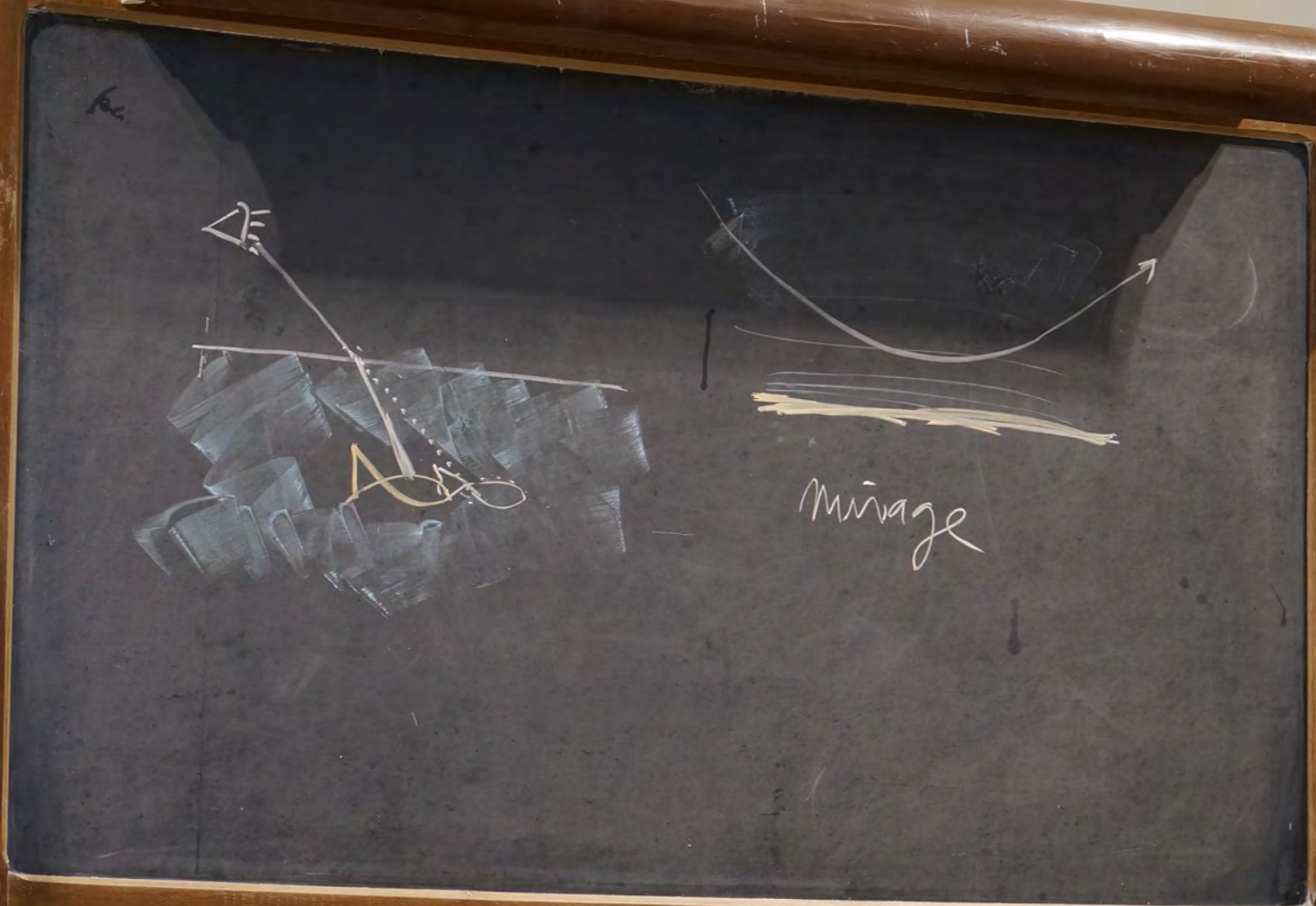


Prof. Ulf Leonhardt  
"The Geometry of Light"

Slides of the Lecture  
Wednesday 2 July 2014

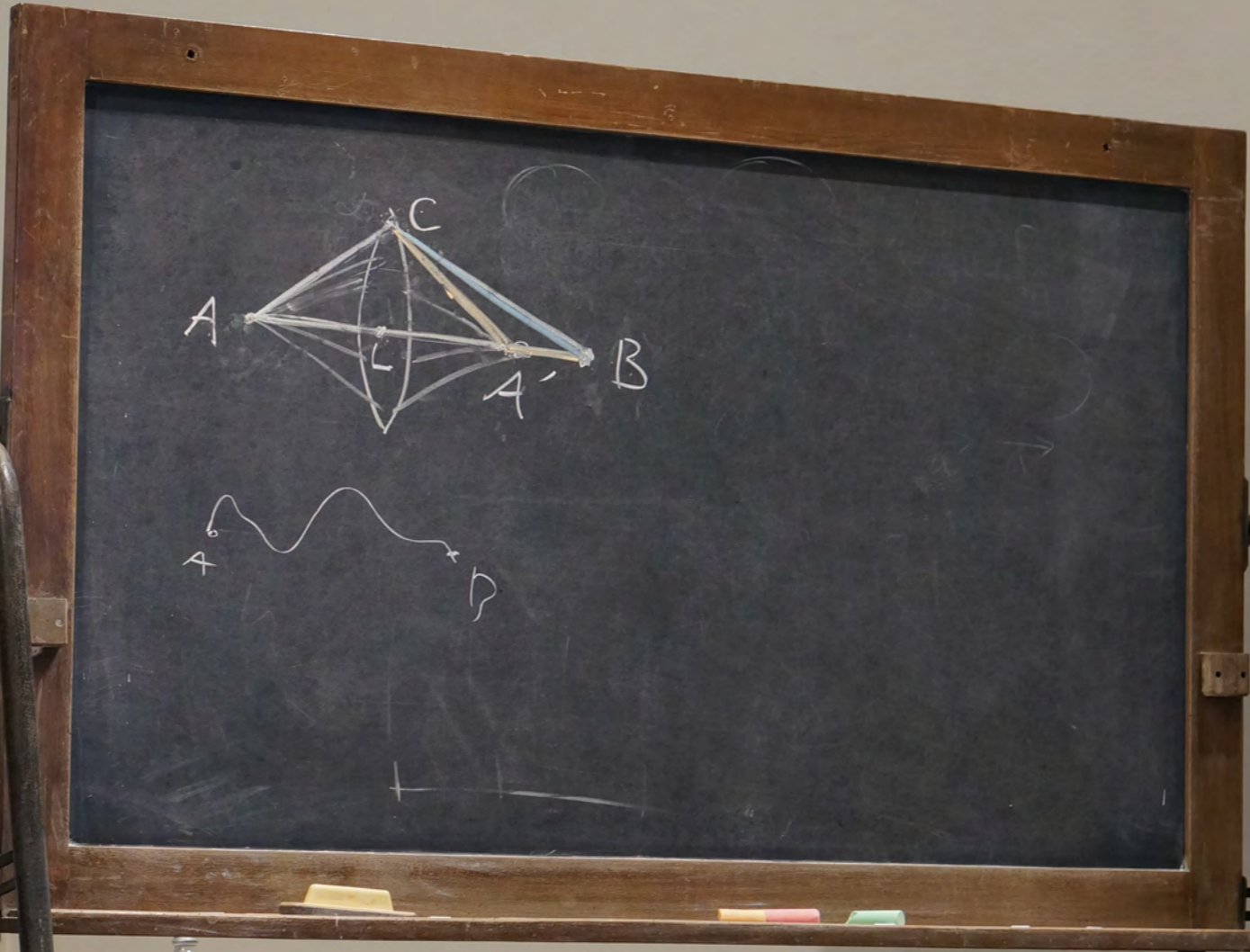
International School of Physics Enrico Fermi  
Varenna, Lake Como  
Course 190 -Frontiers in Modern Optics

Slides Armat...



Mirage

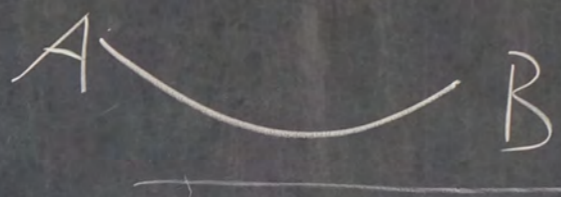




# GEOMETRY & LIGHT

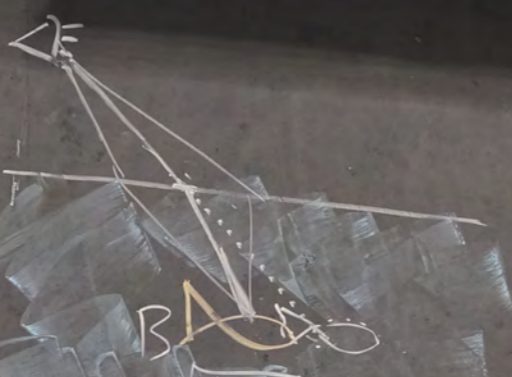
Down, 2010

- 1662 Fermat's principle



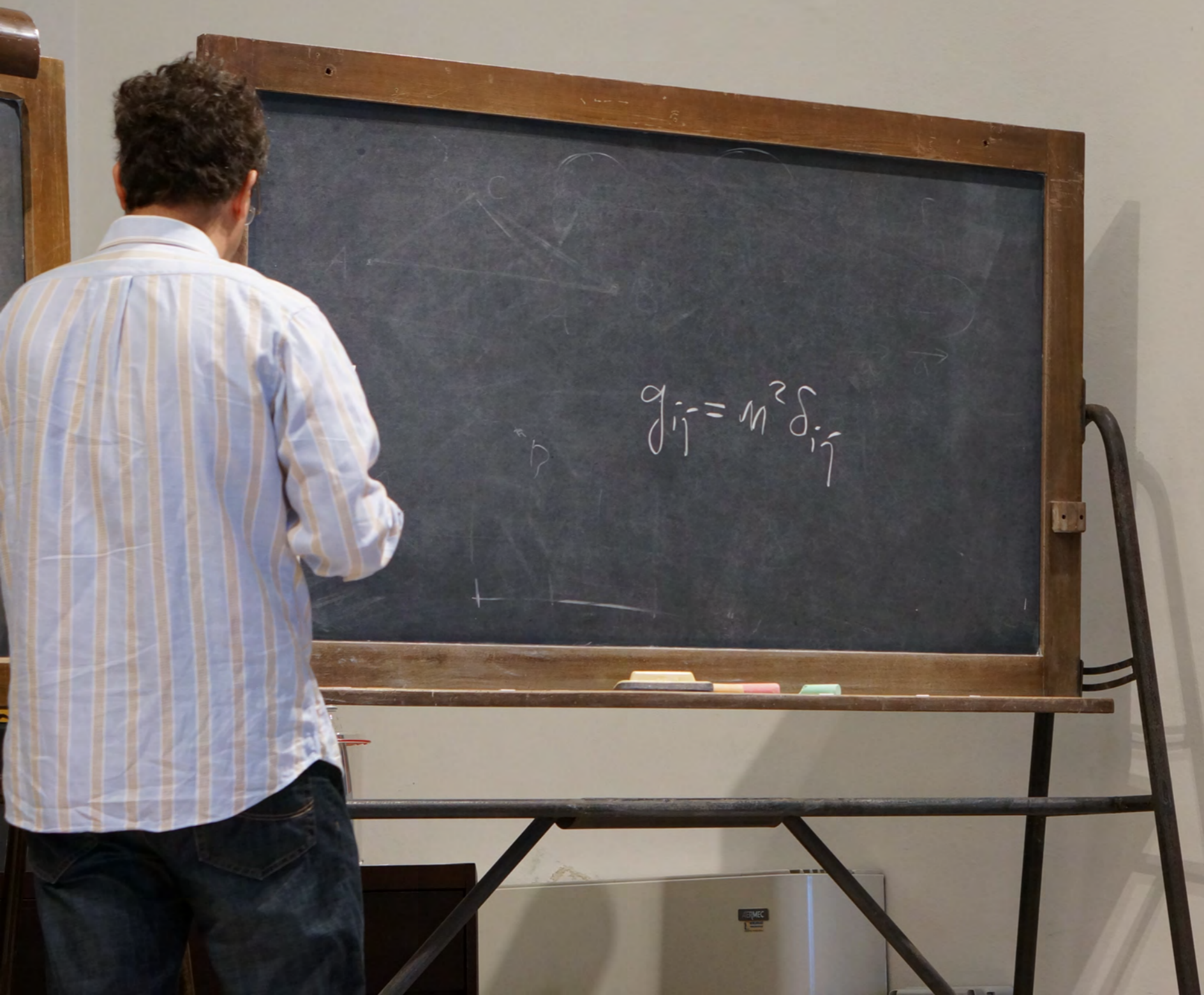
A diagram showing two points, A and B, connected by a curved line. A horizontal line is drawn below the curve, representing a straight path. The curve is labeled 'A' at the start and 'B' at the end.

$$\text{extremum}_{B \rightarrow A} = \int_A^B n \, dl$$
$$= \int_A^B n \sqrt{dx^2 + dy^2 + dz^2}$$



$$S = \int_A^B \sqrt{n^2(dx^2 + dy^2 + dz^2)} = \int \sqrt{\sum_{ij} g_{ij} dx^i dx^j}$$





$$g_{ij} = m^2 \delta_{ij}$$

## 2. Spatial geometry

$$ds^2 = \sum_{ij} g_{ij} dx^i dx^j, \quad s = \int ds$$

$g_{ij}$ : metric tensor,  $x^i$ : coordinates

$$x^1 = x, \quad x^2 = y, \quad x^3 = z \quad \text{Cartesian}$$

$$x^1 = r, \quad x^2 = \theta, \quad x^3 = \phi$$

$$\begin{aligned}
 ds^2 &= dx^2 + dy^2 + dz^2 \\
 &= dr^2 + r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2
 \end{aligned}$$

$$g_{ij} = \text{diag}(1, r^2, r^2 \sin^2 \theta)$$

$$g^{ij} \equiv (g_{ij})^{-1} \text{ matrix inverse}$$

$$g^{ij} = \text{diag}\left(1, \frac{1}{r^2}, \frac{1}{r^2 \sin^2 \theta}\right)$$

$$g = \det(g_{ij})$$

$\sqrt{|g|}$ : 3D volume element



• coordinate transformation

$$dx^i = \sum_{i''} \frac{\partial x^i}{\partial x^{i''}} dx^{i''} = \sum_{i''} A^i_{i''} dx^{i''}$$

$$dx^{i''} = \sum_i \frac{\partial x^{i''}}{\partial x^i} dx^i = \sum_i A^{i''}_i dx^i$$

$$\sum_{i''} A^i_{i''} A^{i''}_j = \delta^i_j$$

$\Gamma_{ij}^k dx^i dx^j$   
 $\Gamma_{ij}^k dx^i dx^j$

$$\partial_i \equiv \frac{\partial}{\partial x^i} = \sum_j \frac{\partial x'^j}{\partial x^i} \partial'_j = \sum_j \Lambda^j_i \partial'_j$$

differentials are transformed with  $\Lambda^j_i$   
 derivations with  $\Lambda^j_i$

vectors and one-forms  
 $x^i$

$F_i$  like  $\partial_i$   
 $F^i$  like  $dx^i$

Lowering and raising of indices

$$F_i = \sum_j g^{ij} F_j, \quad F^i = \sum_j g_{ij} F^j$$

$$\begin{aligned} ds^2 &= \sum_{i,j} g_{ij} dx^i dx^j \\ &= \sum_{i,j} g_{ij} \delta^i_{i'} \delta^j_{j'} dx^{i'} dx^{j'} \\ &= \sum_{i',j'} g'_{i'j'} dx^{i'} dx^{j'} \end{aligned}$$

ndian  
 $g_{ij} F^j$   
 $dx^i dx^j$

$$g'_{i'j'} = \sum_{ij} g_{ij} \Lambda^i_{i'} \Lambda^j_{j'} = (\Lambda^i_{i'})^T (g_{ij}) (\Lambda^j_{j'})$$
$$\sum_{ij} \Lambda^i_{i'} g_{ij} \Lambda^j_{j'} \quad A' = T A T$$
$$g'_{i'j'} = (\Lambda^i_{i'}) (g_{ij}) (\Lambda^j_{j'})^T$$

Lowering and raising of indices

$$F^i = \sum_j g^{ij} F_j, \quad F_i = \sum_j g_{ij} F^j$$

$$g_{ij} = M^2 \delta_{ij}, \quad p_i = \partial_i S = \hbar \partial_i \varphi, \quad \partial_i \varphi = k_i$$

$$k = M \frac{\omega}{c}$$

Minkowski metric

$$p = \hbar m \frac{\omega}{c}$$

$F^i$   
 $\partial_i \varphi = k_i$   
 $\varphi$   
 Abraham

$$p^i = m \frac{dx^i}{dt}$$

$$\left( p^\alpha = m \frac{dx^\alpha}{dx^0} \right)$$

$$p^i = \sum_j g^{ij} p_j$$

$$p^i = \frac{1}{m^2} P_i, \quad p^i = \frac{h\nu}{c\lambda}$$

$$P_i = \sum_j \left( \frac{1}{m^2} \delta_{ij} \right) p_j = \frac{1}{m^2} P_i$$
 Abraham momentum



• Maxwell's equations

$$\nabla \cdot \vec{E} = 0, \quad \nabla \times \vec{E} = -\partial_t \vec{B}$$

$$\nabla \cdot \vec{B} = 0, \quad \nabla \times \vec{B} = \frac{1}{c^2} \partial_t \vec{E}$$

from differential geometry

$$\nabla \cdot \vec{E} = \frac{1}{\sqrt{g}} \sum_{ij} \partial_i \sqrt{g} g^{ij} E_j$$

$$g = \det(g_{ij})$$

$$g = \det(g_{ij})$$

$$g_{ij} = \text{diag}(1, n^2, n^2 \sin^2 \theta), \quad g = n^4 \sin^2 \theta$$

$$g^{ij} = \text{diag}\left(1, \frac{1}{n^2}, \frac{1}{n^2 \sin^2 \theta}\right)$$

$$\nabla \cdot \vec{E} = \frac{1}{n^2 \sin^2 \theta} \left( \partial_r n^2 \sin^2 \theta E_r + \partial_\theta \sin^2 \theta E_\theta \right.$$

$$E_\theta = 0, E_\phi = 0$$

$$\left. + \partial_\phi \frac{1}{\sin^2 \theta} E_\phi \right)$$

$$\nabla \cdot \vec{E} = \frac{1}{n^2} \left( \partial_r n^2 E_r = \frac{\partial E_r}{\partial r} + \frac{2}{r} E_r \right)$$



$$\vec{E} = -\nabla \varphi$$

$$\operatorname{div} \vec{E} = \operatorname{div} \nabla \varphi = \nabla^2 \varphi + \frac{2}{r} \frac{\partial \varphi}{\partial r}$$

$$(\nabla \times \vec{E})^i = \pm \sum_{j,k} \frac{[ijk]}{\sqrt{g}} \partial_j E_k \quad (\nabla \times E)^i =$$

$[ijk]$  antisymmetric

$$[123] = 1$$

$$[jik] = -[ijk]$$

$$= \begin{pmatrix} \partial_2 E_3 - \partial_3 E_2 \\ \partial_3 E_1 - \partial_1 E_3 \\ \partial_1 E_2 - \partial_2 E_1 \end{pmatrix}$$

$\pm$ : right/left-handed +

$$\begin{aligned} \nabla \times \mathbf{E} &= \\ \begin{vmatrix} \partial_x E_3 - \partial_3 E_2 \\ \partial_y E_3 - \partial_2 E_1 \\ \partial_z E_2 - \partial_2 E_1 \end{vmatrix} &= -[\dot{\gamma}^k] \end{aligned}$$

$$\sum_{ij} \partial_i \sqrt{g} g^{ij} E_j = 0$$

$$\sum_i \partial_i D^i = 0, \quad D^i = \pm \epsilon \sum_j \sqrt{g} g^{ij} E_j$$

$$\vec{D} = \epsilon_0 \epsilon \vec{E}, \quad \epsilon = \pm \sqrt{g} g^{ij}$$

Abraham's moment

$$(\nabla \times \vec{E})_i = \pm \sum_{jk} \frac{[ijk]}{\sqrt{g}} \partial_j E_k \quad (\nabla \times \vec{E})_i =$$

$$\pm \sum_{jk} \frac{[ijk]}{\sqrt{g}} \partial_j B_k = \frac{1}{c^2} \partial_t \sum_j g^{ij} E_j$$

$$\sum_{jk} [ijk] \partial_j B_k = \frac{1}{c^2} \partial_t \sum_j (\pm \sqrt{g} g^{ij}) E_j$$

$$= \frac{1}{c^2 \epsilon_0} \partial_t \frac{D^i}{\sqrt{g}} E_i$$

$$\sum_j a_j \sqrt{g} g^{ij} E_j = 0$$

$$\sum_i a_i D^i = 0, \quad D^i = \pm \epsilon \sum_j \sqrt{g} g^{ij} E_j$$

$$D^i = \epsilon_0 \vec{E}^i, \quad E = \pm \sqrt{g} g^{ij}$$

$$H^i = \epsilon_0 c^2 B^i$$

Abraham's momentum

$$\sum_{ij} [\epsilon_{ijk}] \partial_j H_i = \partial_k D'$$

$$\nabla \times \vec{H} = \partial_t \vec{D}$$

$$D'_i = \pm \epsilon_0 \sum_j \sqrt{g} g^{ij} E_j$$

$$\vec{D} = \epsilon_0 \epsilon \vec{E}$$

$$H'_i = \frac{1}{\mu_0 c} B_i$$

$$E = \pm \sqrt{g} g^{ij}$$

Abraham's formula

$$\sum_{ij} [\epsilon_{ijk}] \partial_j H_i = \partial_k D'$$

$$\nabla \times \vec{H} = \partial_t \vec{D}$$

$$D'_i = \pm \epsilon_0 \sum_j \sqrt{g} g^{ij} E_j$$

$$B'_i = \pm \mu_0 \sum_j \sqrt{g} g^{ij} H_j$$

• Maxwell's equations

$$\begin{aligned} \nabla \cdot \vec{E} &= 0, & \nabla \times \vec{E} &= -\partial_t \vec{B} & \text{in general} \\ \nabla \cdot \vec{B} &= 0, & \nabla \times \vec{B} &= \frac{1}{c^2} \partial_t \vec{E} & \text{coordinates \&} \\ & & & & \text{geometry} \end{aligned}$$

---

from differential geometry  $\vec{a} = \text{field of } \vec{a}$

$$\begin{aligned} \nabla \cdot \vec{D} &= 0, & \nabla \times \vec{E} &= -\partial_t \vec{B} & \text{in Cartesian} \\ \nabla \cdot \vec{B} &= 0, & \nabla \times \vec{H} &= \partial_t \vec{D} & \text{coordinates} \end{aligned}$$



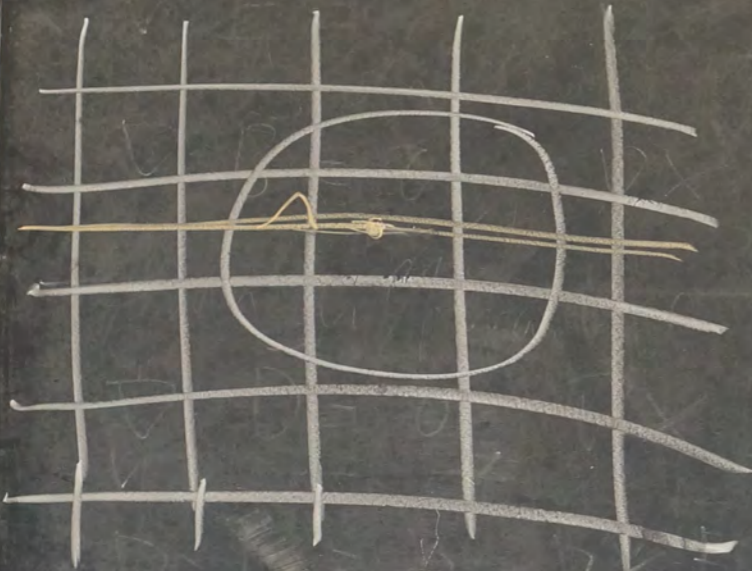
$$\vec{D} = \epsilon_0 \epsilon \vec{E}, \quad \vec{B} = \mu_0 \mu \vec{H}, \quad \epsilon_0 \mu_0 = \frac{1}{c^2}$$

$$\epsilon_{ij} = \mu_{ij} = \pm \sqrt{|g|} g_{ij}$$

spatial geometry  $\Leftrightarrow$  impedance-matched medium

$$\det(\epsilon_{ij}) = \pm g^{3/2} \frac{1}{g} = \pm \sqrt{|g|}, \quad g_{ij} = \frac{\epsilon_{ij}}{\det(\epsilon_{ij})}$$

examples, cloaking  
virtual space



physical  
physical space  $g_{ij} = m^2 \delta_{ij}$

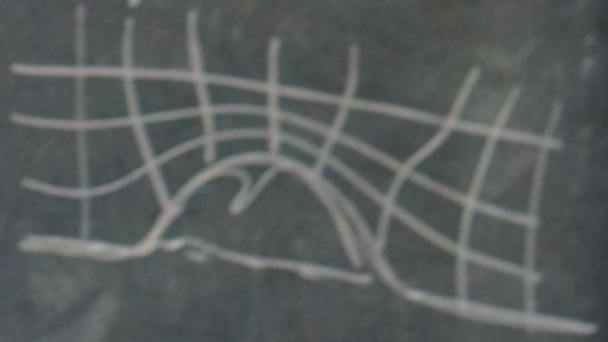




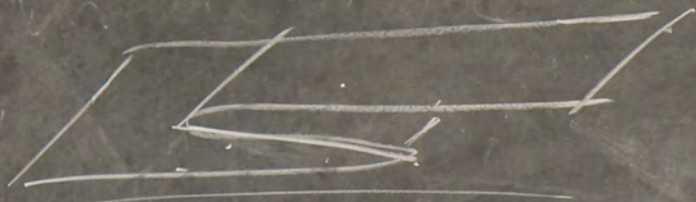
internal



physical



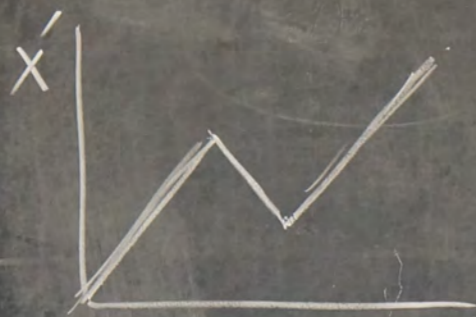
• example: perfect imaging with negative refraction



virtual  $x'$



perfect copy of em field  
aka perfect lens



$$\frac{dx'}{dx} = -1$$

$$\begin{aligned} ds^2 &= dx'^2 + dy'^2 + dz'^2 \\ &= \left(\frac{dx'}{dx}\right)^2 dx^2 + dy^2 + dz^2 \\ &= dx^2 + dy^2 + dz^2 \end{aligned}$$

$$\epsilon = \mu = -1$$

$$n^2 = \epsilon \mu$$

negative refraction  
left-handed metamaterial  
Veselago

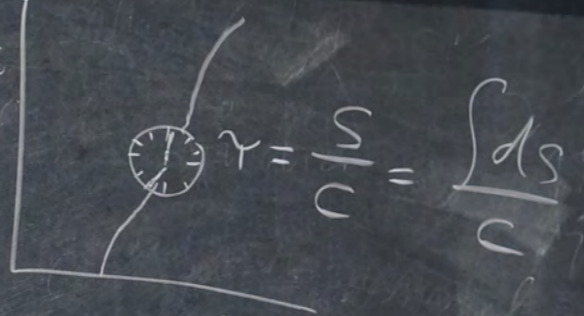
point: perfect lens without negative refraction

medium

$$\frac{\epsilon \vec{r}}{\det(\epsilon)}$$

$$ct = x^0, \quad g_{\alpha\beta}$$

$$ds^2 = \sum_{\alpha\beta} g_{\alpha\beta} dx^\alpha dx^\beta$$



Minimalkan  $ds^2 = c^2 dt^2 - dx^2 - dy^2 - dz^2$

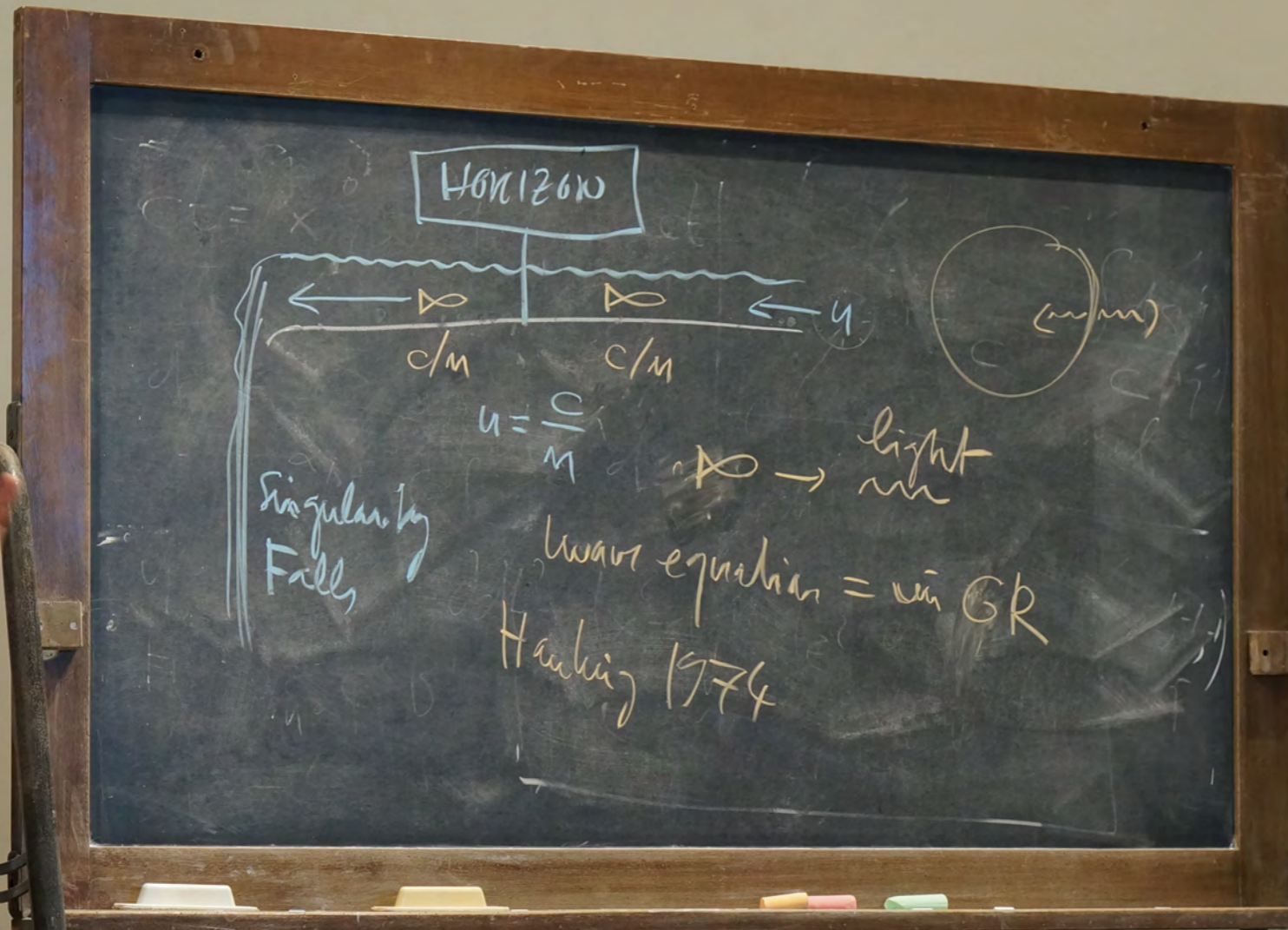
$g_{\alpha\beta} = \text{diag}(1, -1, -1, -1)$

$$\vec{D} = \epsilon_0 \epsilon \vec{E} + \vec{\nabla} \times \vec{H}$$

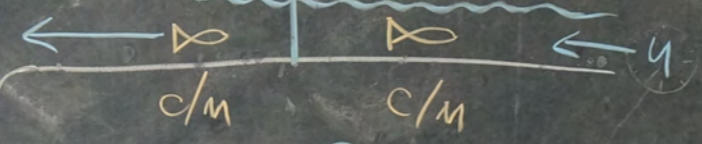
$$\vec{B} = \mu_0 \mu \vec{H} - \vec{\nabla} \times \vec{E}$$

$$\epsilon = \mu = \mp \sqrt{\frac{-g}{g_{00}}} g^{ij}, \quad w_j = \frac{g_{0i}}{g_{00}}$$

impedance-matched moving medium



HORIZON



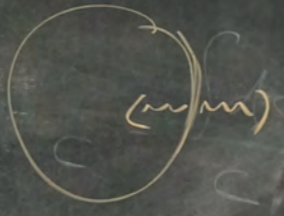
$$u = \frac{c}{m}$$

$\infty \rightarrow$  light  
m

Singularity  
Falls

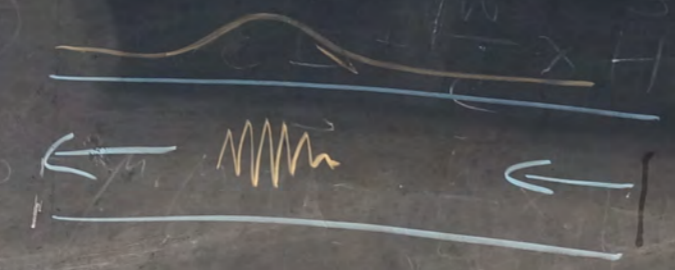
Wave equation = in GR

Hawking 1974





for  
(FD)



$$u = \frac{c}{m}$$

$$n = n_0 + \delta n, \quad \delta n \propto I(z, t)$$