Higher-order Kuramoto model and dynamics of topological signals

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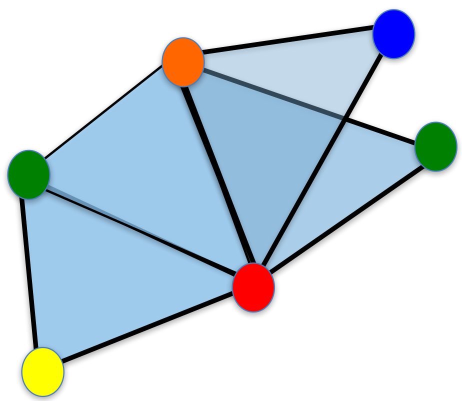
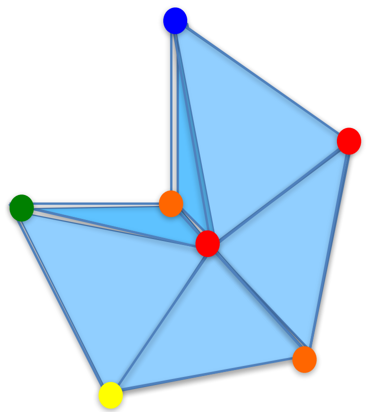
Higher-order networks

Higher-order network are characterising the interaction between two ore more nodes and are formed by nodes, links, triangles, tetrahedra etc.



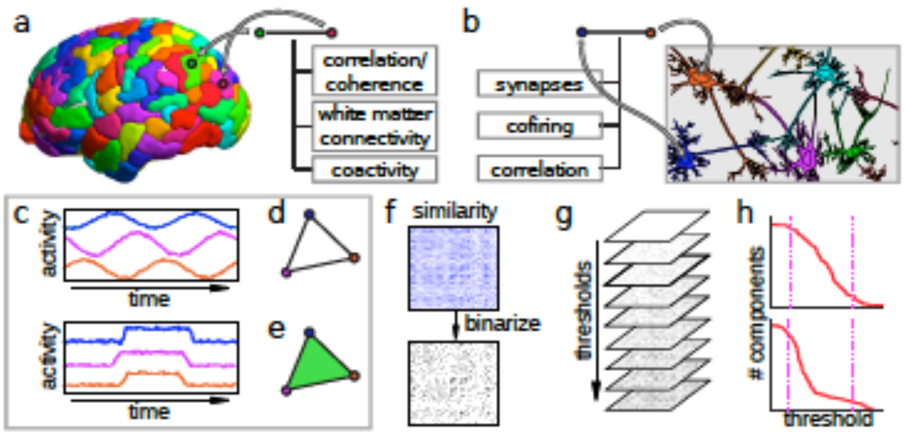
d=2 simplicial complex



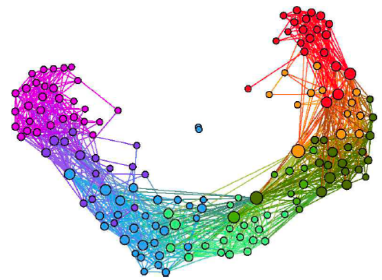
d=3 simplicial complex

Higher-order network data

Brain data

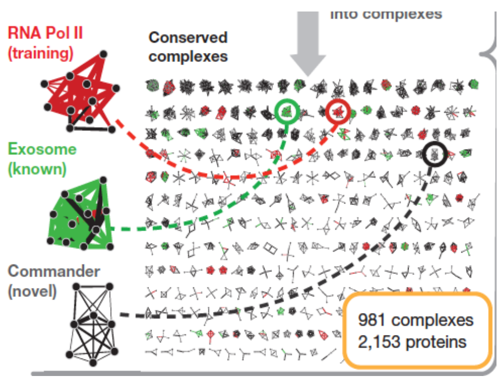


Face-to-face interactions

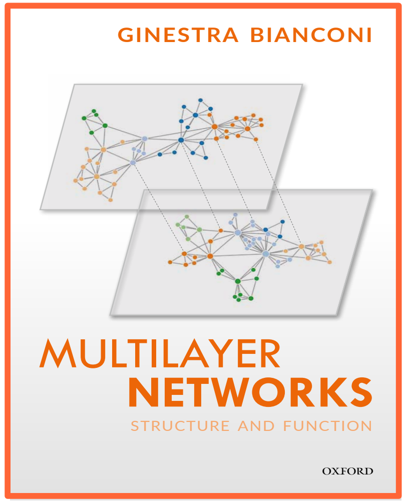


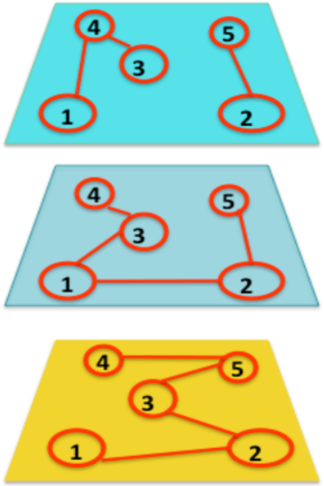
Collaboration networks

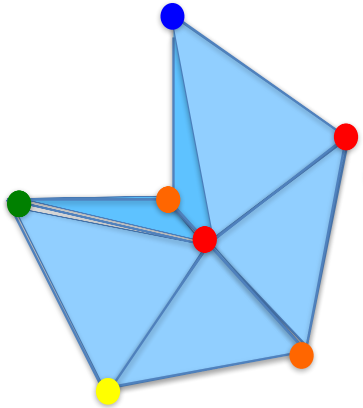
Protein interactions

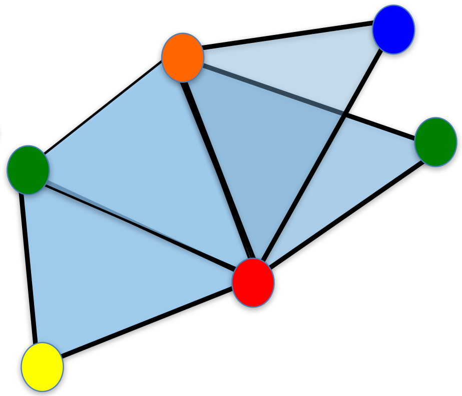


Generalized network structures









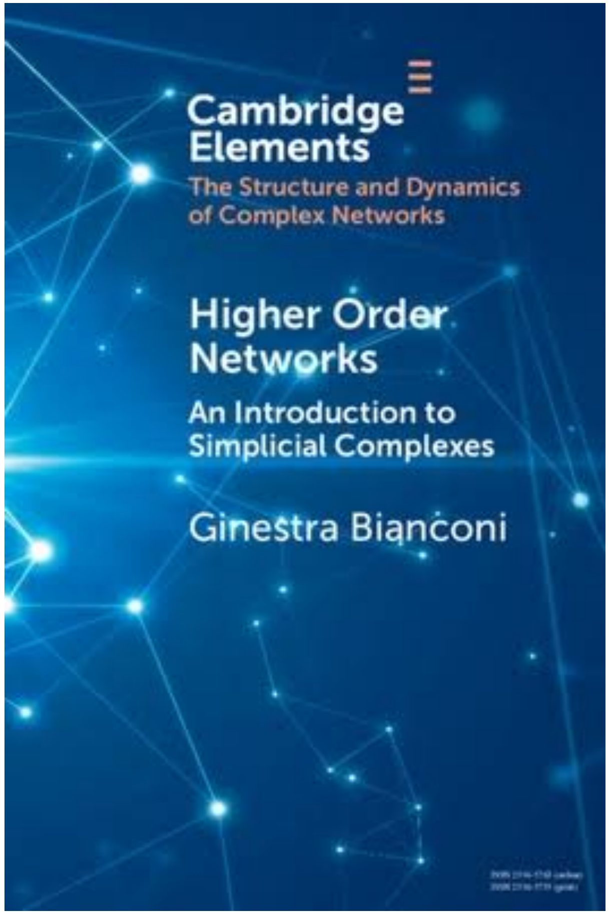
Going beyond the framework of simple networks

is of fundamental importance

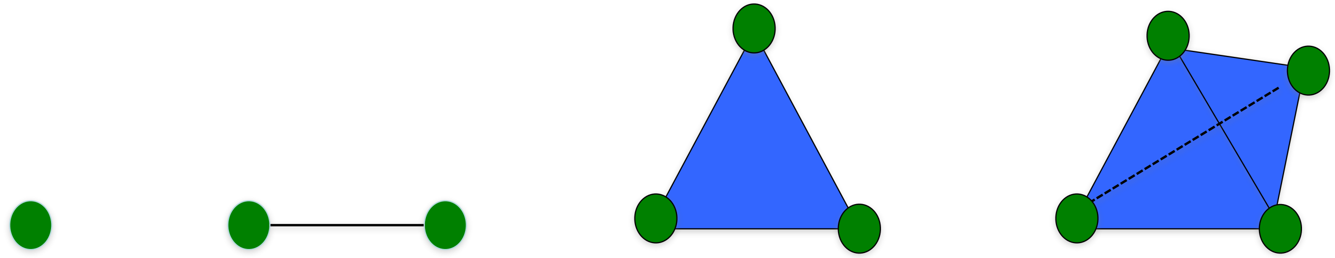
for understanding the relation between structure and

dynamics in complex systems

Forthcoming book



Simplices



0-simplex 1-simplex 2-simplex 3-simplex

A simplex of dimension d is a set of d+1 nodes

$$\alpha = [i_0, i_1, i_2, \ldots i_d]$$

-it indicates the interactions between the nodes

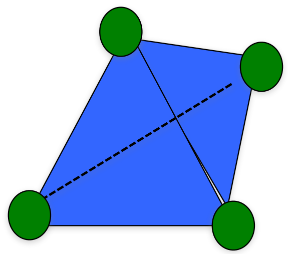
-it admits a topological and geometrical interpretation

Faces of a simplex

A face of a d-dimensional simplex

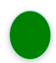
is a δ-dimensional simplex (with δ<d)

formed by a non-empty subset of its nodes



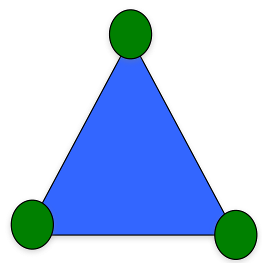
3-simplex

Faces



4 0-simplices

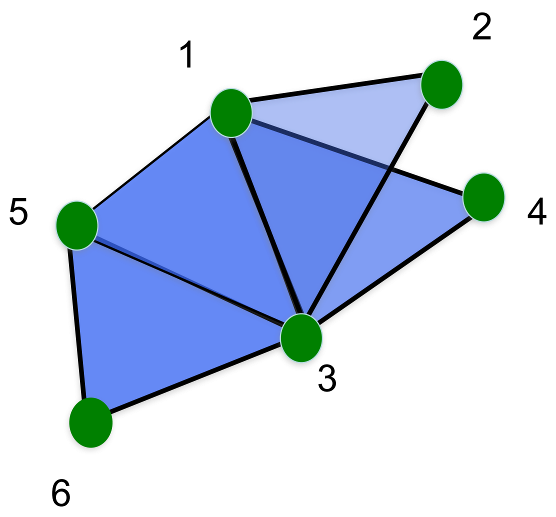
6 1-simplices



4 2-simplices

Simplicial complex

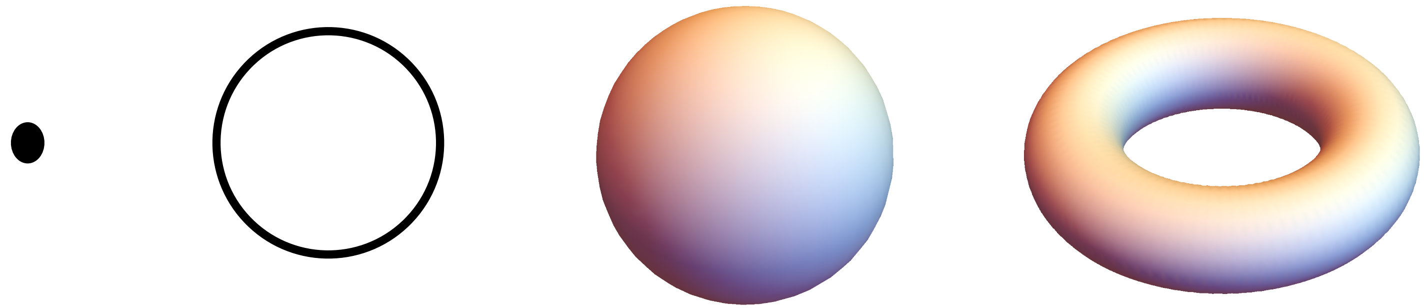
A simplicial complex $\mathscr{K}$ is a set of simplices closed under the inclusion of the faces of every simplex



If a simplex α belongs to the simplicial complex $\mathscr{K}$ then every face of α must also belong to $\mathscr{K}$

$\mathscr{K}$ = {[1], [2], [3], [4], [5], [6],
[1,2], [1,3], [1,4], [1,5], [2,3],
[3,4], [3,5], [3,6], [5,6],
[1,2,3], [1,3,4], [1,3,5], [3,5,6]}

Betti numbers

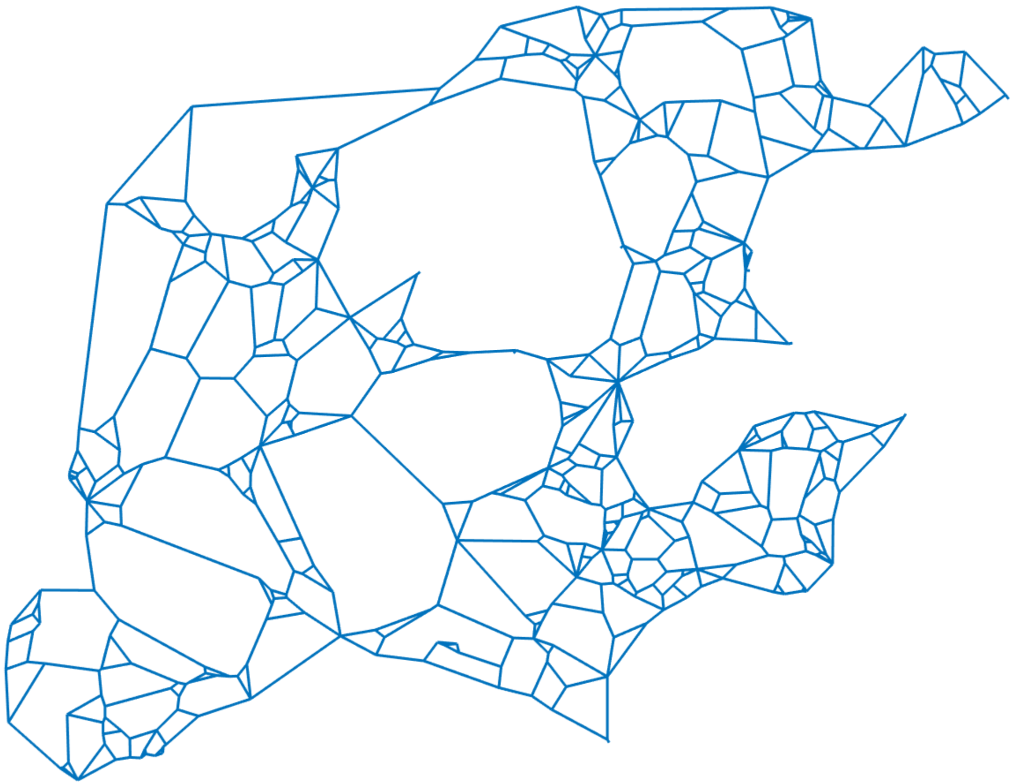


Point	Circle	Sphere	Torus
$\beta_0 = 1$	$\beta_0 = 1$	$\beta_0 = 1$	$\beta_0 = 1$
$\beta_1 = 0$	$\beta_1 = 1$	$\beta_1 = 0$	$\beta_1 = 2$
$\beta_2 = 0$	$\beta_2 = 0$	$\beta_2 = 1$	$\beta_2 = 1$

Euler characteristic

$$\chi = \sum_n (-1)^n \beta_n$$

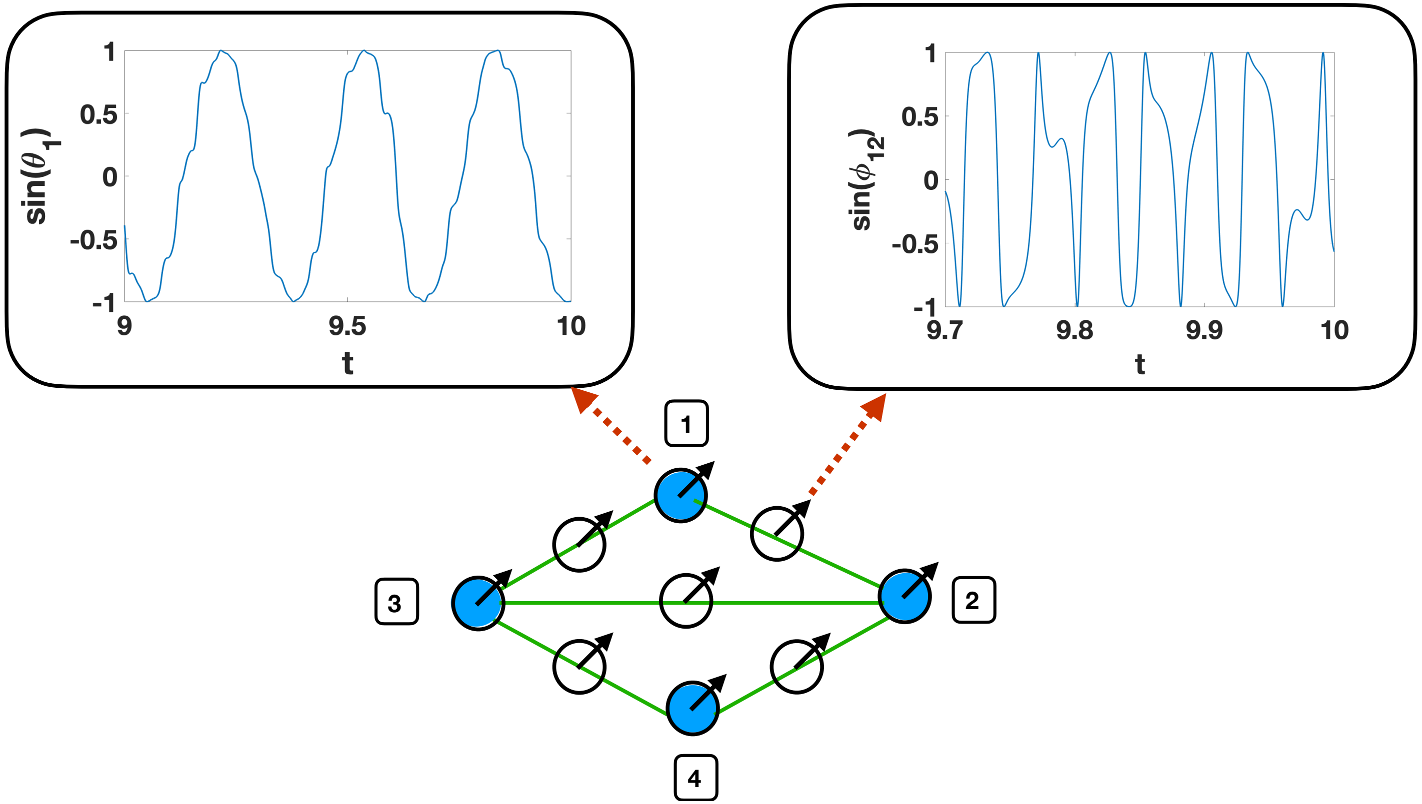
Betti number 1



Fungi network from Sang Hoon Lee, et. al. Jour. Compl. Net. (2016)

Topological signals

Simplicial complexes and networks can sustain dynamical variables (signals) not only defined on nodes but also defined on higher order simplices these signals are called *topological signals*

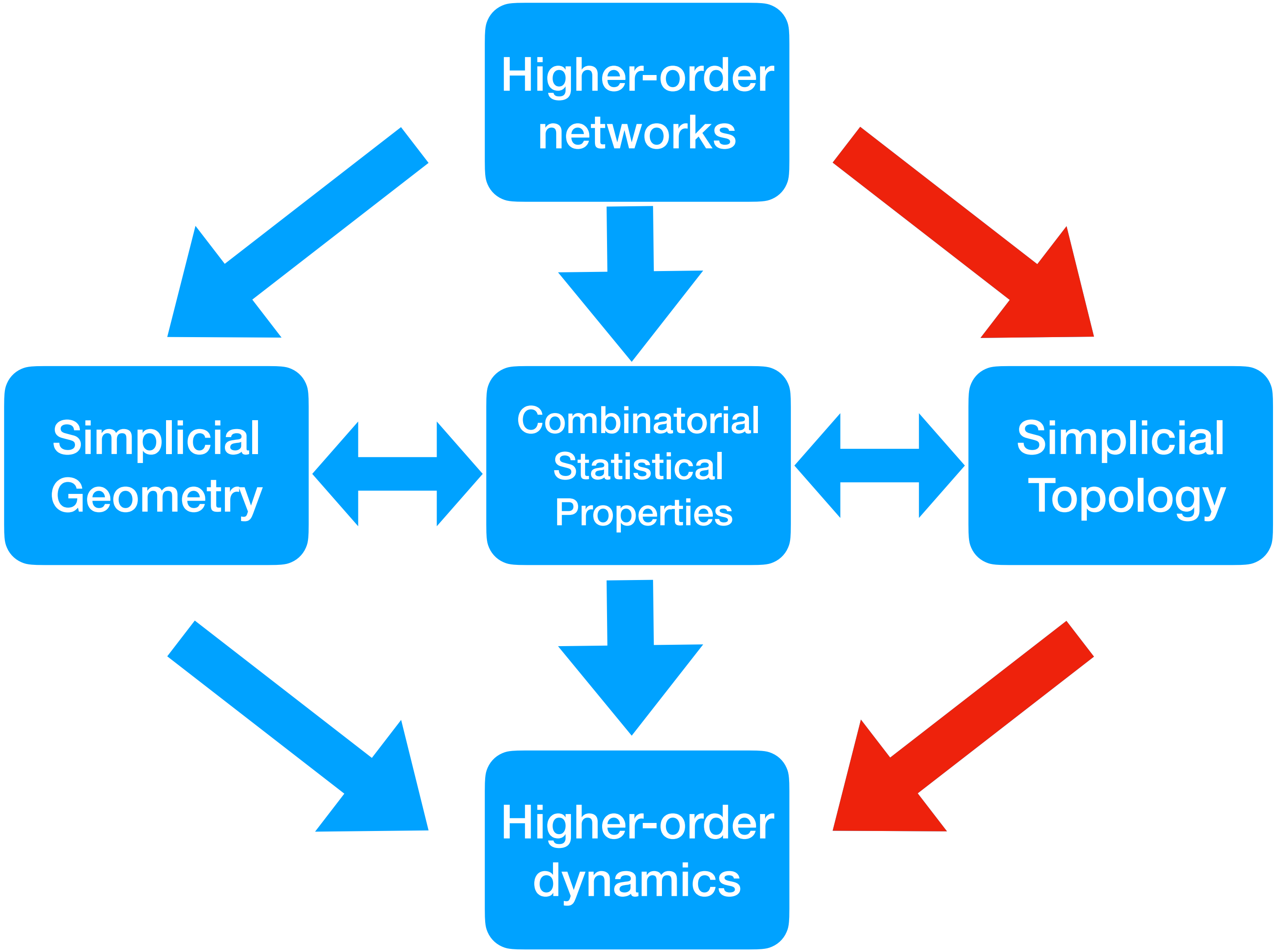


Topological signals

- Citations in a collaboration network
- Speed of wind at given locations
- Currents at given locations in the ocean
- Fluxes in biological transportation networks
- Synaptic signal
- Edge signals in the brain

Topological signals
are co-chains or vector fields

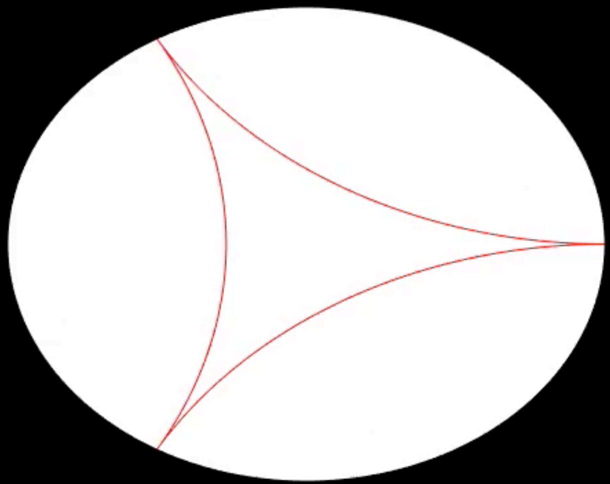
Higher-order structure and dynamics

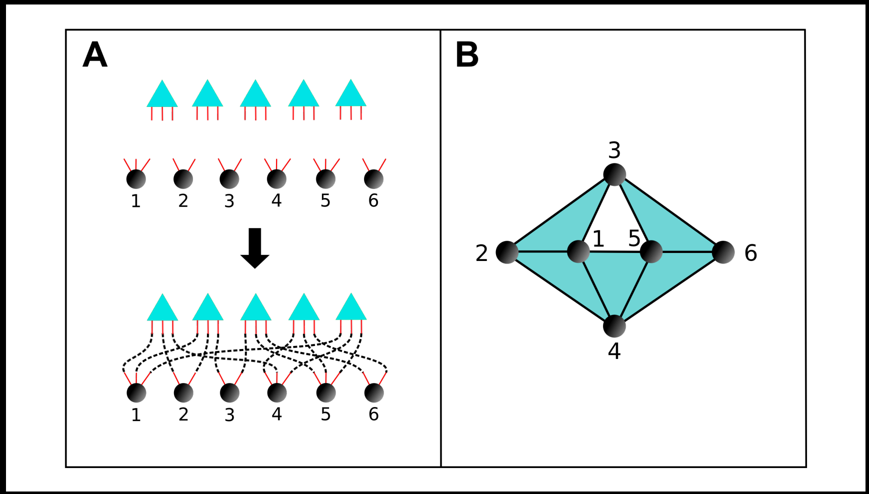


Simplicial complex models

Emergent Geometry
Network Geometry with Flavor (NGF)
[Bianconi Rahmede ,2016 & 2017]

Maximum entropy model
Configuration model
of simplicial complexes
[Courtney Bianconi 2016]

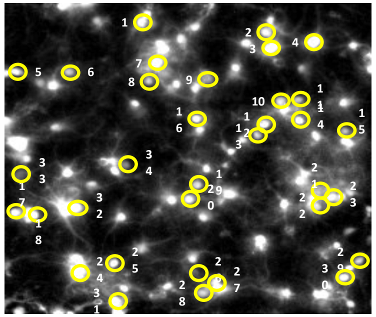


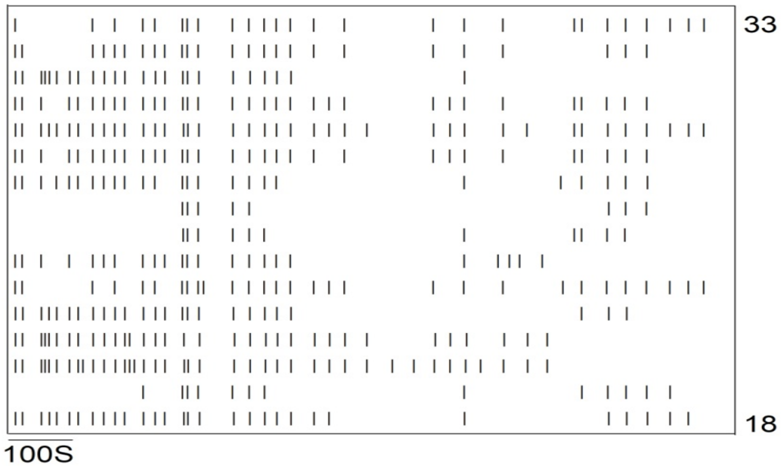


Kuramoto model

Synchronization is a fundamental dynamical process

NEURONS





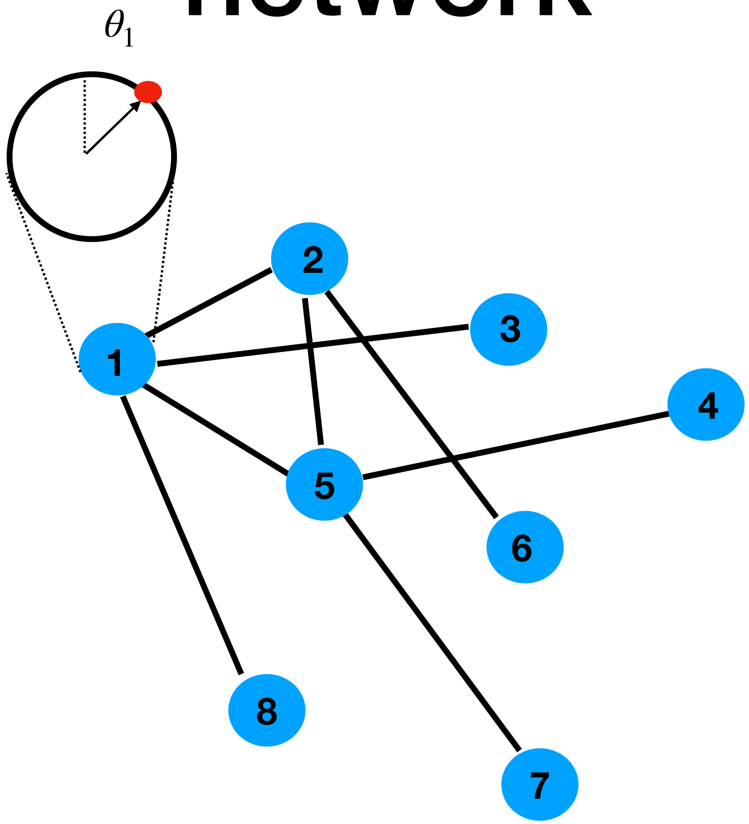
FIREFLIES

Synchronization pioners

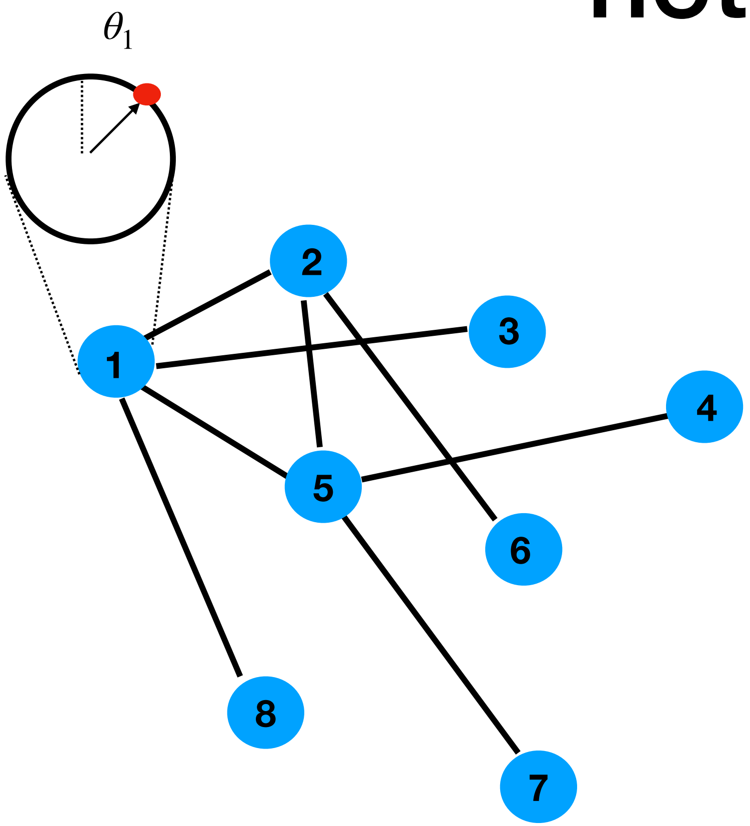
Christiaan Huygens 1665

Yoshiki Kuramoto 1974

Kuramoto model on a network



Kuramoto model on a network



Given a network of N nodes
defined by an adjacency matrix a
we assign to each node a phase obeying

$$\dot{\theta}_i = \omega_i + \sigma \sum_{j=1}^{N} a_{ij} \sin\left(\theta_j - \theta_i\right)$$

where the internal frequencies of the nodes
are drawn randomly from

$$\omega \sim \mathcal{N}(\Omega, 1)$$

and the coupling constant is σ

Order parameter for synchronization

- We consider the global order parameter

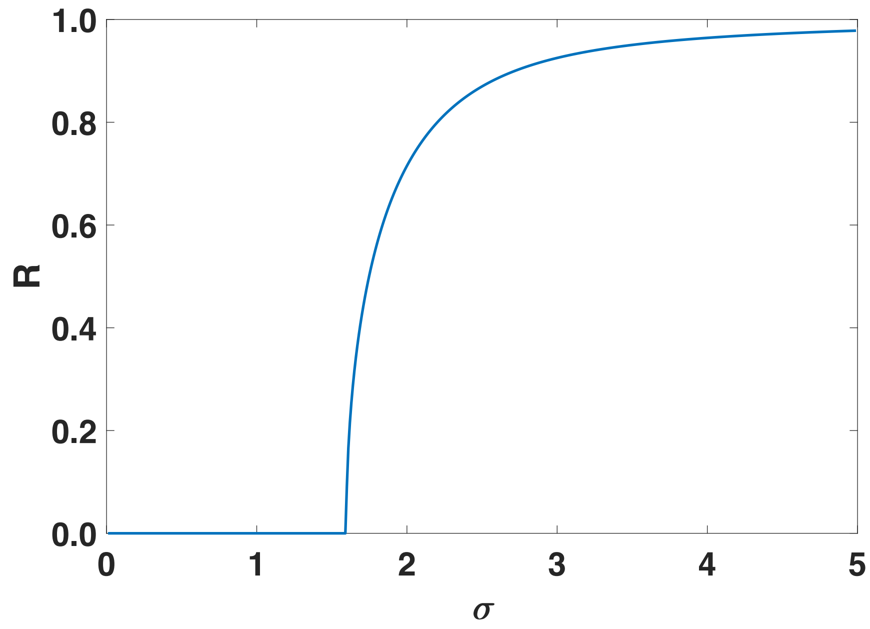
$$R = \frac{1}{N}\left|\sum_{i=1}^{N} e^{\mathrm{i}\theta_i}\right|$$

- The role of R is to indicate the synchronisation transition

$$\begin{array}{ll} R \simeq 0 & \textbf{for } \sigma < \sigma_c \\ R \textbf{ finite} & \textbf{for } \sigma \geq \sigma_c \end{array}$$

Kuramoto Model

In an infinite fully connected network
we have



Network topology
and
Higher-order Laplacians

Orientation of the simplex

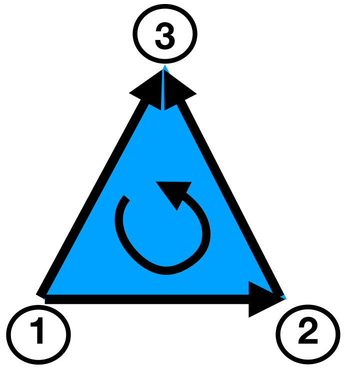
Each simplex

$$\alpha = [i_0, i_1, \ldots, i_n] .$$

has an orientation

Therefore we have

$$[i, j] = -[j, i]$$



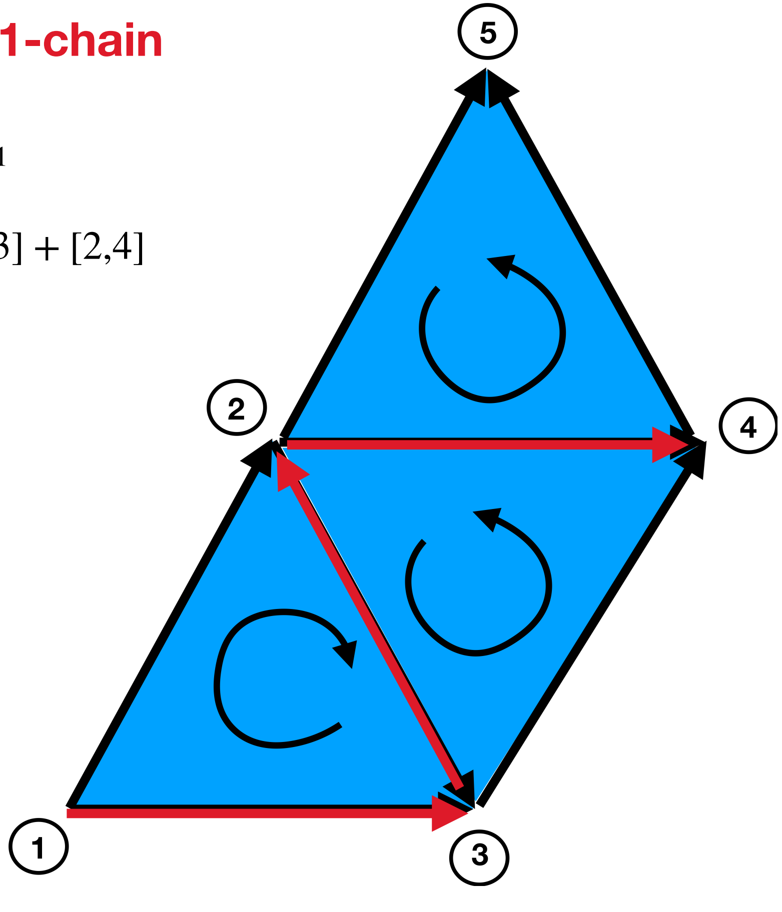
$$[i, j, k] = [j, k, i] = [k, i, j] = -[j, i, k] = -[k, j, i] = -[i, k, j]$$

Oriented simplicial complex and n-chains

Example of 1-chain

$a \in \mathscr{C}_1$

$a = [1,3] - [2,3] + [2,4]$



Boundary operator

The boundary map ∂_n is a linear operator

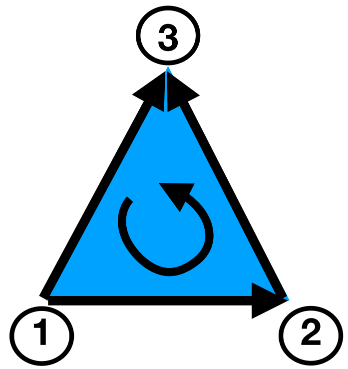
$$\partial_n : \mathscr{C}_n \to \mathscr{C}_{n-1}$$

whose action is determined by the action on each n-simplex of the simplicial complex

$$\partial_n[i_0, i_1 \ldots, i_n] = \sum_{p=0}^{n} (-1)^p [i_0, i_1, \ldots, i_{p-1}, i_{p+1}, \ldots, i_n] \,.$$

Therefore we have

$$\partial_1[1,2] = [2] - [1] \,.$$



$$\partial_2[1,2,3] = [2,3] - [1,3] + [1,2] \,.$$

The boundary of a boundary is null

The boundary operator has the property

$$\partial_n \partial_{n+1} = 0 \quad \forall n \geq 1$$

Which is usually indicated by saying that the boundary of the boundary is null.

This property follows directly from the definition of the boundary, as an example we have

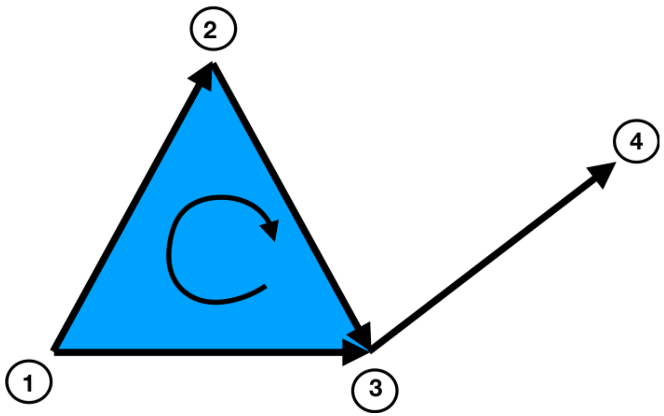
$$\partial_1 \partial_2 [i, j, k] = \partial_1([j, k] - [i, k] + [i, j]) = -[j] + [k] + [i] - [k] - [i] + [j] = 0.$$

Incidence matrices

Given a basis for the n simplices and n-1 simplices
the n-boundary operator

$$\partial_n[i_0, i_1 \ldots, i_n] = \sum_{p=0}^{n} (-1)^p [i_0, i_1, \ldots, i_{p-1}, i_{p+1}, \ldots, i_n] \,.$$

is captured by the incidence matrix $\mathbf{B}_{[n]}$



$$\mathbf{B}_{[1]} = \begin{array}{c|cccc} & [1,2] & [1,3] & [2,3] & [3,4] \\ [1] & -1 & -1 & 0 & 0 \\ [2] & 1 & 0 & -1 & 0 \\ [3] & 0 & 1 & 1 & -1 \\ [4] & 0 & 0 & 0 & 1 \end{array} ,$$

$$\mathbf{B}_{[2]} = \begin{array}{c|c} & [1,2,3] \\ [1,2] & 1 \\ [1,3] & -1 \\ [2,3] & 1 \\ [3,4] & 0 \end{array} .$$

Boundary of the boundary is null

In terms of the incidence matrices the relation

$$\partial_n \partial_{n+1} = 0 \quad \forall n \geq 1$$

Can be expressed as

$$\mathbf{B}_{[n]}\mathbf{B}_{[n+1]} = \mathbf{0} \quad \forall n \geq 1 \qquad \mathbf{B}_{[n+1]}^{\top}\mathbf{B}_{[n]}^{\top} = \mathbf{0} \quad \forall n \geq 1$$

Graph Laplacian in terms of the incidence matrix

The graph Laplacian of elements

$$\left(L_{[0]}\right)_{ij} = \delta_{ij}k_i - a_{ij}$$

Can be expressed in terms of the 1-incidence matrix

as

$$\mathbf{L}_{[0]} = \mathbf{B}_{[1]}\mathbf{B}_{[1]}^{\top}.$$

Higher-order Laplacian

The higher order Laplacians can be defined in terms of the incidence matrices as

$$\mathbf{L}_{[n]} = \mathbf{B}_{[n]}^{\top}\mathbf{B}_{[n]} + \mathbf{B}_{[n+1]}\mathbf{B}_{[n+1]}^{\top}.$$

The dimension of the $\mathrm{ker}(\mathbf{L}_{[n]})$ is the n-Betti number β_n

The higher order Laplacian can be decomposed as

$$\mathbf{L}_{[n]} = \mathbf{L}_{[n]}^{down} + \mathbf{L}_{[n]}^{up},$$

with

$$\mathbf{L}_{[n]}^{down} = \mathbf{B}_{[n]}^{\top}\mathbf{B}_{[n]},$$
$$\mathbf{L}_{[n]}^{up} = \mathbf{B}_{[n+1]}\mathbf{B}_{[n+1]}^{\top}.$$

Hodge decomposition

The Hodge decomposition can be summarised as

$$\mathbb{R}^{D_n} = \textbf{img}(\mathbf{B}_{[n]}^{\top}) \oplus \textbf{ker}(\mathbf{L}_{[n]}) \oplus \textbf{img}(\mathbf{B}_{[n+1]})$$

This means that $\mathbf{L}_{[n]}, \mathbf{L}_{[n]}^{up}, \mathbf{L}_{[n]}^{down}$ are commuting and can be diagonalised simultaneously. In this basis these matrices have the block structure

$$\mathbf{U}^{-1}\mathbf{L}_{[n]}\mathbf{U} = \begin{pmatrix} \mathbf{D}_{[n]}^{down} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{D}_{[n]}^{up} \end{pmatrix} \quad \mathbf{U}^{-1}\mathbf{L}_{[n]}^{down}\mathbf{U} = \begin{pmatrix} \mathbf{D}_{[n]}^{down} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \end{pmatrix} \quad \mathbf{U}^{-1}\mathbf{L}_{[n]}^{up}\mathbf{U} = \begin{pmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{D}_{[n]}^{up} \end{pmatrix}$$

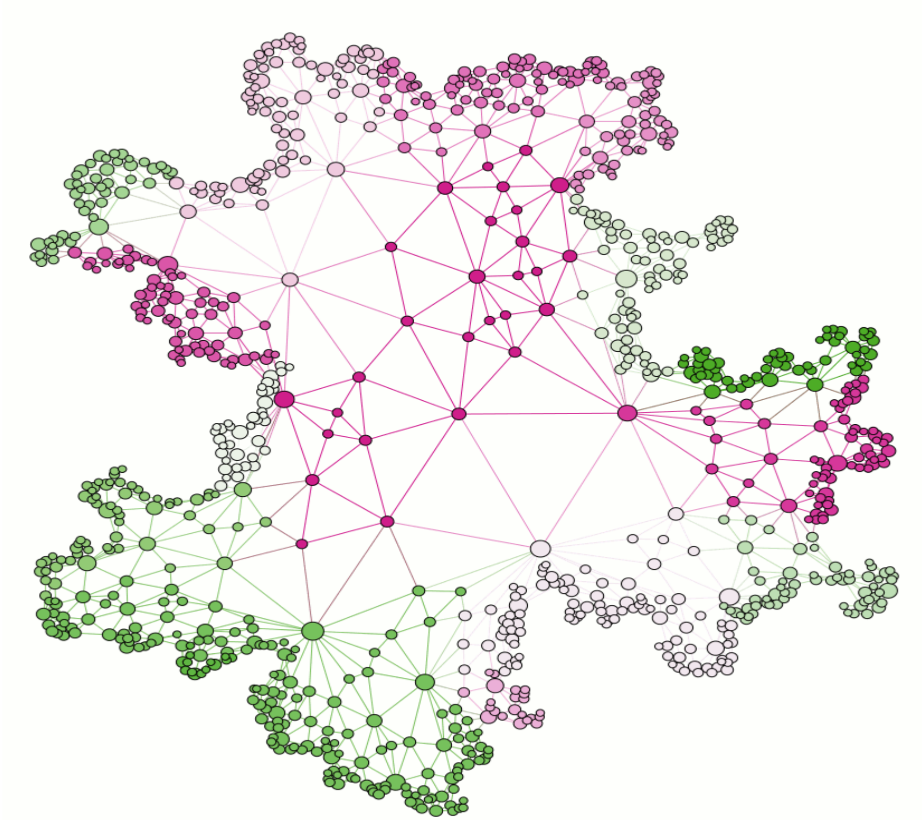
- **Therefore an eigenvector in the ker of $\mathbf{L}_{[n]}$ is also in the ker of both $\mathbf{L}_{[n]}^{up}, \mathbf{L}_{[n]}^{down}$**
- **An eigenvector corresponding to an non-zero eigenvalue of $\mathbf{L}_{[n]}$ is either a non-zero eigenvector of $\mathbf{L}_{[n]}^{up}$ or a non-zero eigenvector of $\mathbf{L}_{[n]}^{down}$**

Explosive higher-order Kuramoto model on simplicial complexes

A. P. Millán, J. J. Torres, and G.Bianconi,
Physical Review Letters, *124*, 218301 (2020)

Topological signals

Simplicial complexes can sustain dynamical variables (signals)
not only defined on nodes but also defined on higher order simplices
these signals are called *topological signals*



Standard Kuramoto model in terms of incidence matrices

The standard Kuramoto model, can be expressed in terms

of the incidence matrix $\mathbf{B}_{[1]}$ as

$$\dot{\boldsymbol{\theta}} = \boldsymbol{\omega} - \sigma \mathbf{B}_{[1]} \sin \mathbf{B}_{[1]}^{\top} \boldsymbol{\theta}$$

where we have defined the vectors

$$\boldsymbol{\theta} = (\theta_1, \theta_2, \ldots, \theta_i \ldots)^{\top}$$
$$\boldsymbol{\omega} = (\omega_1, \omega_2, \ldots, \omega_i \ldots)^{\top}$$

and we use the notation $\sin \mathbf{x}$

to indicates the column vector where the sine function is taken element wise

Topological signals

We associate to each

n-dimensional simplex α a phase ϕ_α

For instance for n=1 we might associate to each link a oscillating flux

The vector of phases is indicated by

$$\boldsymbol{\phi} = (\ldots, \phi_\alpha \ldots)^\top$$

Higher-order Kuramoto model

We propose to study the higher-order Kuramoto model

defined as

$$\dot{\boldsymbol{\phi}} = \hat{\boldsymbol{\omega}} - \sigma \mathbf{B}_{[n+1]} \sin \mathbf{B}_{[n+1]}^{\top} \boldsymbol{\phi} - \sigma \mathbf{B}_{[n]}^{\top} \sin \mathbf{B}_{[n]} \boldsymbol{\phi},$$

where is the vector of phases associated to n-simplices

and the topological signals ad their internal frequencies are indicated by

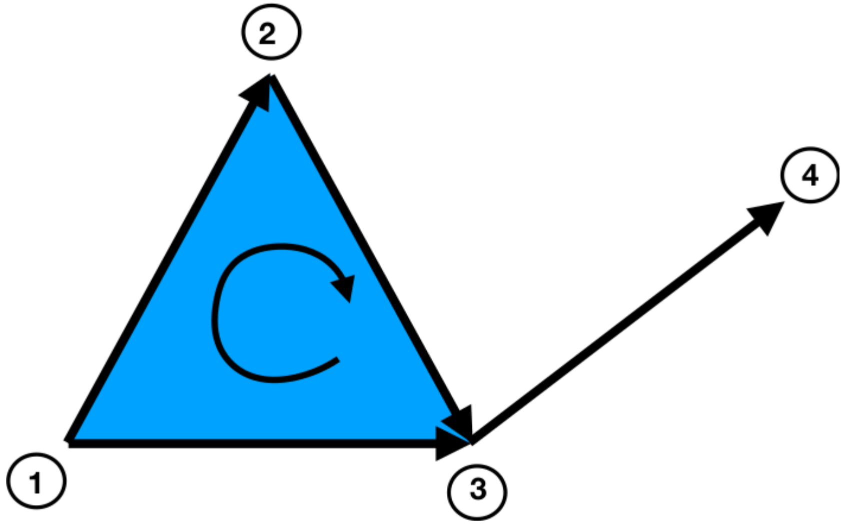
$$\boldsymbol{\phi} = (\ldots, \theta_{\alpha} \ldots)^{\top}$$

$$\hat{\boldsymbol{\omega}} = (\ldots, \hat{\omega}_{\alpha} \ldots)^{\top}$$

with the internal frequencies

$$\hat{\omega}_{\alpha} \sim \mathcal{N}(\Omega, 1)$$

Topologically induced many-body interactions



$$\dot{\phi}_{[12]} = \hat{\omega}_{[12]} - \sigma \sin(\phi_{[23]} - \phi_{[13]} + \phi_{[12]}) - \sigma \left[\sin(\phi_{[12]} - \phi_{[23]}) + \sin(\phi_{[13]} + \phi_{[12]})\right],$$

$$\dot{\phi}_{[13]} = \hat{\omega}_{[13]} + \sigma \sin(\phi_{[23]} - \phi_{[13]} + \phi_{[12]}) - \sigma \left[\sin(\phi_{[13]} + \phi_{[12]}) + \sin(\phi_{[13]} + \phi_{[23]} - \phi_{[34]})\right],$$

$$\dot{\phi}_{[23]} = \hat{\omega}_{[23]} - \sigma \sin(\phi_{[23]} - \phi_{[13]} + \phi_{[12]}) - \sigma \left[\sin(\phi_{[23]} - \phi_{[12]}) + \sin(\phi_{[13]} + \phi_{[23]} - \phi_{[34]})\right],$$

$$\dot{\phi}_{[34]} = \hat{\omega}_{[34]} - \sigma \left[\sin(\phi_{[34]}) - \sin(\phi_{[13]} + \phi_{[23]} - \phi_{[34]})\right],$$

If we define a higher-order Kuramoto model on

n-simplices,

(let us say links, n=1) a key question is:

What is the dynamics induced

on (n-1) faces and (n+1) faces?

i.e. what is the dynamics induced on nodes and triangles?

Projected dynamics on n-1 and n+1 faces

A natural way to project the dynamics is to use the incidence matrices obtaining

$$\boldsymbol{\phi}^{[+]} = \mathbf{B}_{[n+1]}^{\top}\boldsymbol{\phi}$$ **Discrete curl**

$$\boldsymbol{\phi}^{[-]} = \mathbf{B}_{[n]}\boldsymbol{\phi}$$ **Discrete divergence**

Projected dynamics on n-1 and n+1 faces

Thanks to Hodge decomposition,

the projected dynamics

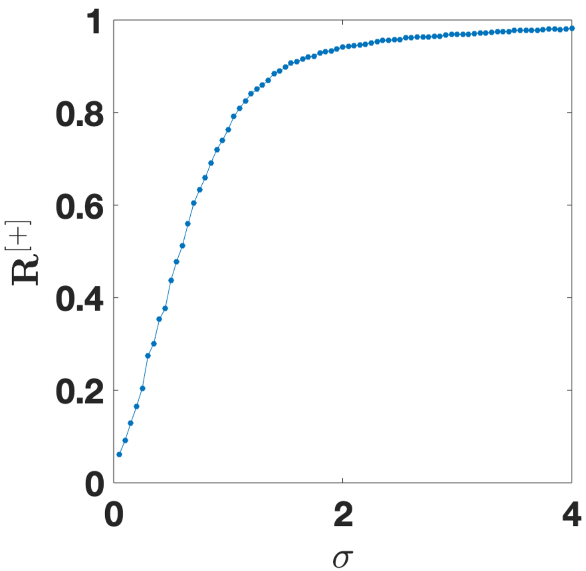
on the (n-1) and (n+1) faces

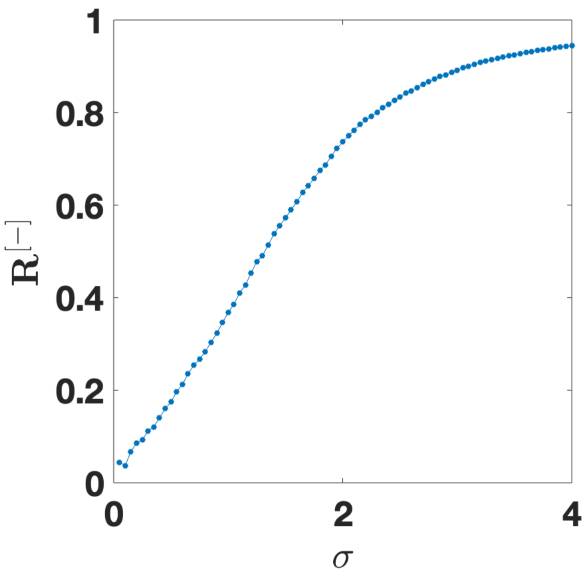
decouple

$$\dot{\boldsymbol{\phi}}^{[+]} = \mathbf{B}_{[n+1]}^{\top}\hat{\boldsymbol{\omega}} - \sigma \mathbf{L}_{[n+1]}^{[down]} \sin(\boldsymbol{\phi}^{[+]})$$
$$\dot{\boldsymbol{\phi}}^{[-]} = \mathbf{B}_{[n]}\hat{\boldsymbol{\omega}} - \sigma \mathbf{L}_{[n-1]}^{[up]} \sin(\boldsymbol{\phi}^{[-]})$$

Synchronization transition

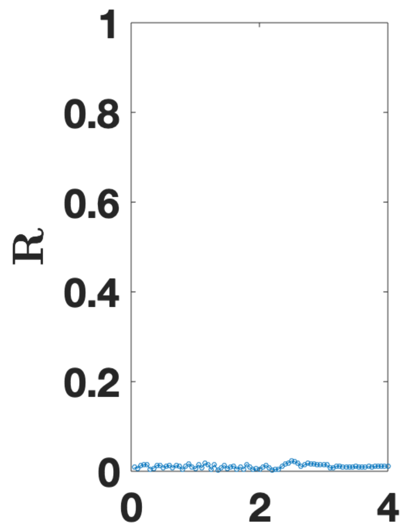
$$R^{[+]} = \frac{1}{N_{n+1}} \left| \sum_{\alpha=1}^{N_{n+1}} e^{\,i\phi_\alpha^{[+]}} \right| \qquad R^{[-]} = \frac{1}{N_{n-1}} \left| \sum_{\alpha=1}^{N_{n-1}} e^{\,i\phi_\alpha^{[-]}} \right|$$





Order parameters using the n-dimensional phases

$$R = \frac{1}{N_n} \left| \sum_{\alpha=1}^{N_n} e^{i \phi_\alpha} \right|$$



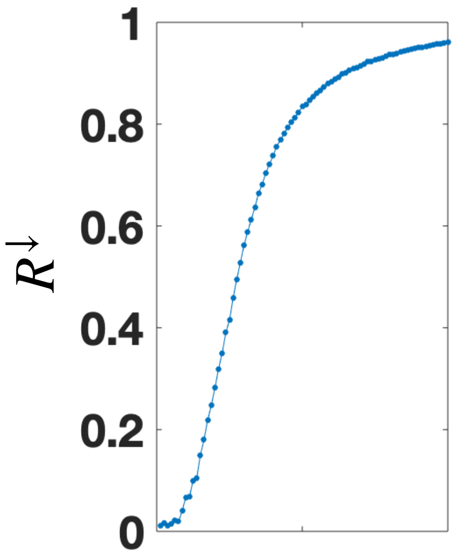
Order parameters using the n-dimensional phases

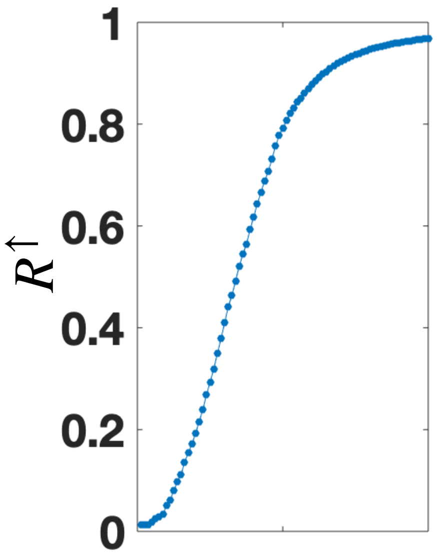
$$\boldsymbol{\phi}^{\downarrow} = \mathbf{L}_{[n]}^{down}\boldsymbol{\phi}$$

$$R^{\downarrow} = \frac{1}{N_n}\left|\sum_{\alpha=1}^{N_n} e^{i\,\phi_\alpha^{\downarrow}}\right|$$

$$\boldsymbol{\phi}^{\uparrow} = \mathbf{L}_{[n]}^{up}\boldsymbol{\phi}$$

$$R^{\uparrow} = \frac{1}{N_n}\left|\sum_{\alpha=1}^{N_n} e^{i\,\phi_\alpha^{\uparrow}}\right|$$





Only if we perform

the correct topological filtering

of the topological signal

we can reveal higher-order synchronisation

Explosive higher-order synchronisation

We propose the Explosive Higher-order Kuramoto model

defined as

$$\dot{\boldsymbol{\phi}} = \hat{\boldsymbol{\omega}} - \sigma R^{[-]}\mathbf{B}_{[n+1]} \sin \mathbf{B}_{[n+1]}^{\top}\boldsymbol{\phi} - \sigma R^{[+]}\mathbf{B}_{[n]}^{\top} \sin \mathbf{B}_{[n]}\boldsymbol{\phi}$$

Projected dynamics

The projected dynamics on

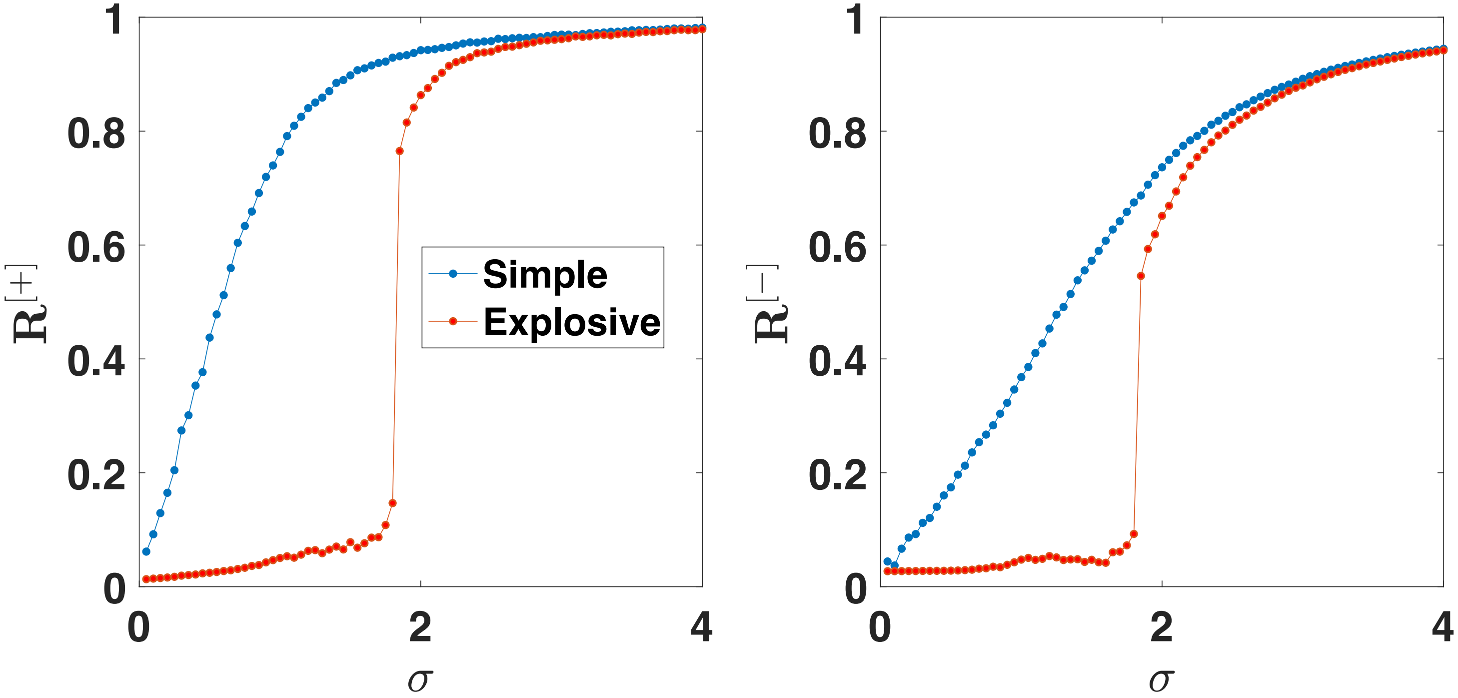
(n+1) and (n-1) are now coupled

by their order parameters

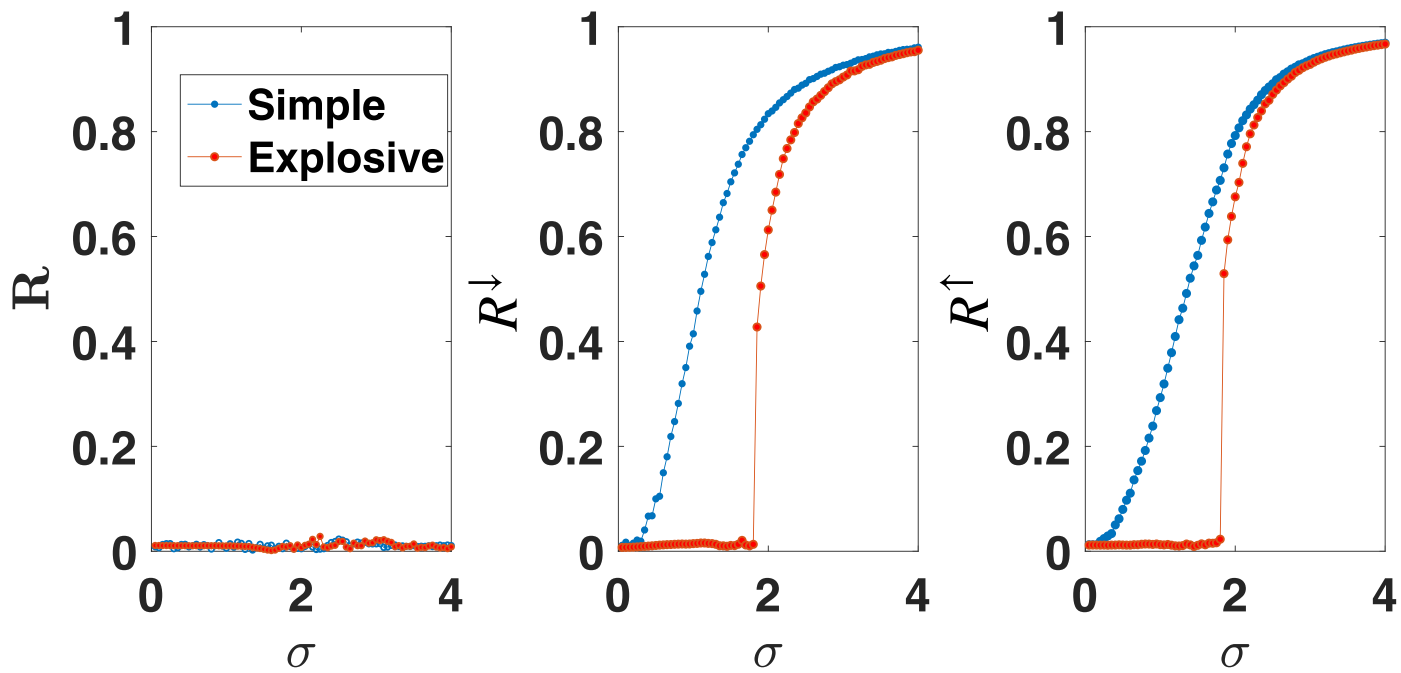
$$\dot{\boldsymbol{\phi}}^{[+]} = \mathbf{B}_{[n+1]}^{\top}\hat{\boldsymbol{\omega}} - \sigma R^{[-]}\mathbf{L}_{[n+1]}^{[down]}\sin(\boldsymbol{\phi}^{[+]})$$

$$\dot{\boldsymbol{\phi}}^{[-]} = \mathbf{B}_{[n]}\hat{\boldsymbol{\omega}} - \sigma R^{[+]}\mathbf{L}_{[n-1]}^{[up]}\sin(\boldsymbol{\phi}^{[-]})$$

The synchronisation transition is discontinuous



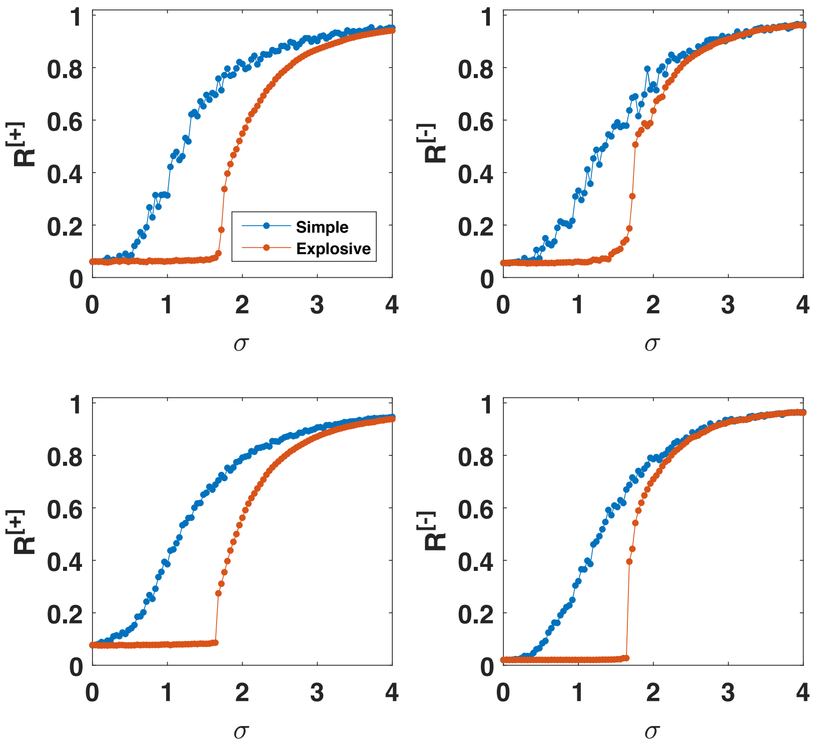
Order parameters associated to n-faces



Higher-order synchronisation on real Connectomes

Homo sapiens Connectome

C.elegans Connectome



Take home messages

- **Hodge theory is combined with the theory of dynamical systems to shed light on higher-order synchronization.**
- **With our theoretical framework we can treat synchronization of topological dynamical signals associated to links, like fluxes, or to triangles or other higher-order simplices.**
- **The simple higher-order Kuramoto model of n-dimensional topological signals induces a dynamics on n+1 and n-1 faces that is uncoupled and synchronises at a continuous synchronisation transition with $\sigma_c=0$.**
- **The explosive higher-order Kuramoto model couples the projected dynamics on n+1 and n-1 simplices inducing a discontinuous transition.**

Take home messages

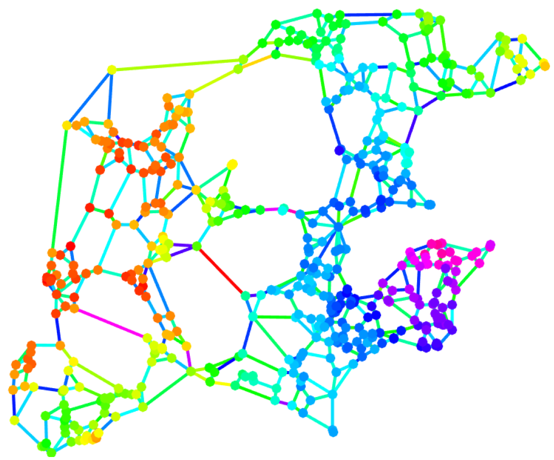
We have shown that topological signals can undergo a synchronization transition, but this synchronization can be unnoticed if the correct topological transformations are not performed.

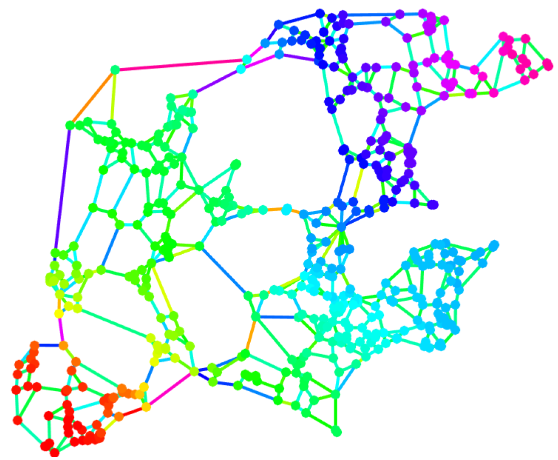
What we propose here is the equivalent of a Fourier transform for topological signals that can reveal this transition in real systems such as biological transport networks and the brain.

Topological Dirac operator

How to treat the interaction between topological signals of different dimensions coexisting in the same network topology?

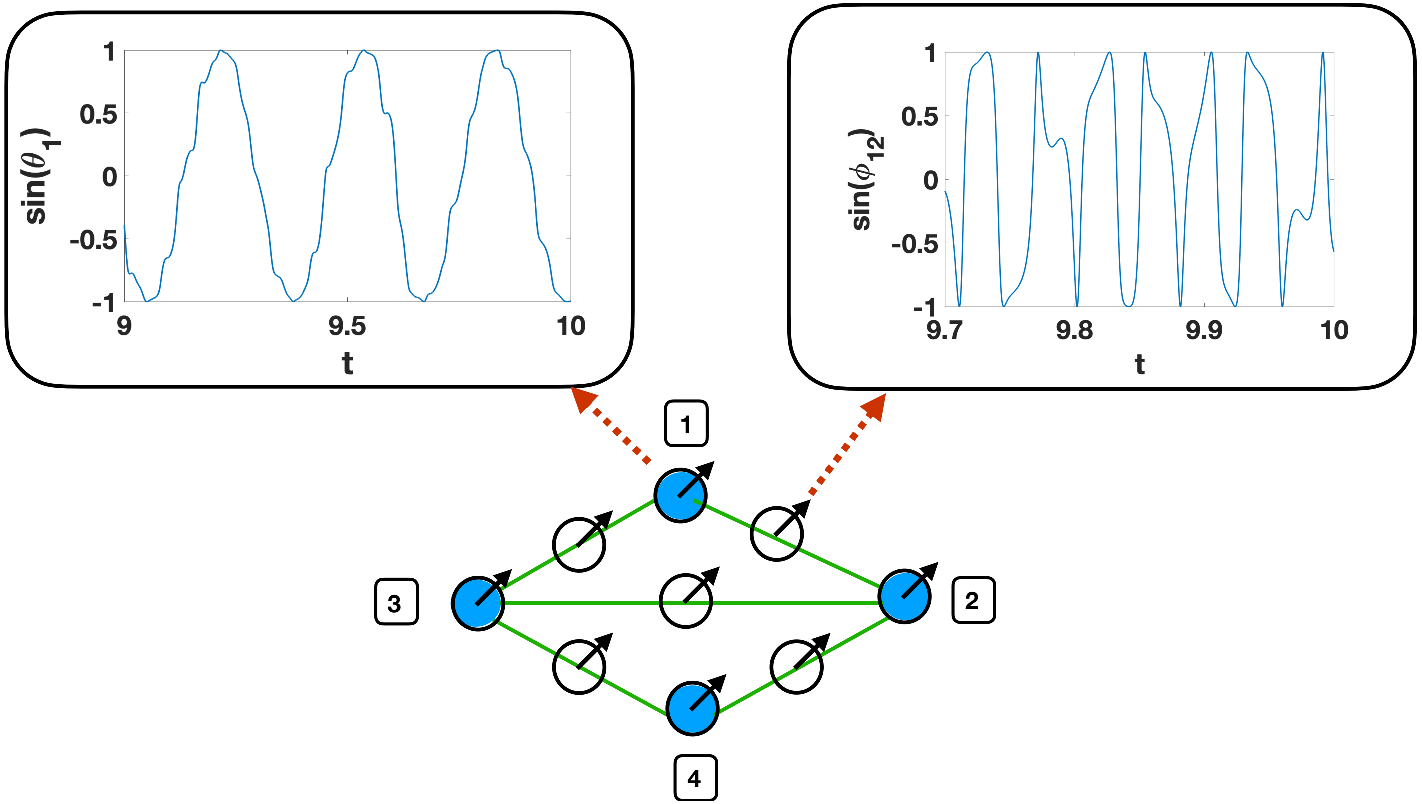
G. Bianconi,
Topological Dirac equation on networks and simplicial complexes (2021)





Topological sinchronization

In a network topological synchronisation locally couples
topological signals defined on nodes and links



Dirac operator of a network

- The Dirac operator of a network can be defined as

$$\mathbf{D} = \begin{pmatrix} \mathbf{0} & \mathbf{B}_{[1]} \\ \mathbf{B}_{[1]}^{\top} & \mathbf{0} \end{pmatrix}$$

- Acting on a vector formed by topological signal of nodes and links (a vector whose block structure is formed by a 0-cochain and a 1-cochain)

$$\boldsymbol{\Phi} = \begin{pmatrix} \boldsymbol{\theta} \\ \boldsymbol{\phi} \end{pmatrix}$$

Dirac operator is the "square root" of the Laplacian

- The square of the Dirac operator of a network

$$\mathbf{D} = \begin{pmatrix} \mathbf{0} & \mathbf{B}_{[1]} \\ \mathbf{B}_{[1]}^{\top} & \mathbf{0} \end{pmatrix}$$

- is the higher-order Laplacian matrix

$$\mathbf{D}^2 = \mathbf{L} = \begin{pmatrix} \mathbf{L}_{[0]} & \mathbf{0} \\ \mathbf{0} & \mathbf{L}_{[1]} \end{pmatrix}$$

Synchronization of uncoupled topological signals

- The uncoupled dynamics of nodes and links of a network

$$\dot{\boldsymbol{\theta}} = \boldsymbol{\omega} - \sigma \mathbf{B}_{[1]} \sin \mathbf{B}_{[1]}^{\top} \boldsymbol{\theta} \qquad \dot{\boldsymbol{\phi}} = \hat{\boldsymbol{\omega}} - \sigma \mathbf{B}_{[1]}^{\top} \sin \mathbf{B}_{[1]} \boldsymbol{\phi}$$

- can be expressed by a single equation

$$\boxed{\dot{\boldsymbol{\Phi}} = \boldsymbol{\Omega} - \sigma \mathbf{D} \sin \mathbf{D} \boldsymbol{\Phi}}$$

- Where we have defined the Dirac operator **D** and the vectors $\boldsymbol{\Phi}$ and $\boldsymbol{\Omega}$ as

$$\mathbf{D} = \begin{pmatrix} \mathbf{0} & \mathbf{B}_{[1]}^{\top} \\ \mathbf{B}_{[1]} & \mathbf{0} \end{pmatrix} \qquad \boldsymbol{\Phi} = \begin{pmatrix} \boldsymbol{\theta} \\ \boldsymbol{\phi} \end{pmatrix} \qquad \boldsymbol{\Omega} = \begin{pmatrix} \boldsymbol{\omega} \\ \hat{\boldsymbol{\omega}} \end{pmatrix}$$

Phase-lags

We introduce a phase-lag for the dynamics of the nodes that depends on the dynamics of the nearby links

We introduce a phase-lag for the dynamics of the link that depends on the dynamics of the nearby nodes

Topological synchronisation

- With the following notation

$$\mathbf{\Phi} = \begin{pmatrix} \boldsymbol{\theta} \\ \boldsymbol{\phi} \end{pmatrix} \qquad \mathbf{\Omega} = \begin{pmatrix} \boldsymbol{\omega} \\ \hat{\boldsymbol{\omega}} \end{pmatrix} \qquad \mathbf{D} = \begin{pmatrix} \mathbf{0} & \mathbf{B}_{[1]} \\ \mathbf{B}_{[1]}^{\top} & \mathbf{0} \end{pmatrix}$$

- Topological synchronisation follows the equation

$$\dot{\mathbf{\Phi}} = \mathbf{\Omega} - \sigma \mathbf{D} \sin[(\mathbf{D} - \boldsymbol{\gamma} \boldsymbol{\mathscr{K}}^{-1} \mathbf{L})\mathbf{\Phi}]$$

- Where we have defined

$$\mathbf{D}^2 = \mathbf{L} = \begin{pmatrix} \mathbf{L}_{[0]} & \mathbf{0} \\ \mathbf{0} & \mathbf{L}_{[1]} \end{pmatrix} \qquad \boldsymbol{\gamma} = \begin{pmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{0} & -\mathbf{I} \end{pmatrix} \qquad \boldsymbol{\mathscr{K}} = \begin{pmatrix} \mathbf{K}_{[0]} & \mathbf{0} \\ \mathbf{0} & \mathbf{K}_{[1]} \end{pmatrix}$$

Topological synchronisation

The dynamics of topological signals of nodes and links is dictated by the set of equations

$$\dot{\boldsymbol{\theta}} = \boldsymbol{\omega} - \sigma \mathbf{B}_{[1]}^{\top} \sin(\mathbf{B}_{[1]}\boldsymbol{\theta} + \mathbf{L}_{[1]}\boldsymbol{\phi}/2)$$
$$\dot{\boldsymbol{\phi}} = \hat{\boldsymbol{\omega}} - \sigma \mathbf{B}_{[1]}^{\top} \sin(\mathbf{B}_{[1]}\boldsymbol{\phi} - \mathbf{K}_{[\mathbf{0}]}{}^{-1}\mathbf{L}_{[0]}\boldsymbol{\theta})$$

Order parameters

- The canonical variables for topological synchronisation are

$$\boldsymbol{\alpha} = \boldsymbol{\theta} + \boldsymbol{\psi}/2$$
$$\boldsymbol{\beta} = \boldsymbol{\psi} - \boldsymbol{\theta} + \boldsymbol{\Theta}$$

- Where $\boldsymbol{\psi} = \mathbf{B}_{[1]}\boldsymbol{\phi}$ and $\boldsymbol{\Theta} = \mathbf{K}_{[0]}^{-1}\mathbf{A}\boldsymbol{\theta}$ leading to the two order parameters

$$X_\alpha = R_\alpha e^{i\Psi_\alpha} = \frac{1}{N}\sum_{j=1}^{N} e^{i\alpha_j}$$

$$X_\beta = R_\beta e^{i\Psi_\beta} = \frac{1}{N}\sum_{j=1}^{N} e^{i\beta_j}$$

Stationary state solution on a fully connected network

- We consider a fully connected network
- With parameters given by

$$\sigma \to \sigma/N$$
$$\omega_i \sim \mathcal{N}(\Omega_0, 1)$$
$$\tilde{\omega}_i \sim \mathcal{N}(0, 1/\sqrt{N-1})$$

Assuming stationarity the order parameter follows the same equations of the standard Kuramoto model

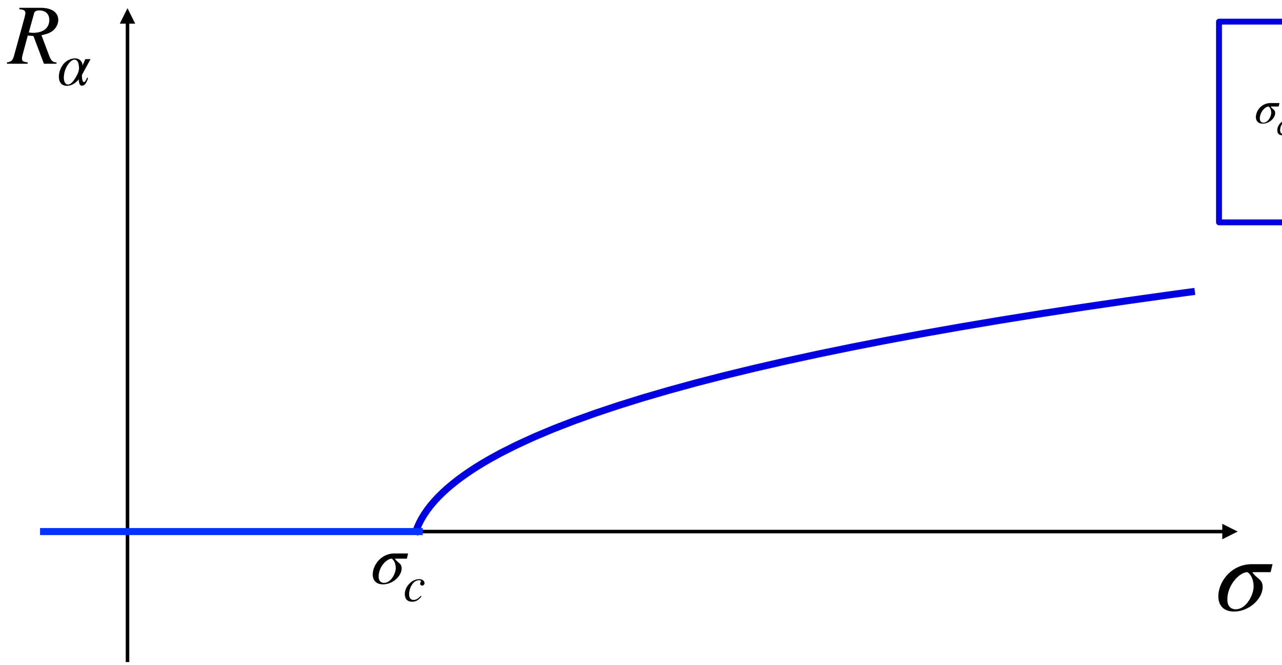
$$R_\alpha = 0$$

Incoherent phase

$$R_\alpha = \frac{1}{N} \int_{|\omega - \Omega_0| \leq \sigma R_\alpha} d\omega G_0(\omega) \sqrt{1 - \left(\frac{\omega - \Omega_0}{\sigma R_\alpha} \right)^2}$$

Coherent phase

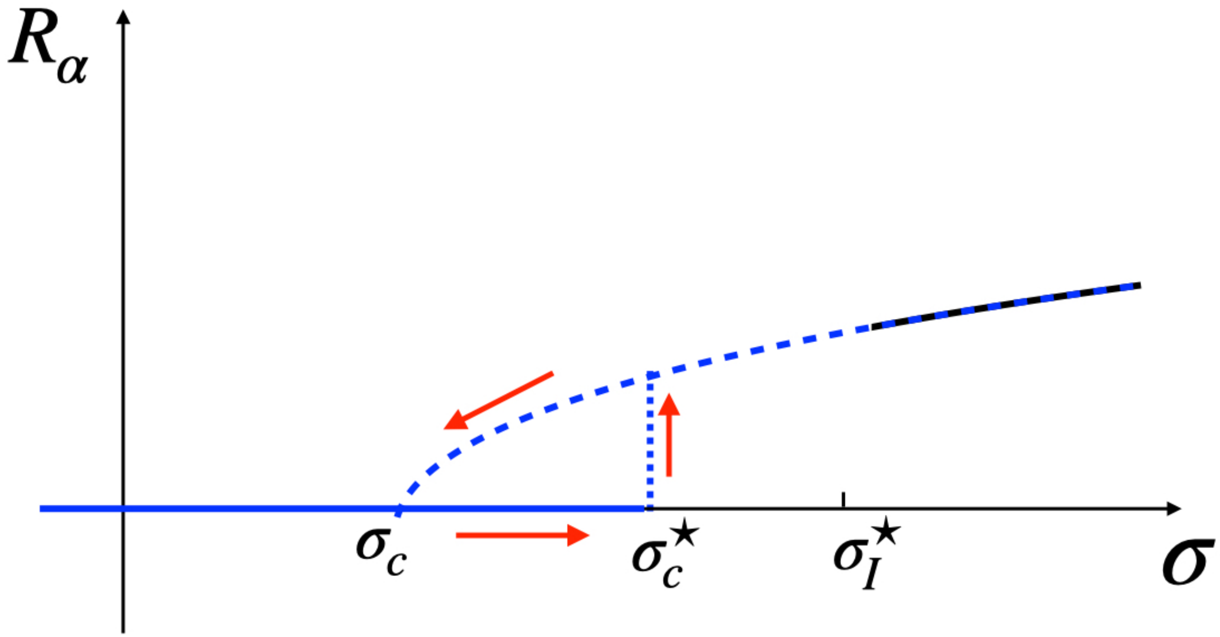
Phase diagram of the standard Kuramoto model



$$\sigma_c = \frac{2}{\pi G(\Omega_0)} = 1.5957\ldots$$

Phase diagram in a fully connected network

Topological synchronisation is explosive



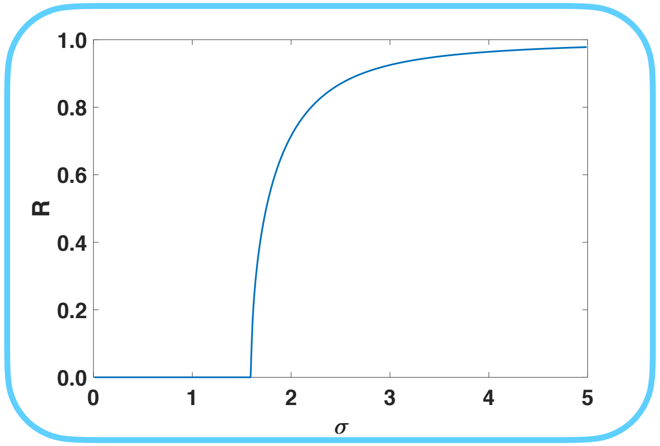
Phase diagram in a fully connected network

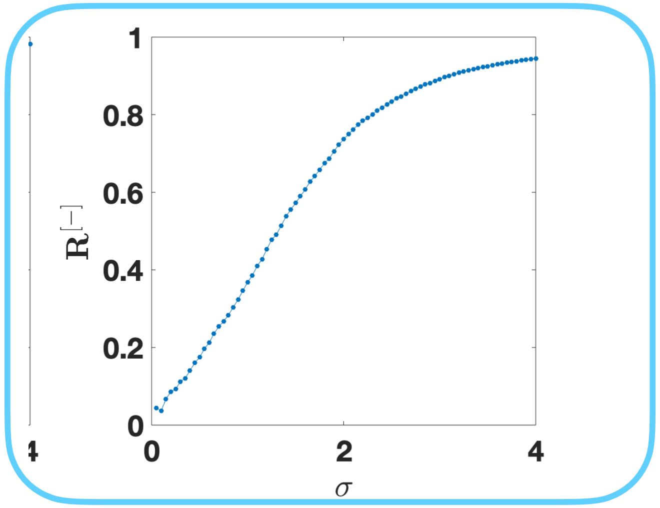
$$\sigma_c = \frac{2}{\pi G(\Omega_0)} = 1.5957\ldots$$

$$\sigma_c^\star = \frac{12}{5}\frac{1}{\pi G(\Omega_0)} = 1.914\ldots$$

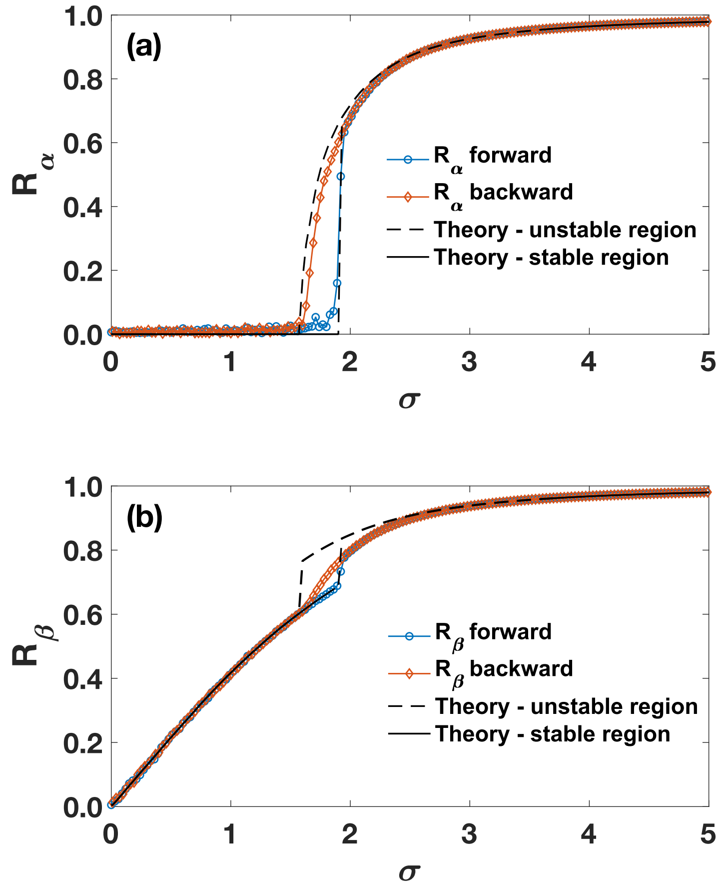
$$\sigma_I^\star = 3.70 \pm 0.05$$

Phase diagram of topological synchronisation

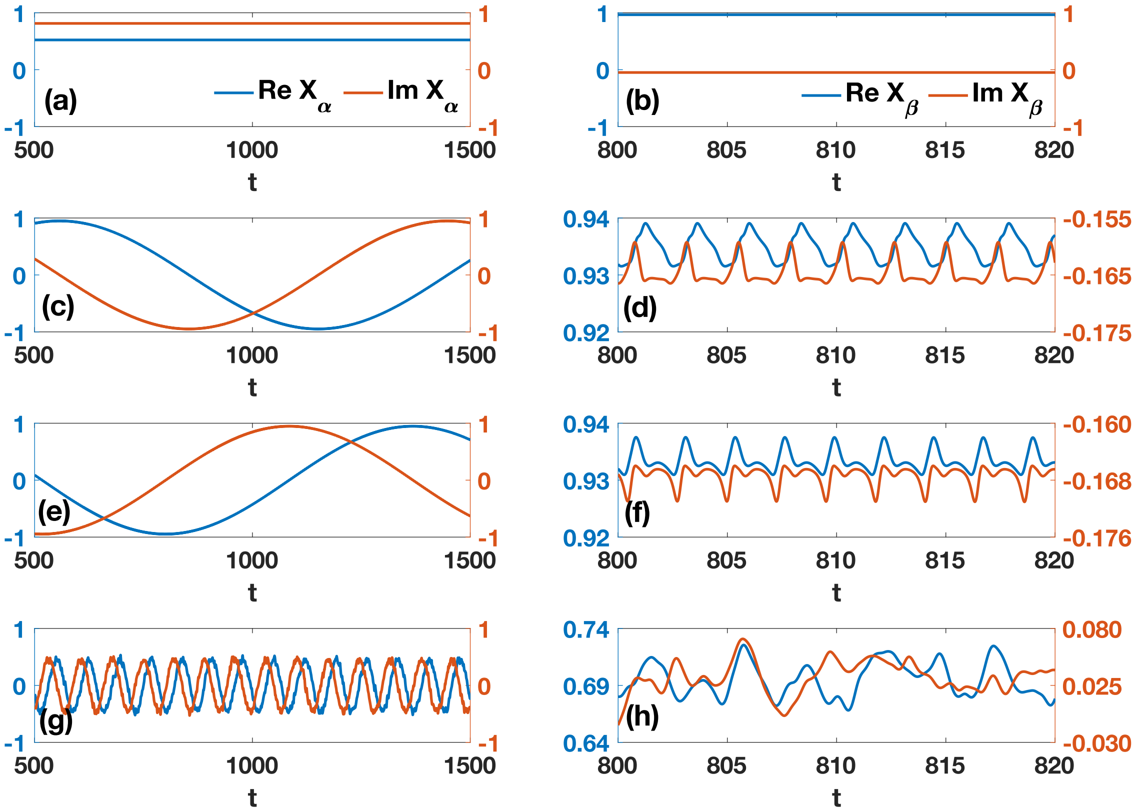


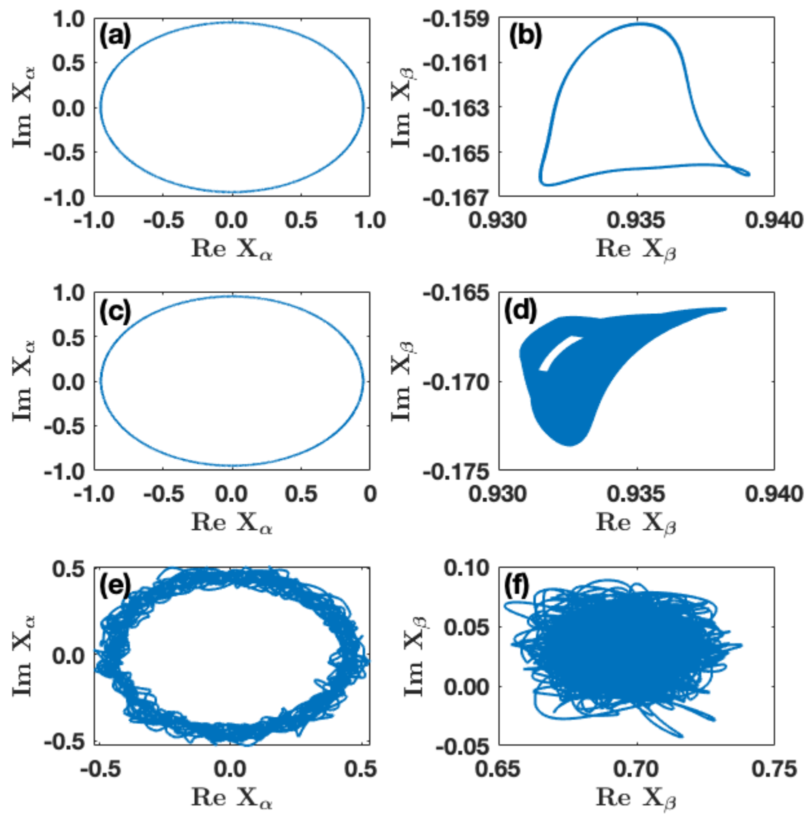


Coupled signals of Topological Synchronization

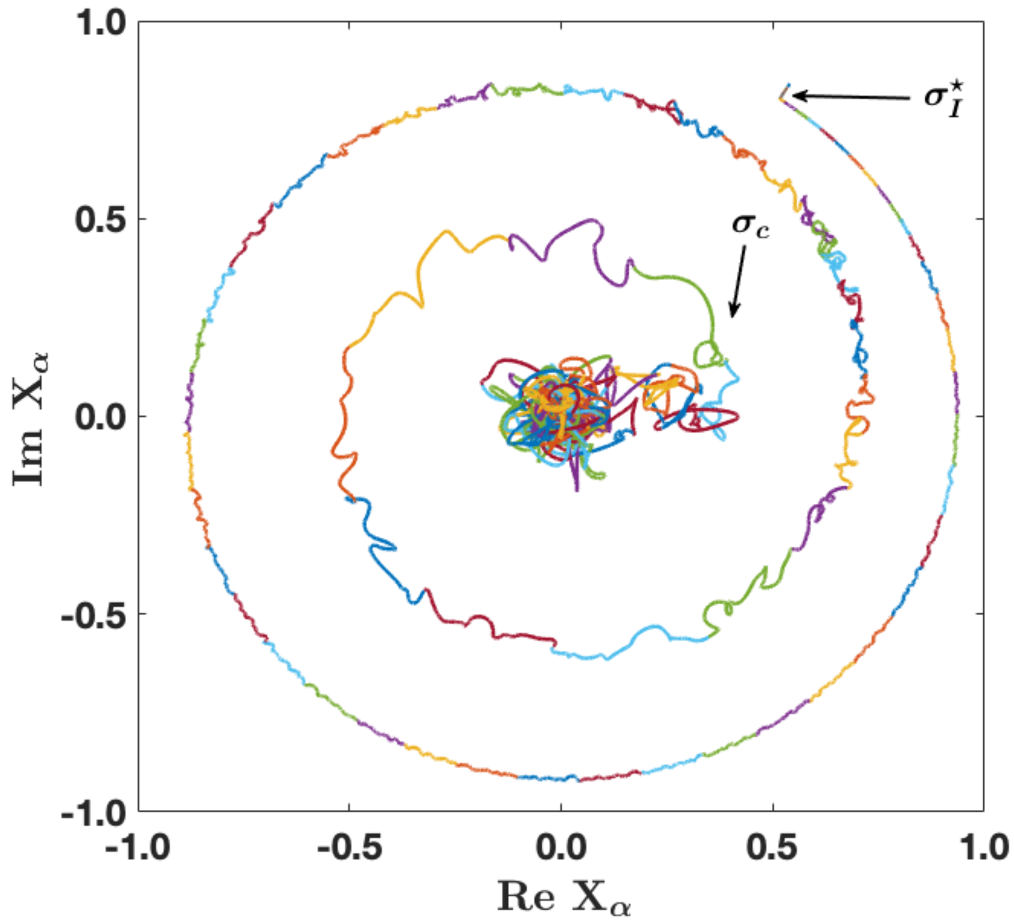


Rhythmic phase





Desynchronization transition in the rhythmic phase



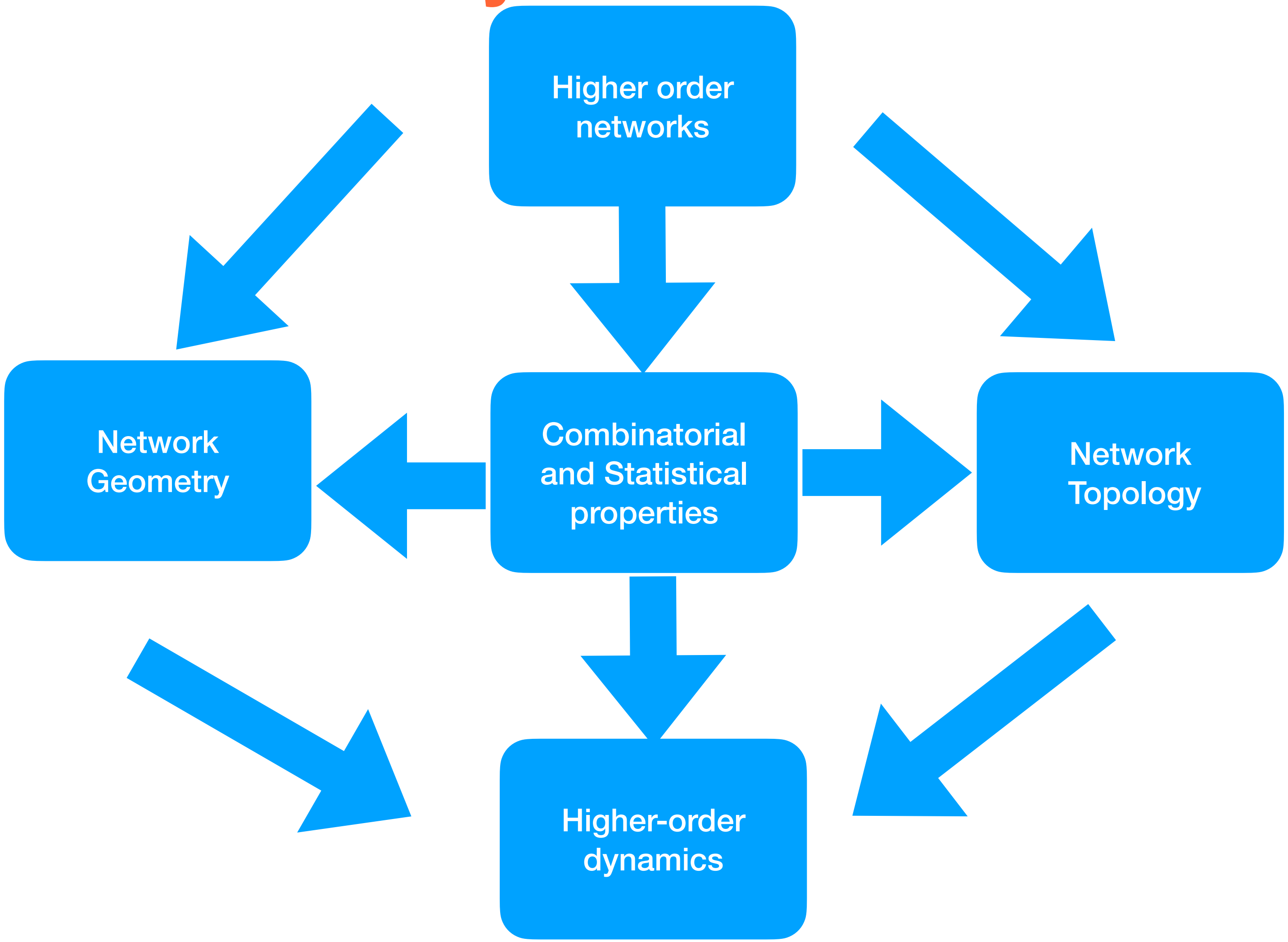
Summary

Topological synchronisation coupling locally topological signals of nodes and links

is explosive

gives rise of rhythmic phases that might be related to brain rhythms

Higher order networks and dynamics



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