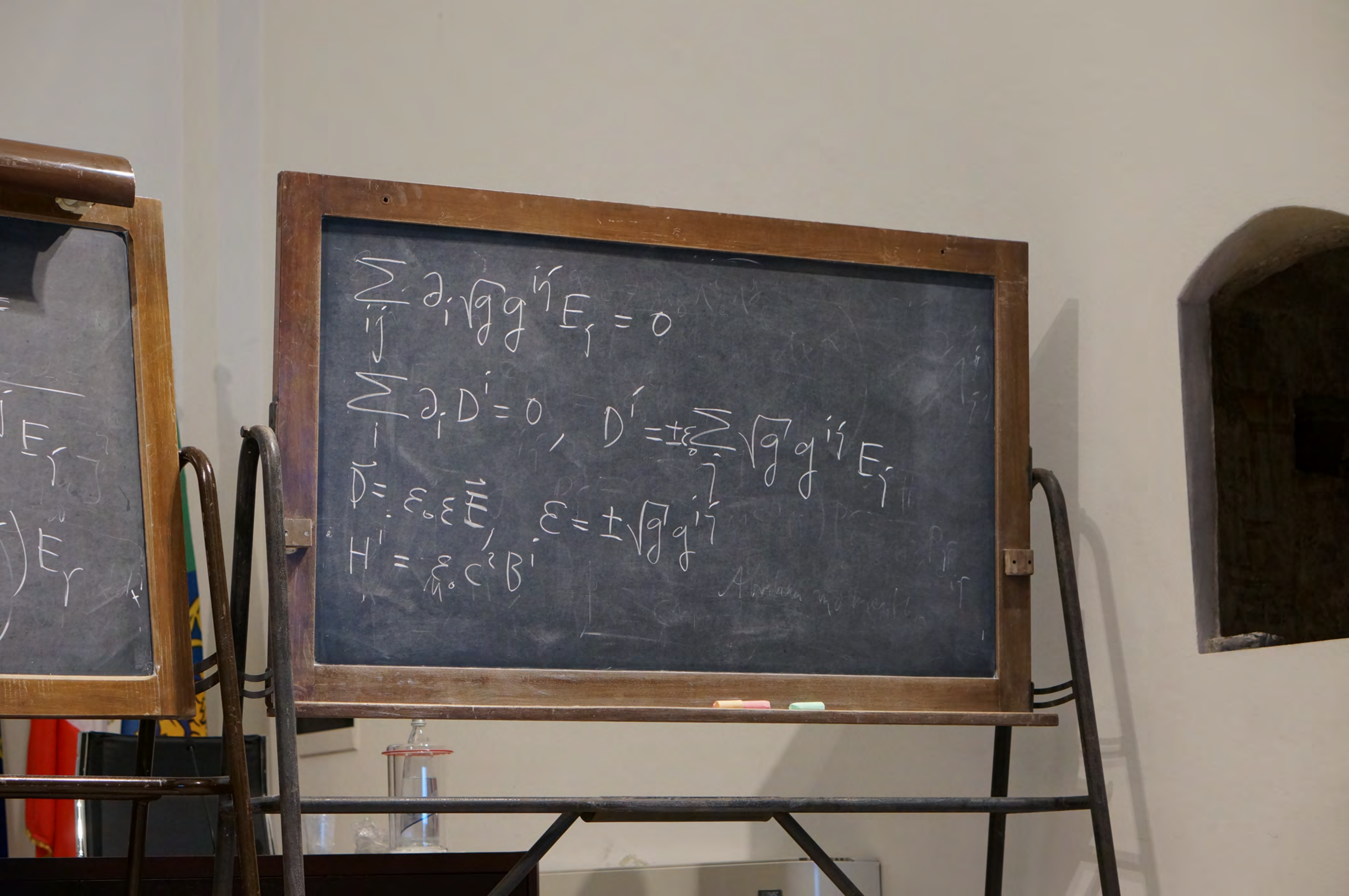


$(\nabla \times E)^i = \pm \sum_{jk} \frac{[ijk]}{\sqrt{g}} \partial_j E_k$
$(\nabla \times E)^1 = \partial_2 E_3 - \partial_3 E_2$
$= \partial_y E_z - \partial_z E_y$
$[ijk]$ antisymmetric
$[123] = 1$
$[jik] = -[ijk]$
$\pm$: right/left-handed

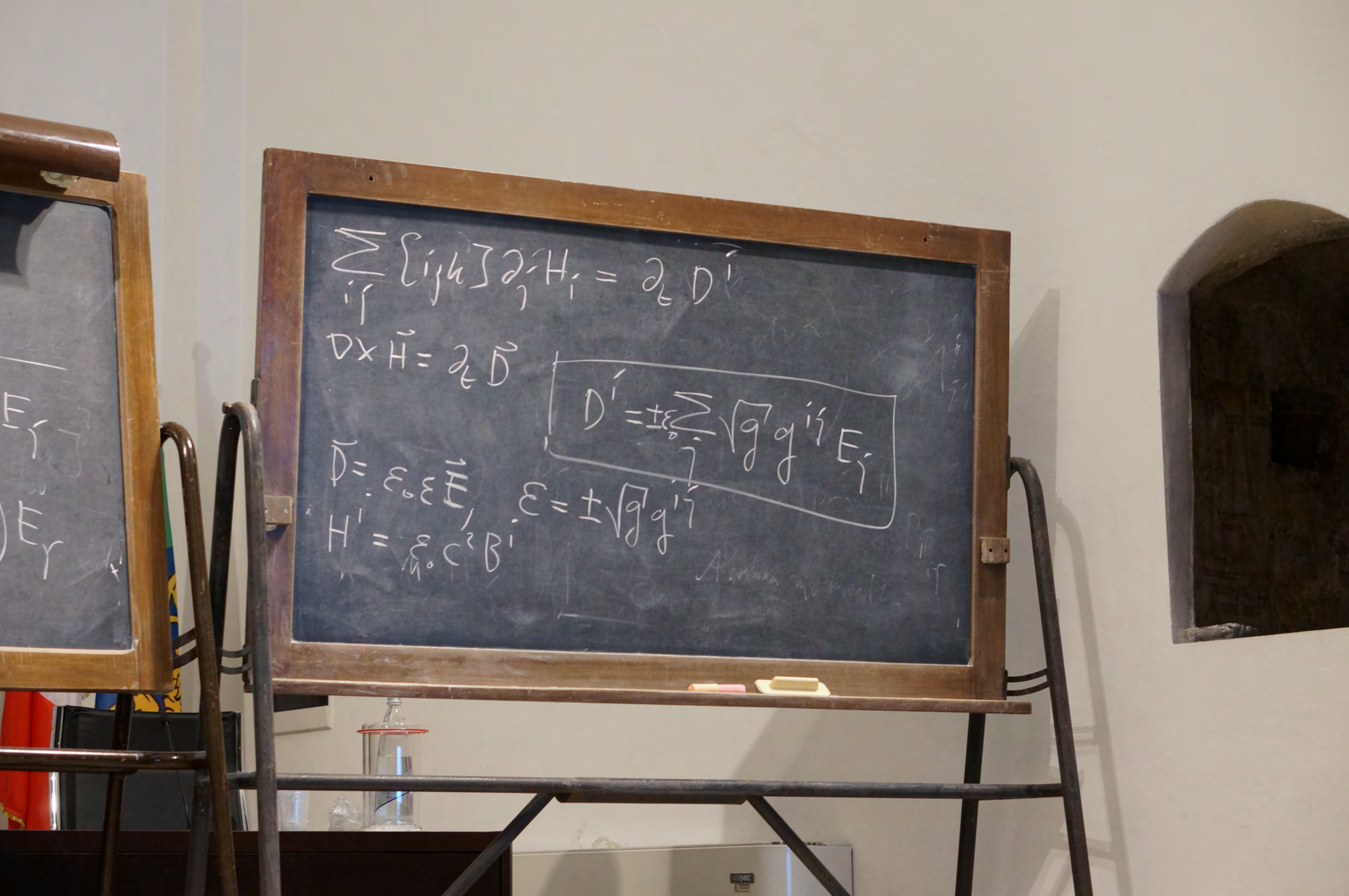


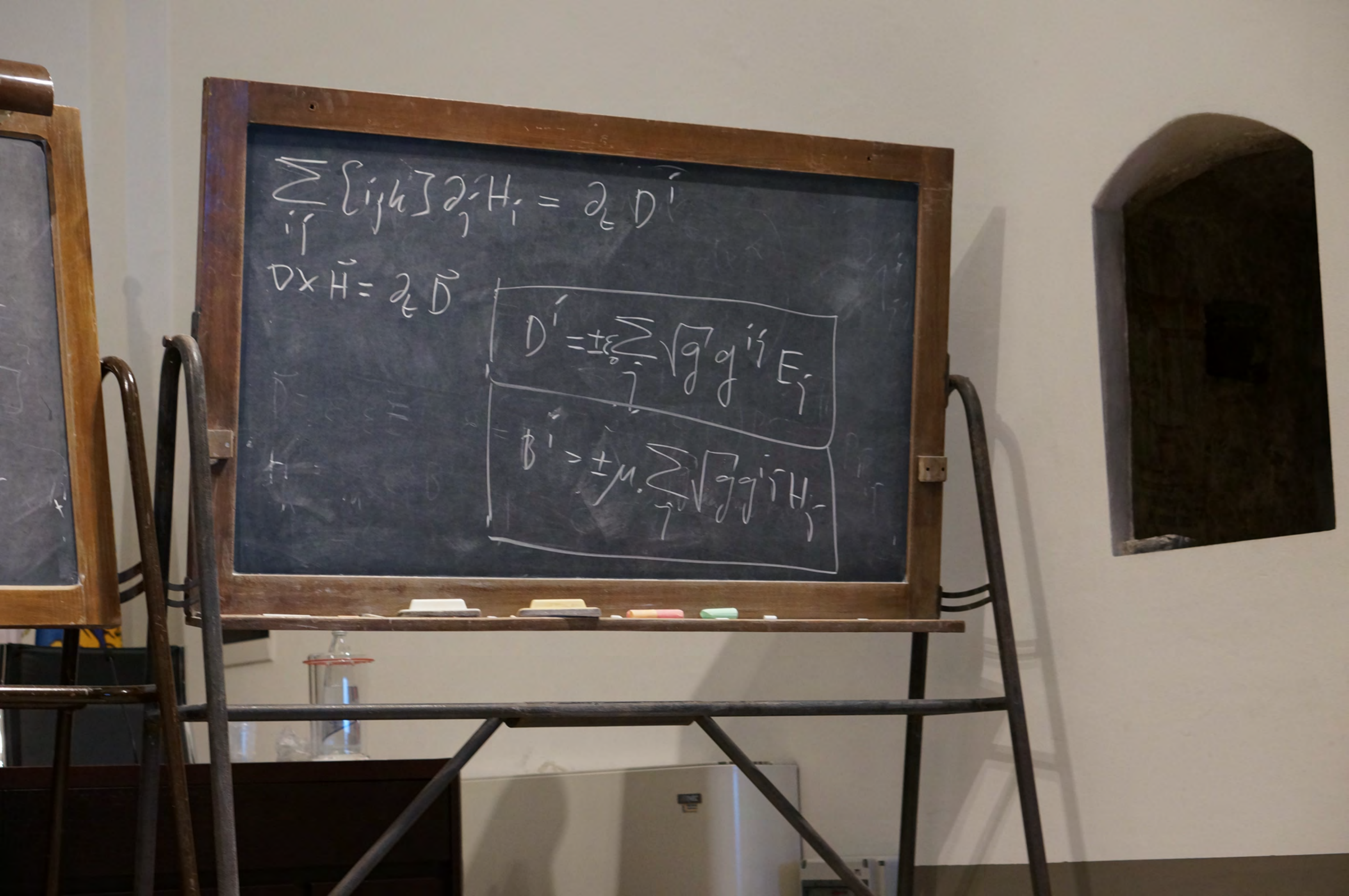
$\sum_{ij} \partial_i \sqrt{g}\, g^{ij} E_j = 0$
$\sum_i \partial_i D^i = 0, \quad D^i = \pm\varepsilon_0 \sum_j \sqrt{g}\, g^{ij} E_j$
$\vec{D} = \varepsilon_0 \varepsilon \vec{E}, \quad \varepsilon = \pm\sqrt{g}\, g^{ij}$


$(\nabla \times E)^i = \pm \sum_{jk} \frac{[ijk]}{\sqrt{g}} \partial_j E_k \quad (\nabla \times E)^i =$
$\pm \sum_{jk} \frac{[ijk]}{\sqrt{g}} \partial_j B_k = \frac{1}{c^2} \partial_t \sum_j g^{ij} E_j$
$\sum_{jk} [ijk] \partial_j B_k = \frac{1}{c^2} \partial_t \sum_j (\pm \sqrt{g}\, g^{ij}) E_j$
$= \frac{1}{c^2 \varepsilon_0} \partial_t D^i$



$\sum_{ij} \partial_i \sqrt{g}\, g^{ij} E_j = 0$
$\sum_i \partial_i D^i = 0, \quad D^i = \pm\varepsilon \sum_j \sqrt{g}\, g^{ij} E_j$
$\vec{D} = \varepsilon_0 \varepsilon \vec{E}, \quad \varepsilon = \pm\sqrt{g}\, g^{ij}$
$H^i = \varepsilon_0 c^2 B^i$


$\sum_{ij} [ijk] \partial_j H_i = \partial_t D^i$
$\nabla \times \vec{H} = \partial_t \vec{D}$
$D^i = \pm \varepsilon_0 \sum_j \sqrt{g}\, g^{ij} E_j$
$\vec{D} = \varepsilon_0 \varepsilon \vec{E}, \quad \varepsilon = \pm \sqrt{g}\, g^{ij}$
$H^i = \varepsilon_0 c^2 B^i$



$\sum_{ij} [ijk]\, \partial_j H_i = \partial_t D^i$
$\nabla \times \vec{H} = \partial_t \vec{D}$
$D^i = \pm\varepsilon_0 \sum_j \sqrt{g}\, g^{ij} E_j$
$B^i = \pm\mu_0 \sum_j \sqrt{g}\, g^{ij} H_j$

Maxwell's equations

$$\nabla \cdot \vec{E} = 0, \quad \nabla \times \vec{E} = -\partial_t \vec{B}$$

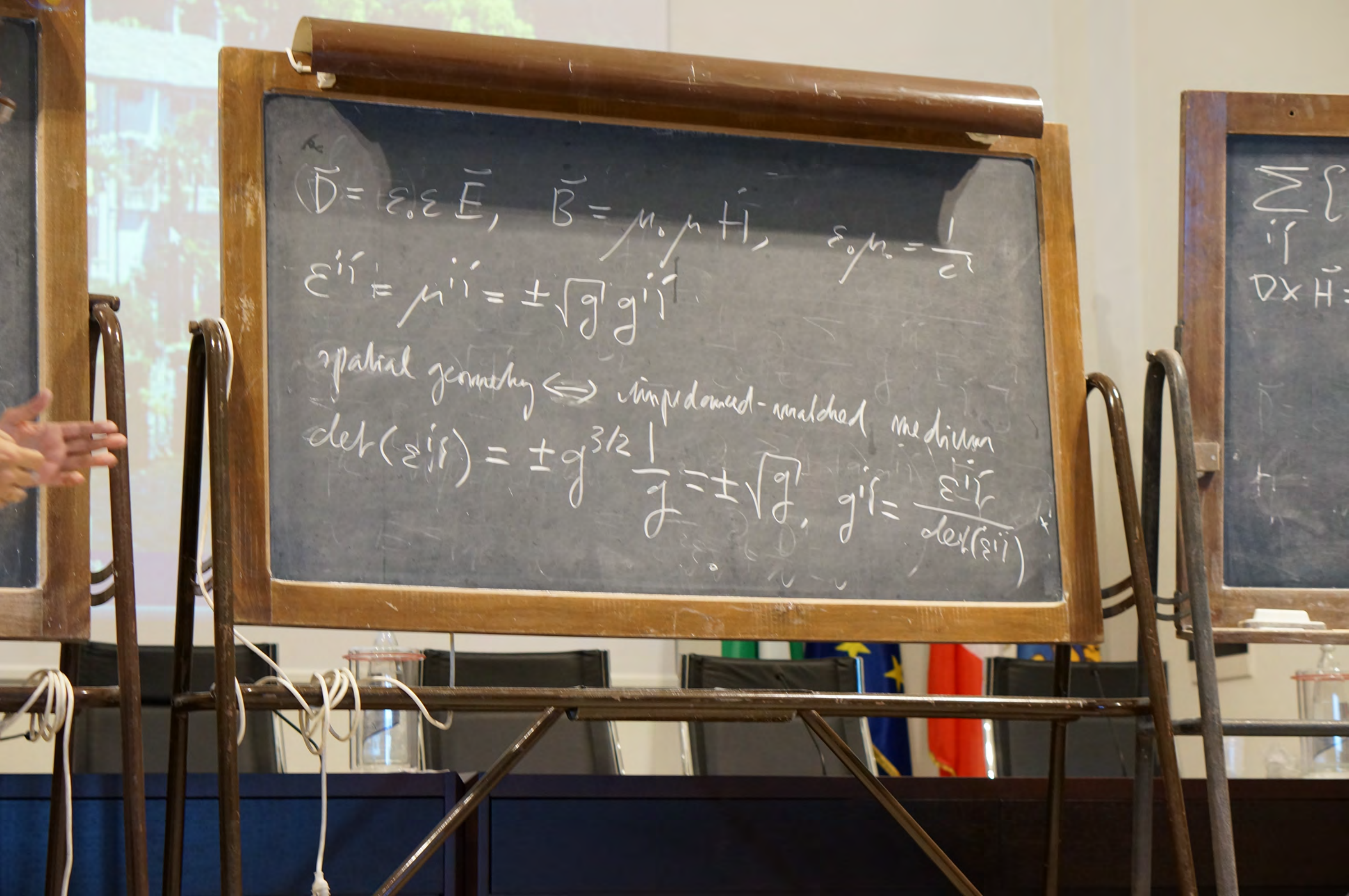
$$\nabla \cdot \vec{B} = 0, \quad \nabla \times \vec{B} = \frac{1}{c^2} \partial_t \vec{E}$$

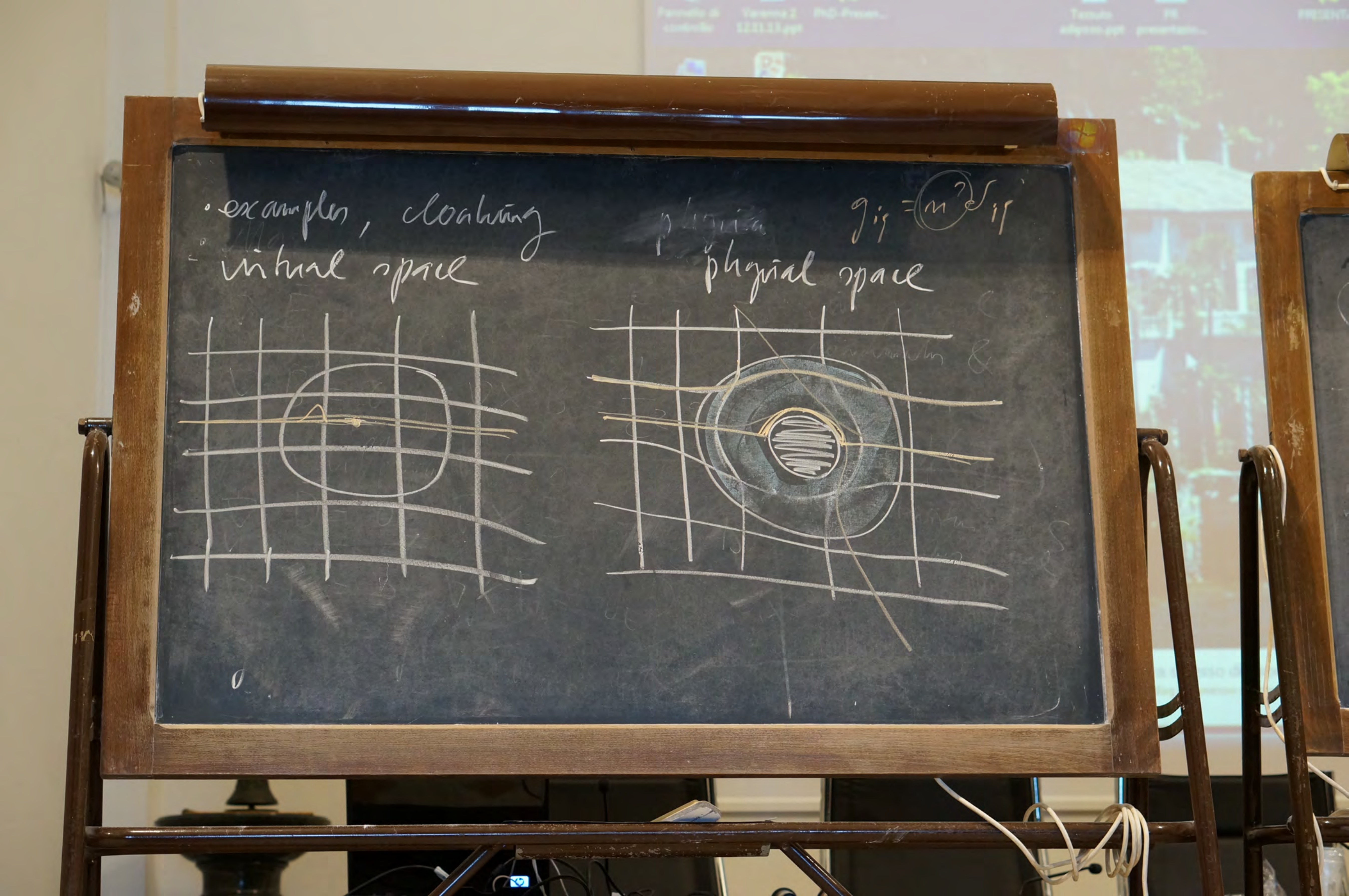
in general coordinates & geometries

$$\nabla \cdot \vec{D} = 0, \quad \nabla \times \vec{E} = -\partial_t \vec{B}$$

$$\nabla \cdot \vec{B} = 0, \quad \nabla \times \vec{H} = \partial_t \vec{D}$$

in Cartesian coordinates

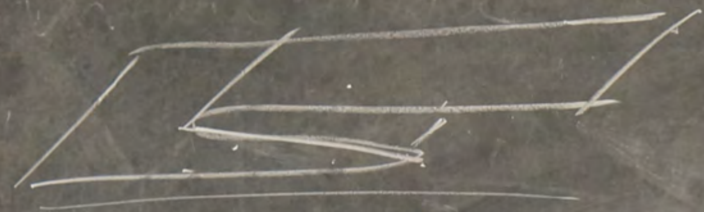

$\vec{D} = \varepsilon_0 \varepsilon \vec{E}$, $\vec{B} = \mu_0 \mu \vec{H}$, $\varepsilon_0 \mu_0 = \frac{1}{c^2}$
$\varepsilon^{ij} = \mu^{ij} = \pm\sqrt{g}\, g^{ij}$
spatial geometry $\Longleftrightarrow$ impedance-matched medium
$\det(\varepsilon^{ij}) = \pm g^{3/2} \frac{1}{g} = \pm\sqrt{g}$, $g^{ij} = \frac{\varepsilon^{ij}}{\det(\varepsilon^{ij})}$



examples, cloaking
virtual space
physical space
$g_{ij} = n^2 \delta_{ij}$

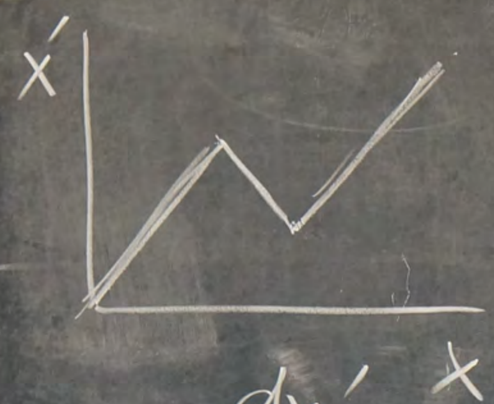

virtual
physical

• example: perfect imaging with negative refraction



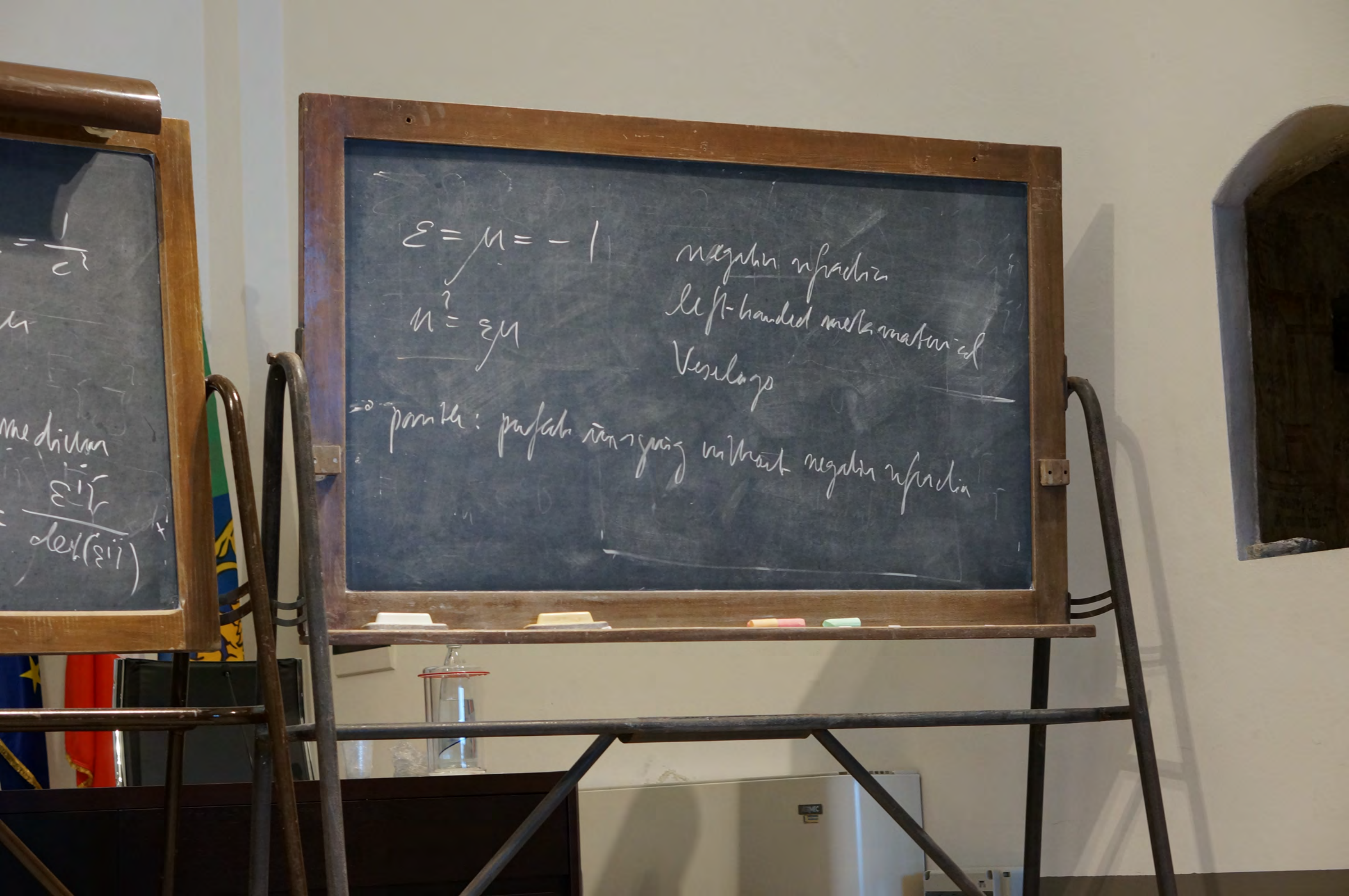
virtual x'

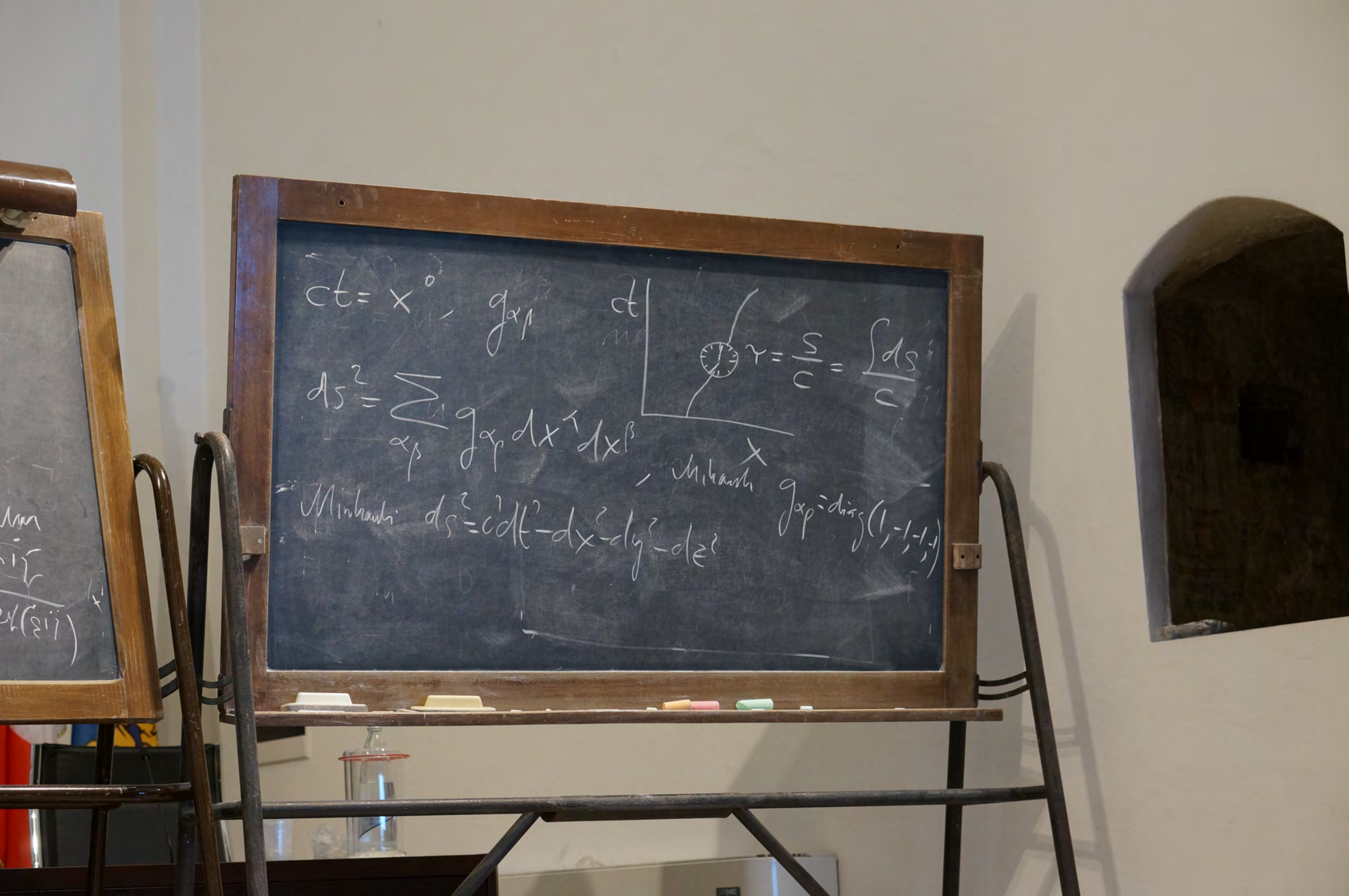
perfect copy of em field
aka perfect lens

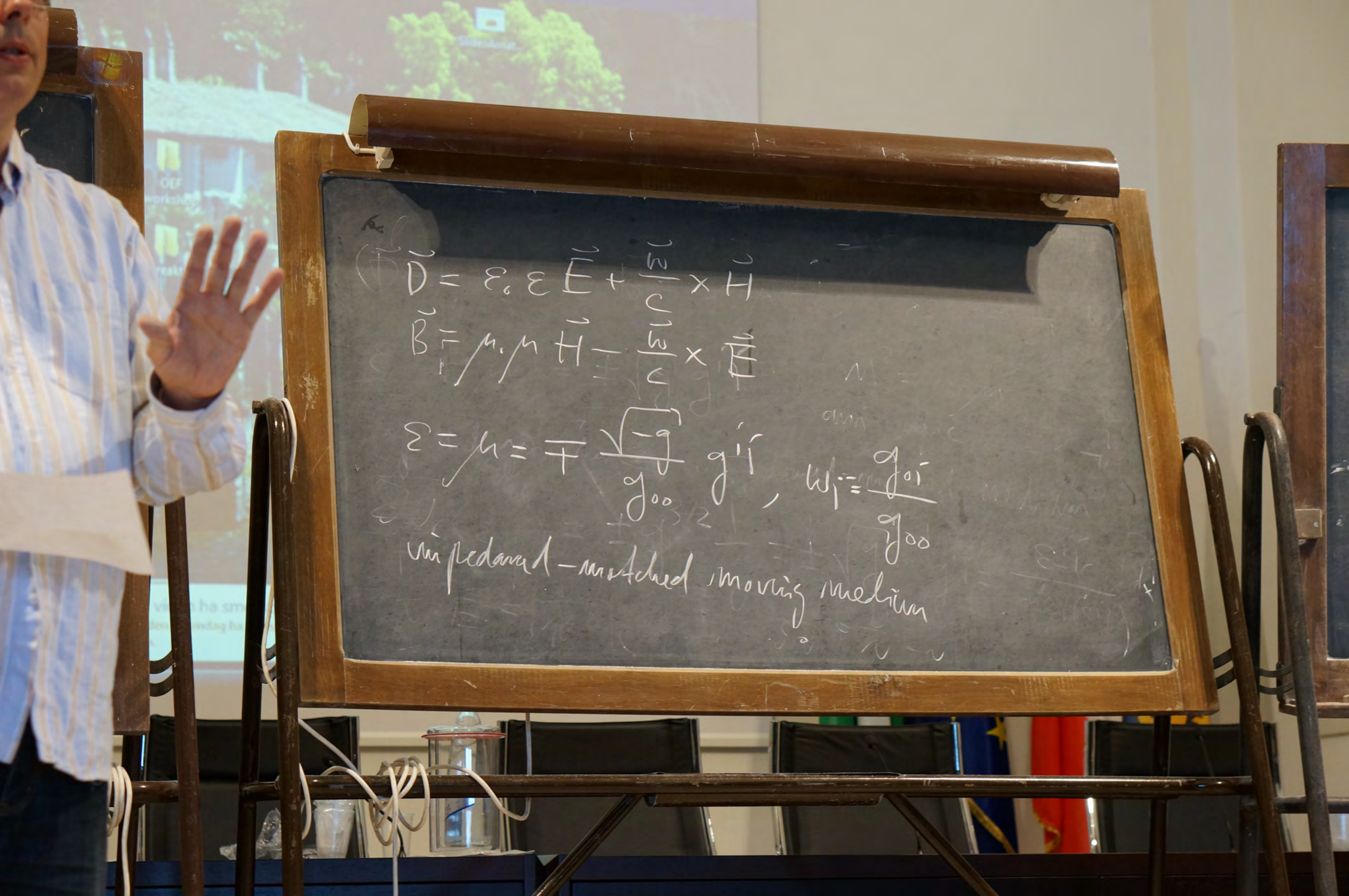


$$\frac{dx'}{dx} = -1$$

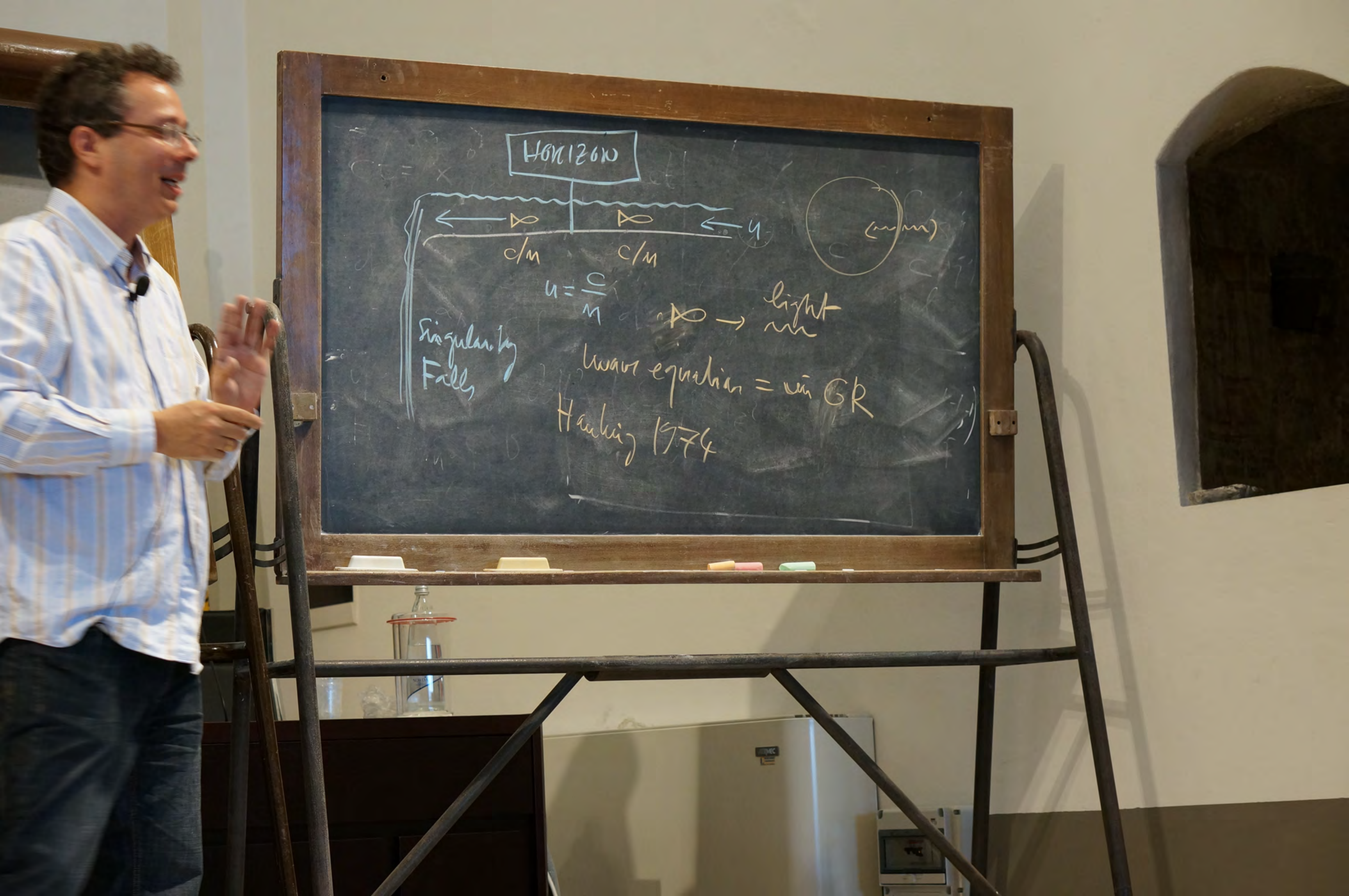
$$ds^2 = dx'^2 + dy'^2 + dz'^2$$
$$= \left(\frac{dx'}{dx}\right)^2 dx^2 + dy^2 + dz^2$$
$$= dx^2 + dy^2 + dz^2$$

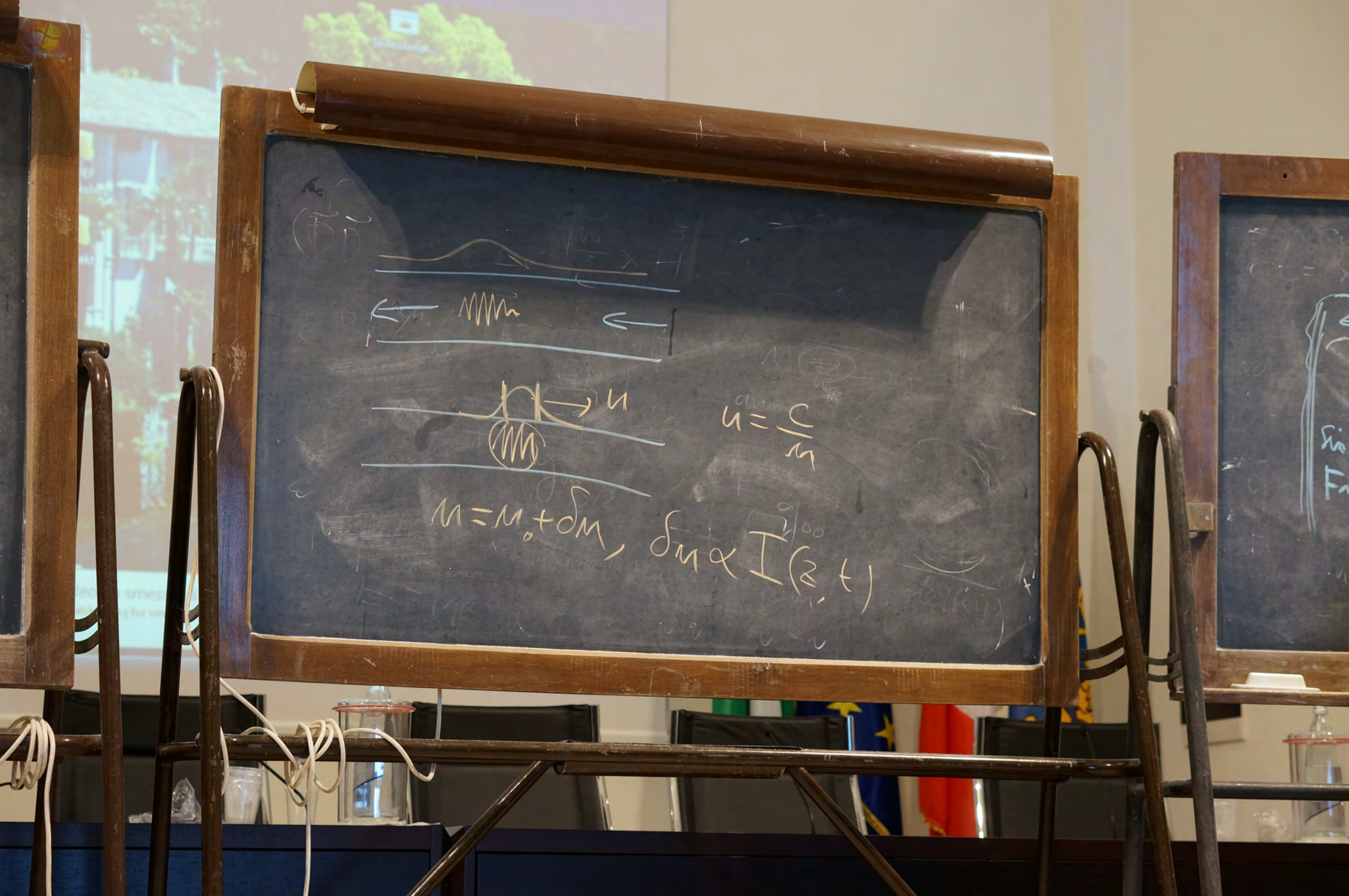

ε = μ = −1
n ≟ εμ
negative refraction
left-handed metamaterial
Veselago
perfect imaging without negative refraction


$ct = x^0$, $g_{\alpha\beta}$
$ds^2 = \sum_{\alpha\beta} g_{\alpha\beta}\, dx^\alpha dx^\beta$
ct
x
$\tau = \frac{s}{c} = \frac{\int ds}{c}$
Minkowski
Minkowski $ds^2 = c^2dt^2 - dx^2 - dy^2 - dz^2$
$g_{\alpha\beta} = \mathrm{diag}(1,-1,-1,-1)$



$\vec{D} = \varepsilon_0 \varepsilon \vec{E} + \frac{\vec{w}}{c} \times \vec{H}$
$\vec{B} = \mu_0 \mu \vec{H} - \frac{\vec{w}}{c} \times \vec{E}$
$\varepsilon = \mu = \mp \frac{\sqrt{-g}}{g_{00}} g^{ii}, \quad w_i = -\frac{g_{0i}}{g_{00}}$
impedance-matched moving medium


HORIZON
c/n
c/n
u = c/n
u
light
Fall
wave equation
GR
Hawking 1974


u
u = c/n
n = n₀ + δn, δn ∝ I(z, t)