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Majorana Fermions in Wire Networks as non-Abelian Anyons

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Quantum Matter at Ultralow Temperatures

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Outline:

Exchange and statistics

Majorana fermions as non-Abelian anyons

“Making” Majorana fermions (Kitaev wire)

Majorana fermions in atomic wires

- “Making” Kitaev wire with cold atoms/molecules
- Braiding protocol

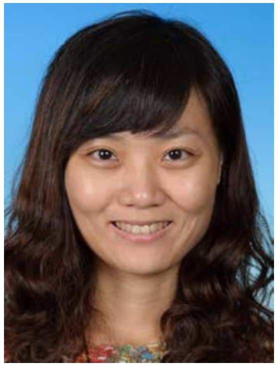
Demonstration of non-Abelian statistics

Using Majorana fermions for QC

- Deutsch-Jozsa algorithm

Conclusion

Innsbruck Majorana Team:



Ying Hu

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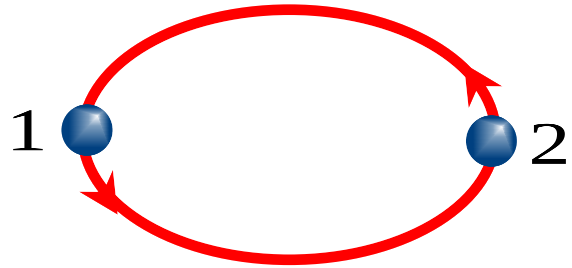
Exchange and statistics

Particle exchange and statistics

Statistics $\Longleftrightarrow$ Behavior of the state (wave function) under the exchange of two identical (quasi)particles

Exchange of two (quasi)particles



$$\Psi(\vec{r}_1,\vec{r}_2,\ldots) \rightarrow \Psi(\vec{r}_2,\vec{r}_1,\ldots) = ?\, \Psi(\vec{r}_1,\vec{r}_2,\ldots)$$

Properties of many-body wave functions:

$$\psi(\vec{R}_1, \vec{R}_2, \ldots; \vec{r}_1, \vec{r}_2, \ldots)$$

$\vec{R}_i$ positions of quasiparticles

$\vec{r}_j$ positions of particles

has to be single-valued with respect to particle coordinates $\vec{r}_j$

but not necessarily with respect to quasiparticle coordinates $\vec{R}_i$

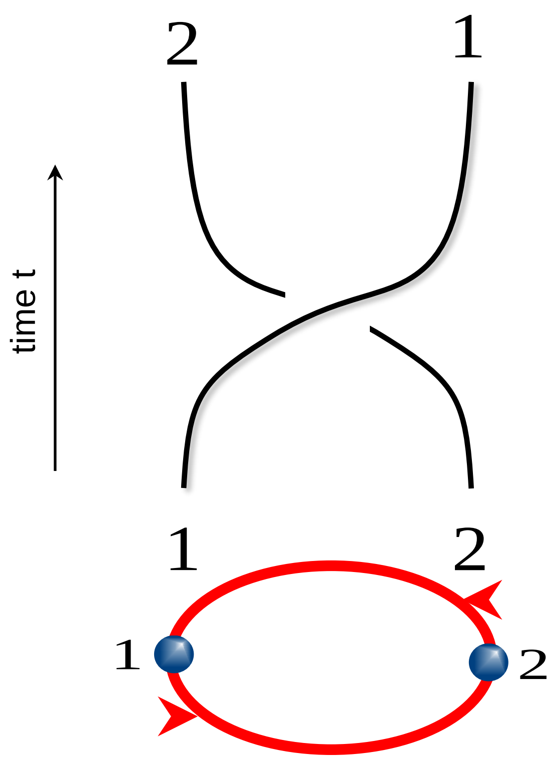
Exchange as adiabatic dynamical evolution
General statements:

1. Adiabatic theorem: States in a (possibly degenerate) energy subspace separated from others by a gap remain in the subspace when the system is changed adiabatically without closing the gap.

2. Change under adiabatic transport (holonomy) = combination of Berry's phase/matrix and transformation of instantaneous energy eigenstate (explicit holonomy)

3. Holonomy is invariant, but Berry's phase/matrix and eigenstate transformation depend on choice of gauge (and can be shifted from one to the other)

Exchange as adiabatic dynamical evolution



For a unique ground state $|\Psi\rangle$ (single-valued)

separated by a gap from excited states

$$\Longrightarrow \qquad |\psi\rangle \rightarrow e^{i\varphi}|\psi\rangle, \qquad \varphi = -\frac{1}{\hbar}\int dt\, E(t) + \alpha$$

↑ dynamical phase

Berry phase $\alpha = i\int dt \langle \psi | \frac{d}{dt}\psi\rangle = \alpha_g(\text{path}) + \vartheta$

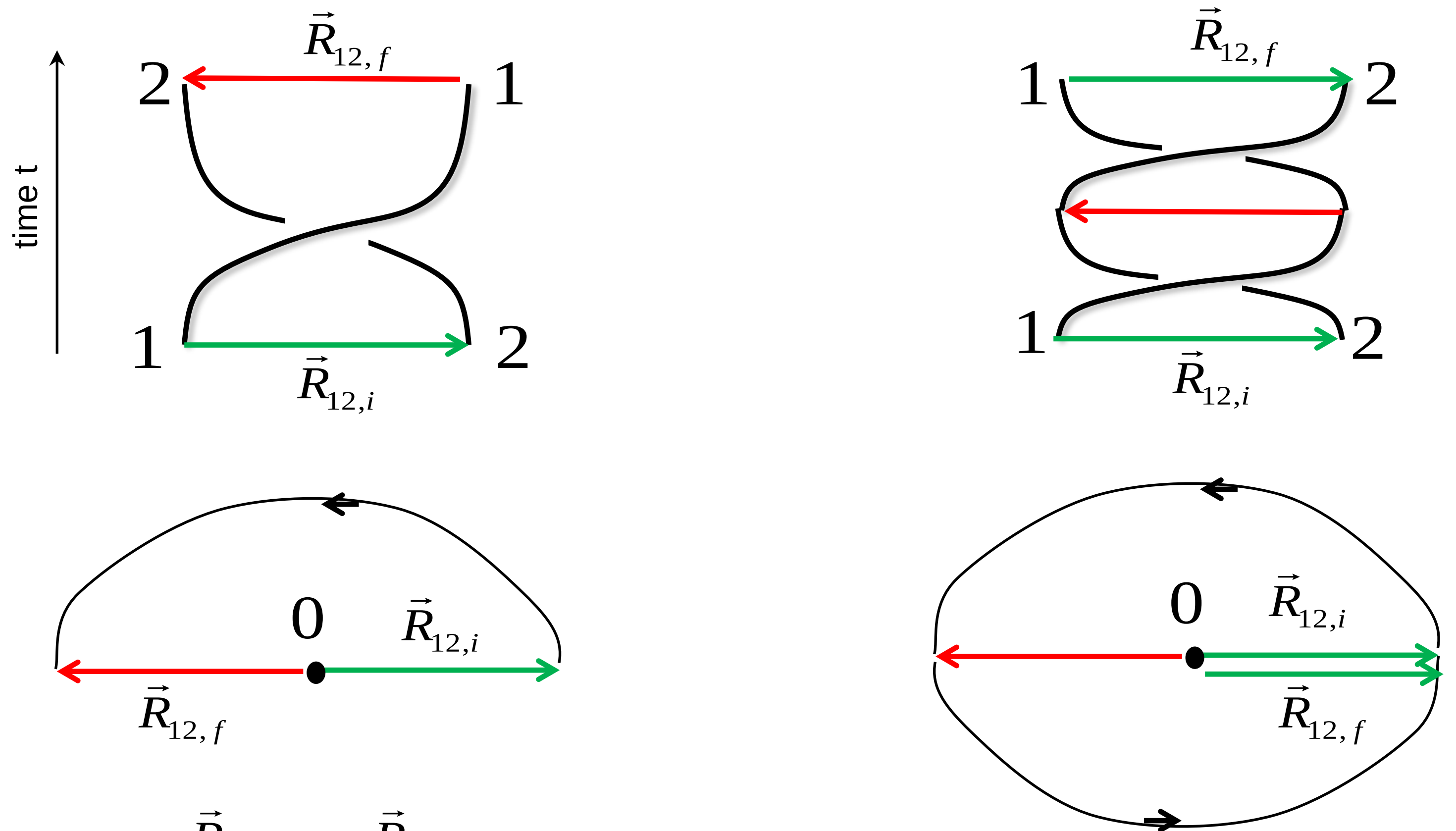
$\alpha_g(\text{path})$ geometrical phase

ϑ **statistical angle** - of interest!

Exchange statistics $\quad |\psi\rangle \rightarrow e^{i\vartheta}|\psi\rangle$

Constraint on the exchange $\vec{R}_{12}$ relative position

exchange $\hat{\sigma}$ double exchange $\hat{\sigma}^2$

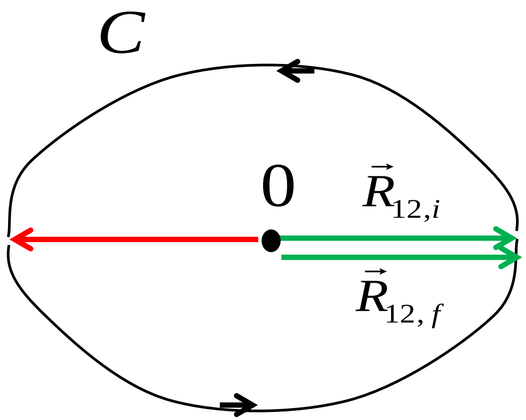


$\vec{R}_{12,f} = -\vec{R}_{12,i}$

$\vec{R}_{12,f} = \vec{R}_{12,i}$

Is $\hat{\sigma}^2$ an identity?

3D case: $\hat{\sigma}^2 = 1$ (identity!)

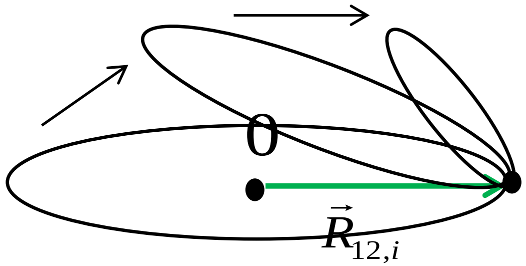


The contour C **can** be deformed

to a point $\vec{R}_{12} = \text{const} \; (= \vec{R}_{12,i})$

(i.e., to the case when nothing happens)

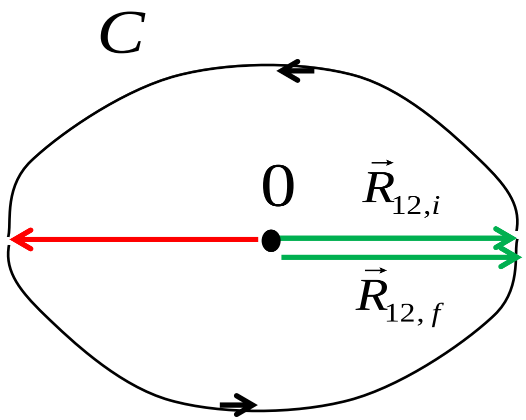
without crossing the origin $\vec{R}_{12} = 0$



Conclusion: in 3D only bosons ($\vartheta = 0$) or fermions ($\vartheta = \pi$)

$$|\psi\rangle \rightarrow \pm |\psi\rangle$$

2D case: $\hat{\sigma}^2 \neq 1$

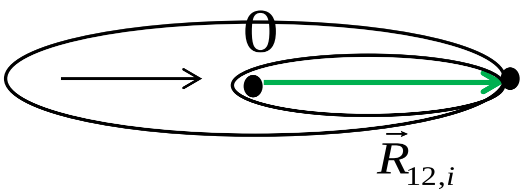


The contour C **cannot** be deformed

to a point $\vec{R}_{12} = \text{const}\ (= \vec{R}_{12,i})$

(i.e., to the case when nothing happens)

without crossing the origin $\vec{R}_{12} = 0$

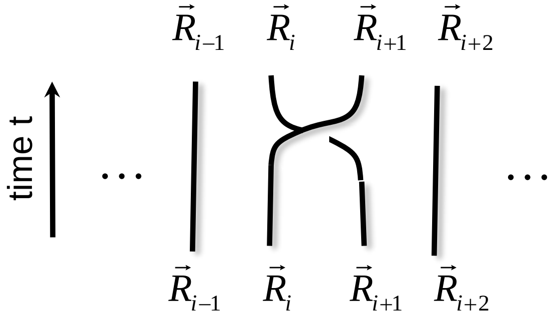


Conclusion: in 2D more possibilities, not only bosons or fermions

Particle exchange in 2D: Braid group (for N particles)

Trajectories that wind around starting from initial positions $\vec{R}_1, \ldots, \vec{R}_N$
to final positions $\vec{R}_1, \ldots, \vec{R}_N$ (the same set – identical particles)

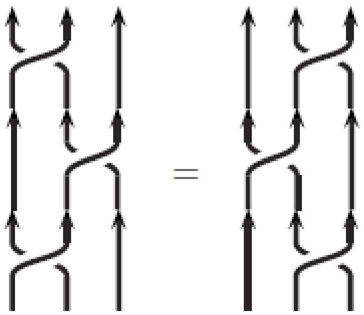
generated by $\hat{\sigma}_i$
(**braiding** of particles i and i+1)



defining relations

$$\hat{\sigma}_i \hat{\sigma}_j = \hat{\sigma}_j \hat{\sigma}_i \quad \text{for } |i-j| \geq 2$$

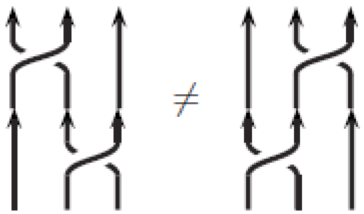
$$\hat{\sigma}_i \hat{\sigma}_{i+1} \hat{\sigma}_i = \hat{\sigma}_{i+1} \hat{\sigma}_i \hat{\sigma}_{i+1}$$



Note:

1. Braid group is infinite dimensional ($\hat{\sigma}^2 \neq 1$!) in contrast to finite-dimensional permutation group ($\hat{p}^2 = 1$)

2. Braid group is non-Abelian $\hat{\sigma}_i \hat{\sigma}_{i+1} \neq \hat{\sigma}_{i+1} \hat{\sigma}_i$



Representations of the braid group: statistics of particles

Elements of the braid group
(trajectories of particles)

⇓ representation

Changes of the states under the evolution
(particle statistics)

1. One-dimensional representations: unique (ground) state

2. Higher-dimensional representations: degenerate (ground) state

1. One-dimensional (Abelian) representations

Unique (ground) state $|\psi\rangle$

Transformation under braiding operation $\hat{\sigma}$

$$|\psi\rangle \xrightarrow{\hat{\sigma}} e^{i\vartheta}|\psi\rangle$$

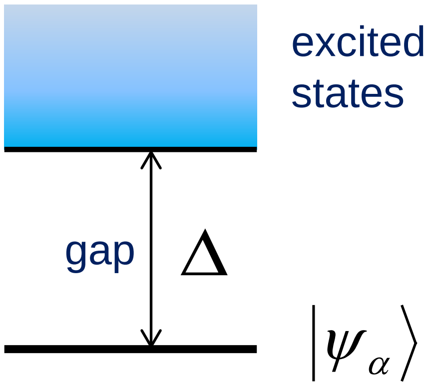
with arbitrary ϑ - Abelian anyons

Examples: 1. bosons ($\vartheta = 0$) and fermions ($\vartheta = \pi$)

2. quasiholes in the at FQHE Laughlin state $\nu = 1/M$

$$\vartheta = \pi / M$$

2. Higher-dimensional representations



Degenerate ground state of N particles with an

with an orthonormal basis $|\psi_\alpha\rangle, \alpha = 1, \ldots, g$

Transformation under braiding operation $\hat{\sigma}$

$$|\psi_\alpha\rangle \xrightarrow{\hat{\sigma}} U(\hat{\sigma})_{\alpha\beta} |\psi_\beta\rangle$$

with **matrix** $U(\hat{\sigma}) = P \exp(i \int dt\, \hat{M})\, \hat{\mathscr{H}},$ $\quad (\hat{M})_{\alpha\beta} = i\langle \psi_\alpha | \frac{d}{dt} \psi_\beta \rangle$

Berry matrix — explicit holonomy

Particles are non-Abelian anyons if $U(\hat{\sigma}_1)_{\alpha\beta} U(\hat{\sigma}_2)_{\beta\gamma} \neq U(\hat{\sigma}_2)_{\alpha\beta} U(\hat{\sigma}_1)_{\beta\gamma}$
for at least two $\hat{\sigma}_1$ and $\hat{\sigma}_2$

(do not commute!)

Example: Majorana fermions (Ising anyons)

Conditions for non-Abelian anyons:

Robust degeneracy of the ground state:

The degeneracy cannot be lifted by local perturbations
(which are needed, i.e., for braiding)

Degenerate ground states cannot be distinguished
by local measurements

$$\langle \psi_\alpha | V_{\mathrm{loc}} | \psi_\beta \rangle = C\delta_{\alpha\beta}$$

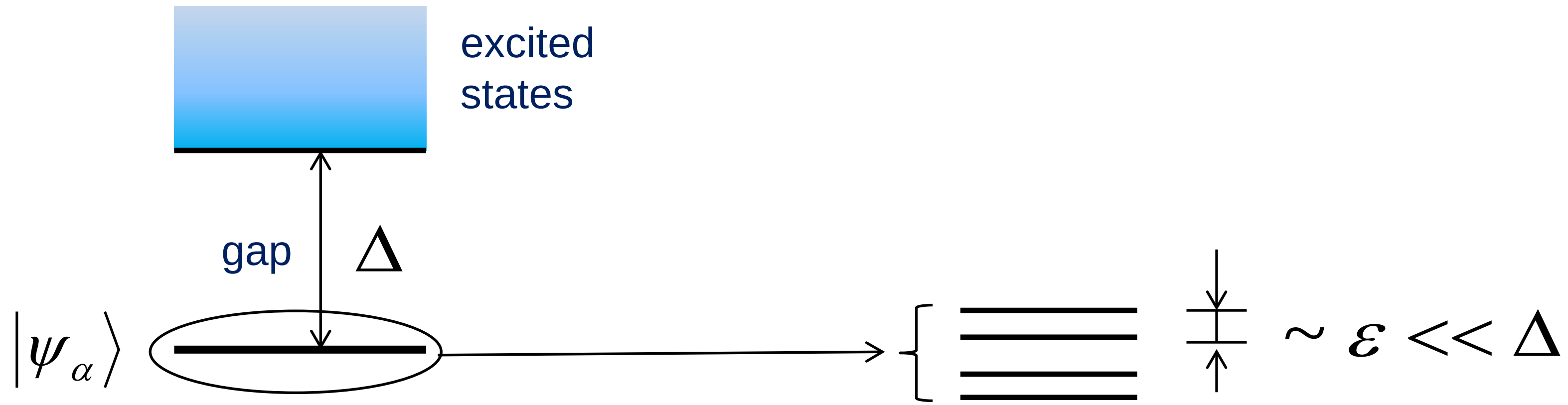
Braiding of identical particles changes state within the degenerate manifold

Nonlocal measurements:
bringing anyons together to know the fusion channel; measuring parity, etc.

Required are highly entangled states and topological field theories

Topology = invariance under continuous deformations

In real world: GS degeneracy is lifted



Condition on the time of operations: $$\frac{\hbar}{\Delta} << T << \frac{\hbar}{\varepsilon}$$

slow enough to be adiabatic

fast enough to NOT resolve the GS manifold

Ettore Majorana,
1906-1938?

Majorana “fermions” as non-Abelian anyons

Introducing Majorana "fermions"

For a (complex) fermionic operators $\hat{a}$ and $\hat{a}^{+}$

with algebra $\{\hat{a}, \hat{a}^{+}\} = 1, \{\hat{a}, \hat{a}\} = \{\hat{a}^{+}, \hat{a}^{+}\} = 0$

two hermitian(!) Majorana operators (Majorana fermions)

$$\gamma_1 = \hat{a} + \hat{a}^{+} = \gamma_1^{+}$$

$$\gamma_2 = (\hat{a} - \hat{a}^{+}) / i = \gamma_2^{+}$$

with algebra $\{\gamma_k, \gamma_l\} = 2\delta_{kl}$ or $\gamma_1^2 = \gamma_2^2 = 1,$ $\gamma_1\gamma_2 = -\gamma_2\gamma_1$

Inverse: $\hat{a} = (\gamma_1 + i\gamma_2)/2$ and $\hat{a}^{+} = (\gamma_1 - i\gamma_2)/2$

One fermionic mode $\Longleftrightarrow$ Two Majoranas

Fermionic states and Majorana fusion

States: $\{|0\rangle, |1\rangle\}: \; a|0\rangle = 0, \; |1\rangle = a^{+}|0\rangle$

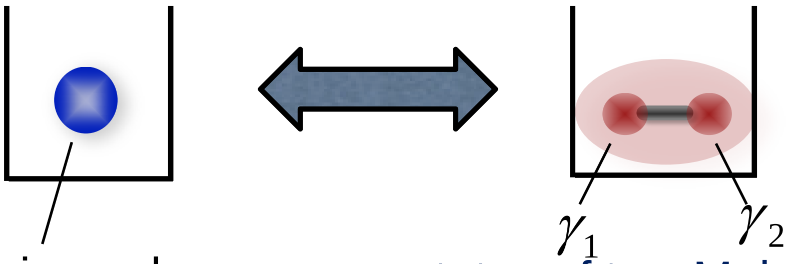
$$\hat{n} = \hat{a}^{+}\hat{a} = \frac{i}{2}\gamma_1\gamma_2 + \frac{1}{2}$$

$$\hat{n}|0\rangle = 0|0\rangle \quad \Rightarrow \quad -i\gamma_1\gamma_2|0\rangle = |0\rangle, \quad |0\rangle \equiv |+\rangle$$

$$-i\gamma_1\gamma_2|1\rangle = -|1\rangle, \quad |1\rangle \equiv |-\rangle$$

fermionic parity

$$P_F = (-1)^{\hat{a}^{+}\hat{a}} = -i\gamma_1\gamma_2$$



fermionic mode
(0 or 1 fermion)

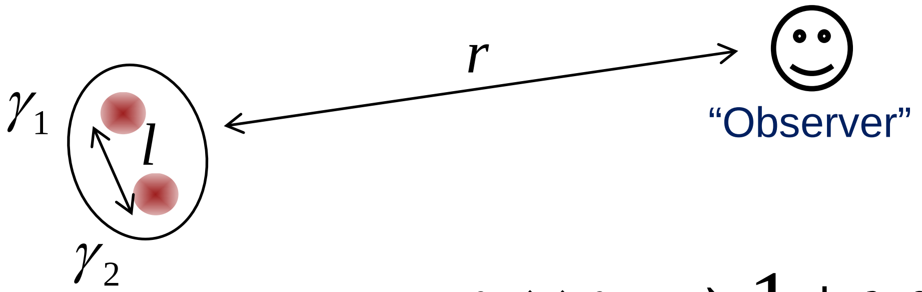
states of two Majoranas
(different fusion channels)

Two-Majorana states: Fusion of Majoranas

State with NO fermion $|0\rangle$ and state with ONE fermion $|1\rangle$
are BOTH described by two Majorana fermions (anyons)

Fusion of two Majoranas γ_1, γ_2 :

how do they behave as a combined object seen from distances much large than the separation between them $r >> l$



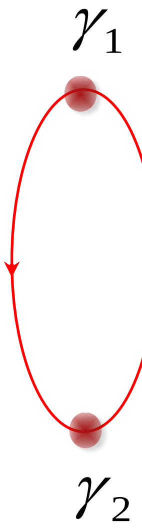
The result is either
fermionic vacuum $|0\rangle\ (=1)$ or
single-fermion $|1\rangle\ (=\psi)$

$$\gamma \times \gamma \rightarrow 1 + \psi$$

- Majorana fusion rules

This "uncertainty" is the origin of their non-Abelian statistics

Braiding and non-Abelian statistics



Braiding (permutation) of Majorana fermions

$$\sigma_{12}: \quad \begin{array}{l} \gamma_1 \rightarrow -\gamma_2 \\ \gamma_2 \rightarrow \gamma_1 \end{array} \quad \begin{array}{l} = U^{-1}\gamma_1 U \\ = U^{-1}\gamma_2 U \end{array}$$

D. Ivanov, PRL **86**, 268 (2000)

J. Alicea et al., Nat. Phys. (2011)

non-Abelian!

σ_{12} is generated by unitary operator (braiding unitary)

$$U_{12} = e^{i\vartheta} \exp\left(-\frac{\pi}{4}\gamma_1\gamma_2 \right) = e^{i\vartheta} \frac{1}{\sqrt{2}}\left(1 - \gamma_1\gamma_2\right)$$

The value for the Abelian phase $\vartheta = \frac{\pi}{8}$ follows from field theory or consistency conditions

Majorana fermions as non-Abelian particles:

- fundamental physical interest
- applications for quantum computation

How to make it useful?

- we need spatially separated Majorana fermions
- we need degenerate ground state (for different fusion channels = for states with different fermionic parity!)

“Making” Majorana fermions

Majorana edge states in Kitaev wire

A.Y. Kitaev, Phys. Usp. (2001)

Kitaev wire: spinless fermions with "p-wave" pairing on a1D chain of size L

$$H = \sum_{j=1}^{L-1} \left(-J\hat{a}_j^+ \hat{a}_{j+1} + \Delta \hat{a}_j \hat{a}_{j+1} + \text{h.c.} - \mu \hat{a}_j^+ \hat{a}_j \right)$$

hopping — pairing — chemical potential

Symmetries: The pairing amplitude Δ breaks the $U(1)$ gauge symmetry

$$a_j \to e^{i\varphi} a_j$$

down to the Z_2 symmetry

$$a_j \to -a_j$$

Parity is a conserved quantum number, not the number of particles

can be measured!

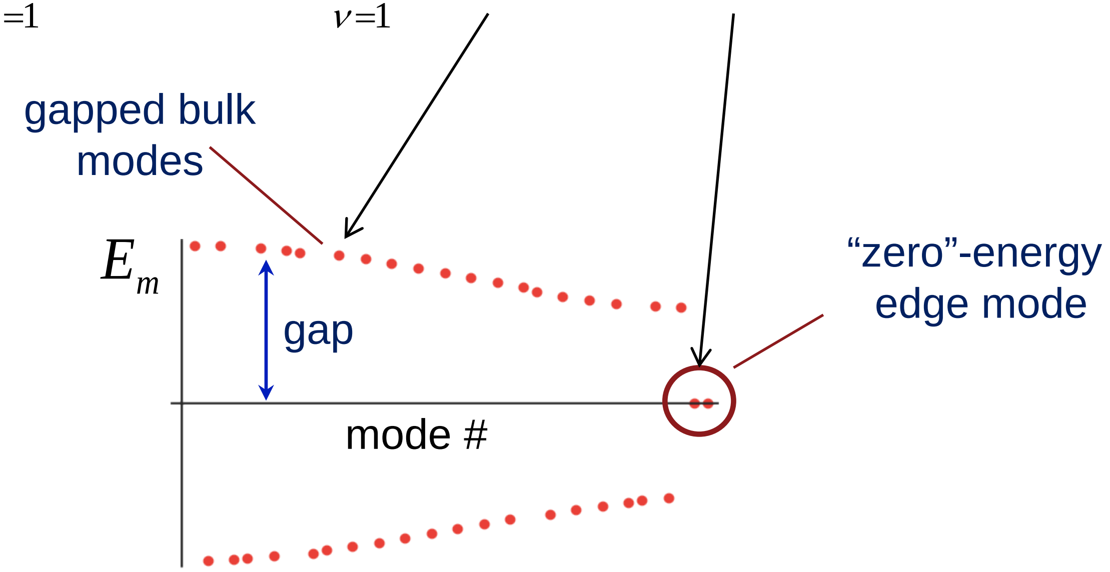
Solving Kitaev wire $\Delta \neq J > 0, |\mu| < 2J$

$$H = \sum_{j=1}^{L-1} \left(-J\hat{a}_j^+ \hat{a}_{j+1} + \Delta \hat{a}_j \hat{a}_{j+1} + \text{h.c.} - \mu \hat{a}_j^+ \hat{a}_j \right)$$

Bogoliubov transformation

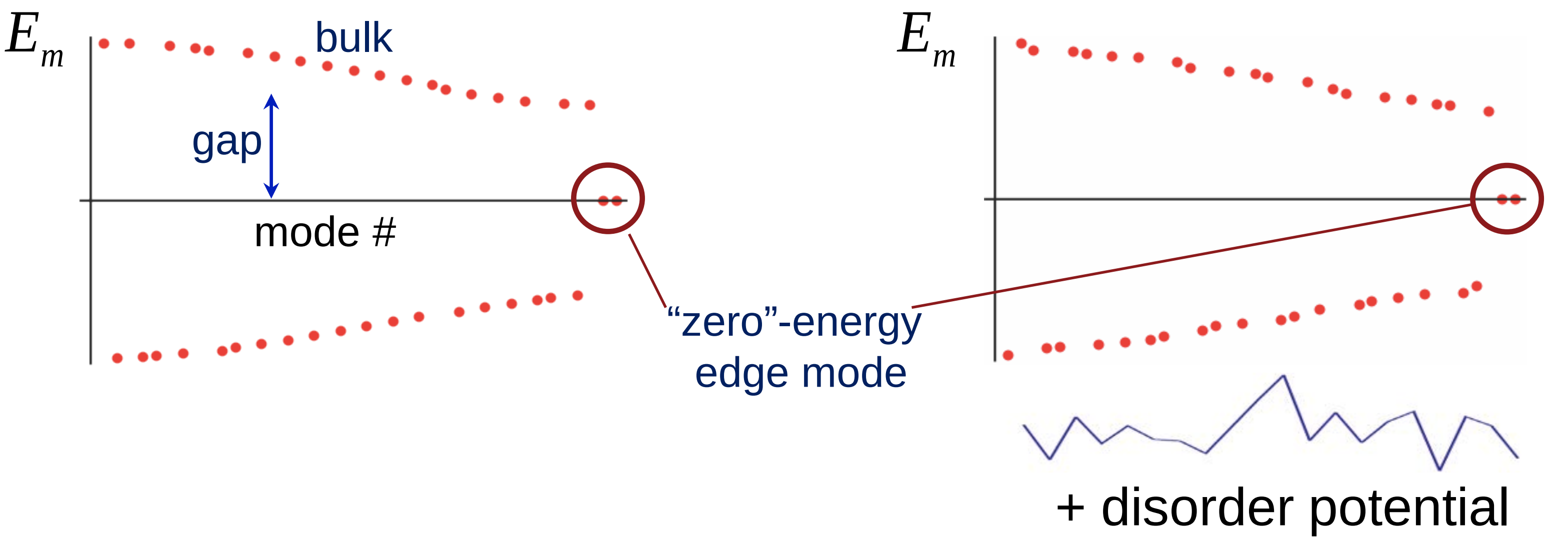
$$\hat{\alpha}_m = \sum_j \left(u_{mj}^* \hat{a}_j + v_{mj}^* \hat{a}_j^+ \right)$$

$$H = \sum_{m=1}^{L} E_m \hat{\alpha}_m^+ \hat{\alpha}_m = \sum_{\nu=1}^{L-1} E_\nu \hat{\alpha}_\nu^+ \hat{\alpha}_\nu + E_M \hat{\alpha}_M^+ \hat{\alpha}_M$$



Robustness

"Zero-energy" eigenvalue is robust against static disorder



This robustness against imperfection is a consequence of the topological order in the bulk – topological protection

Closer look at the “zero-energy” mode

$$\hat{\alpha}_M = (\gamma_L + i\gamma_R)/2$$

γ_L, γ_R - Majorana operators

In Majorana basis $\gamma_{2j-1} = \hat{a}_j + \hat{a}_j^+ \quad \gamma_{2j} = (\hat{a}_j - \hat{a}_j^+)/i$

$$\gamma_L \sim \sum_j (x_+^j - x_-^j)\gamma_{2j-1}$$
$$\gamma_R \sim \sum_j (x_+^{L-j} - x_-^{L-j})\gamma_{2j}$$

$$x_\pm = \frac{-\mu \pm \sqrt{\mu^2 + 4\Delta^2 - 4J^2}}{2(\Delta + J)}$$

$|x_+|, |x_-| < 1$ for $\Delta \neq J > 0, |\mu| < 2J$

γ_L “lives” near the left edge

γ_R “lives” near the right edge

$$x_+^j - x_-^j \sim \exp(-\kappa\, j)$$
$$-\kappa = \ln \min(|x_\pm|)$$

Majorana edge modes

γ_L  γ_R

$\hat{\alpha}_M$ - non-local fermion living on both edges

The energy of the “zero-energy” mode

The energy of the non-local fermion $\hat{\alpha}_M$

$$E_M \sim \Delta \frac{x_+^{L+1} - x_-^{L+1}}{x_+ - x_-} \sim \exp(-\kappa L)$$

is exponentially small with the size of the wire L

The Hamiltonian of the non-local fermion $\hat{\alpha}_M$

$$H_M = E_M \hat{\alpha}_M^+ \hat{\alpha}_M = \frac{i}{2} E_M \gamma_L \gamma_R + \frac{1}{2} E_M$$

$E_M \sim \exp(-\kappa L)$ - coupling between Majorana modes

Quasi degenerate ground state: $|0\rangle$ ($\hat{\alpha}_m |0\rangle = 0$) and

with different fermionic parity $|M\rangle = \hat{\alpha}_M^+ |0\rangle$

have exponentially close energies

In the “ideal” case $\Delta = J > 0,\ \mu = 0$

Zero-energy mode $\hat{\alpha}_M = (\gamma_1 + i\gamma_{2L})/2 = (\hat{a}_1 + \hat{a}_1^+ + \hat{a}_L - \hat{a}_L^+)/2$

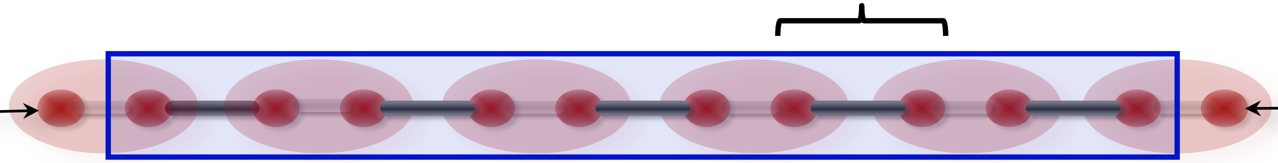
$$E_M = 0$$

Majoranas $\gamma_L = \gamma_1$ and $\gamma_R = \gamma_{2L}$ are completely decoupled

Gapped modes $\hat{\alpha}_\nu = (\gamma_{2\nu} + i\gamma_{2\nu+1})/2 = i(\hat{a}_{\nu+1} + \hat{a}_{\nu+1}^+ - \hat{a}_\nu + \hat{a}_\nu^+)/2$

$$E_\nu = 2J$$

$\tilde{a}_\nu$

$\gamma_L = \gamma_1$  $\gamma_R = \gamma_{2L}$

Degenerate ground state: states $|0\rangle$ ($\hat{\alpha}_m|0\rangle = 0$) and $|M\rangle = \hat{\alpha}_M^+|0\rangle$ have the same energy

Explicit ground state wave functions

$$\left|+\right\rangle = \frac{1}{2^N}\left[1+\sum_{p=1}^{N}\sum_{i_1<\ldots<i_{2p}}^{2N+1} a_{i_{2p}}^{+}\cdots a_{i_1}^{+}\right]\left|\mathrm{vac}\right\rangle$$

$$\left|-\right\rangle = \frac{1}{2^N}\sum_{p=0}^{N}\sum_{i_1<\ldots<i_{2p+1}}^{2N+1} a_{i_{2p+1}}^{+}\cdots a_{i_1}^{+}\left|\mathrm{vac}\right\rangle$$

These states have identical local properties

but different fermionic number parity $\left\langle \pm\right|P_F\left|\pm\right\rangle = \pm 1$

Majorana fermions in atomic wires

“Making” Kitaev wire with cold atoms

System: fermionic atoms in an optical lattice

hopping term $-J\sum_i (a_i^+ a_{i+1} + \text{h.c.})$

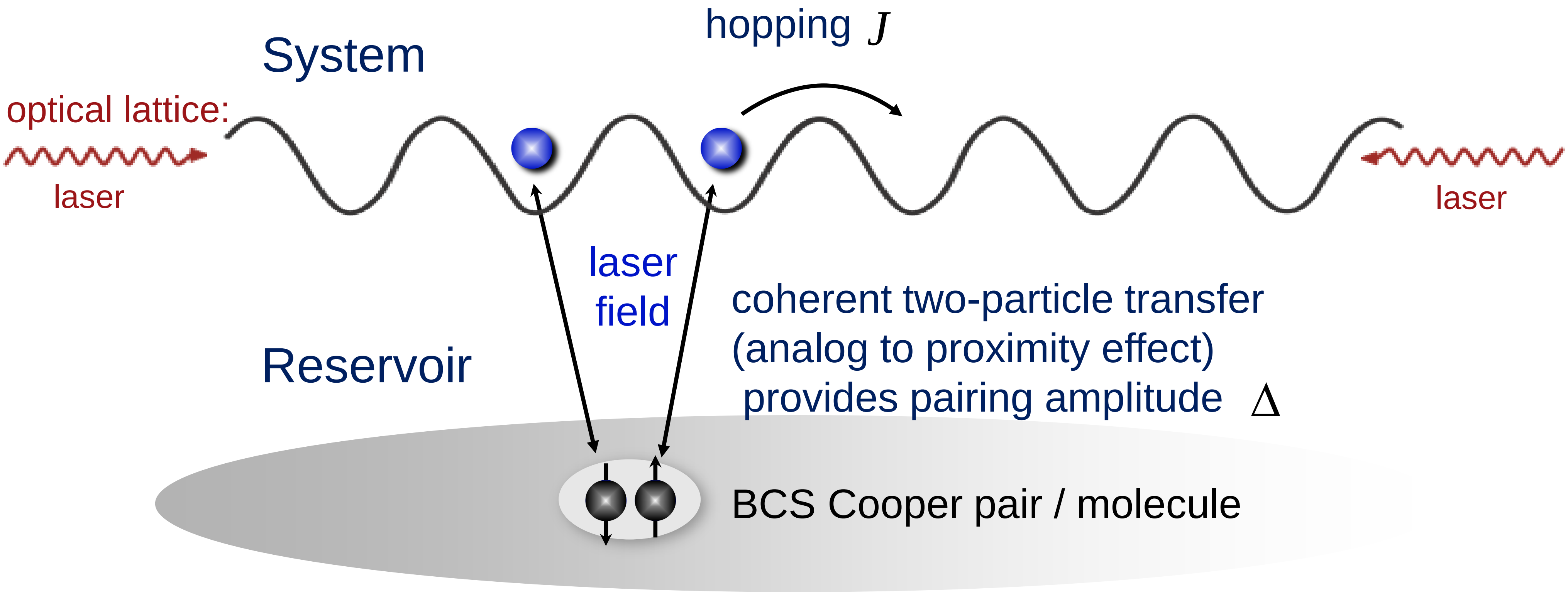
continuous version $-(\hbar^2/2m)\int d\vec{r}\,\hat{\psi}^+ \Delta\hat{\psi}$

Reservoir: molecular BEC (or BCS) cloud

pairing term $\sum_\iota (\Delta a_i^+ a_{i+1}^+ + \text{h.c.})$

continuous version $\Delta_0 \int d\vec{r}\,(\hat{\psi}^+ \nabla \hat{\psi}^+ + h.c.)$

Basic idea



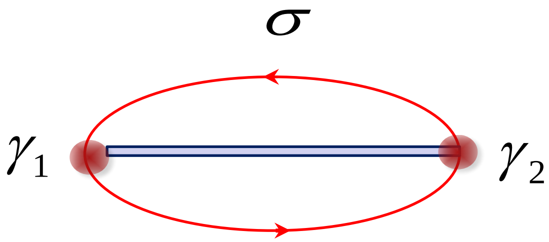
open *Hamiltonian* system.

$$H = \sum_{i=1}^{N-1} \left(-J a_i^{+} a_{i+1} + \Delta a_i a_{i+1} + \text{h.c.} - \mu a_i^{+} a_i \right)$$

L. Jiang, et al, Phys. Rev. Lett. 106, 22042 (2011)

S. Nascimbène, J. Phys. B 46, 134005 (2013)

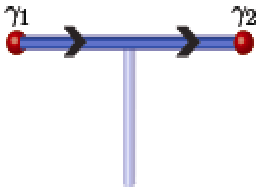
Braiding of Majorana fermions



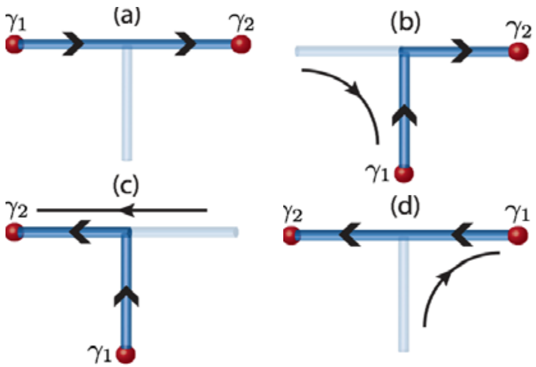
$$\gamma_1 \to -\gamma_2$$
$$\gamma_2 \to \gamma_1$$

How to realize?

T-junction:



J. Alicea et al, Nat. Phys. 7 412 (2011)

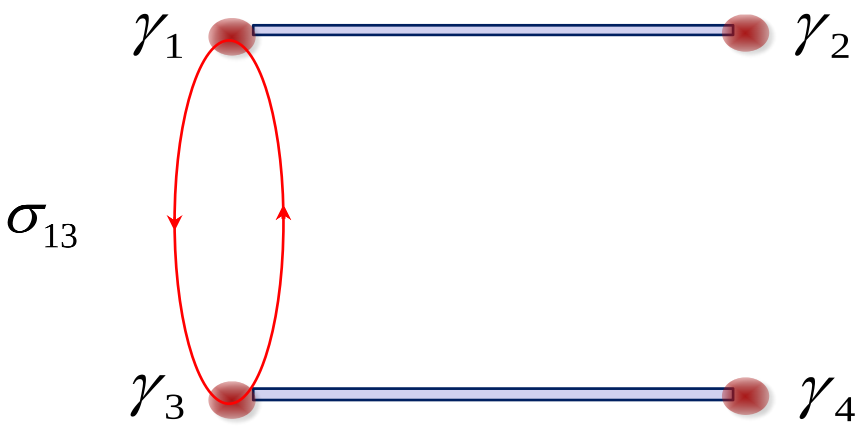


Moving Majoranas around by changing the local potential

Can also be done in atomic wires setup.

Could we do something else?

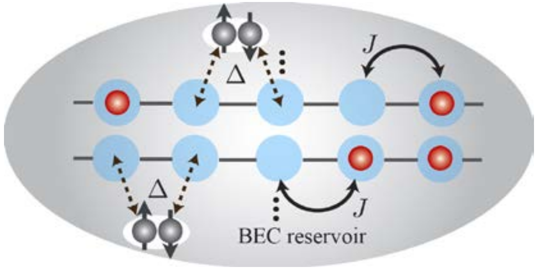
Braiding of Majorana fermions in atomic wires setup



$$\gamma_1 \to -\gamma_3 \;= U_{13}^{-1}\gamma_1 U_{13}$$
$$\gamma_3 \to \gamma_1 \;= U_{13}^{-1}\gamma_3 U_{13}$$

$$U_{13} = e^{i\frac{\pi}{8}} \exp\left(-\frac{\pi}{4}\gamma_1\gamma_3\right) = e^{i\frac{\pi}{8}} \frac{1}{\sqrt{2}}\left(1-\gamma_1\gamma_3\right)$$

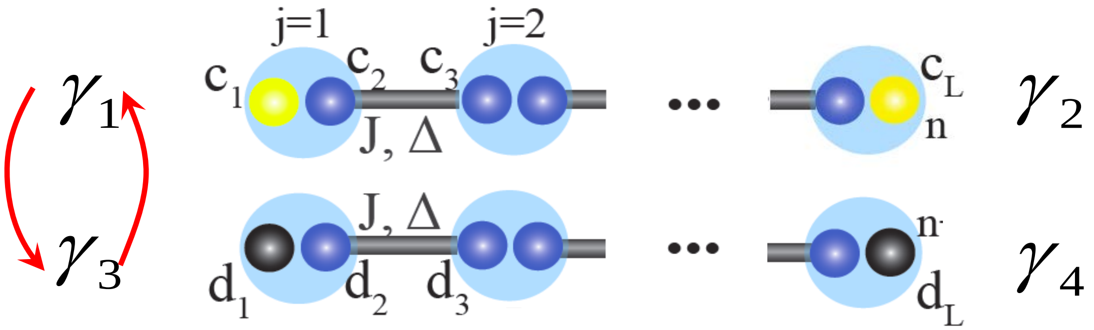
Braiding of Majorana fermions in atomic wires



Two (nearest) Kitaev wires:

$$H = \sum_j \left(-J a^{+}_{u,j} a_{u,j+1} + \Delta a_{u,j} a_{u,j+1} + \text{h.c.} - \mu a^{+}_{u,j} a_{u,j} \right) \quad \leftarrow \quad \text{upper wire}$$

$$+ \sum_j \left(-J a^{+}_{l,j} a_{l,j+1} + \Delta a_{l,j} a_{l,j+1} + \text{h.c.} - \mu a^{+}_{l,j} a_{l,j} \right) \quad \leftarrow \quad \text{lower wire}$$



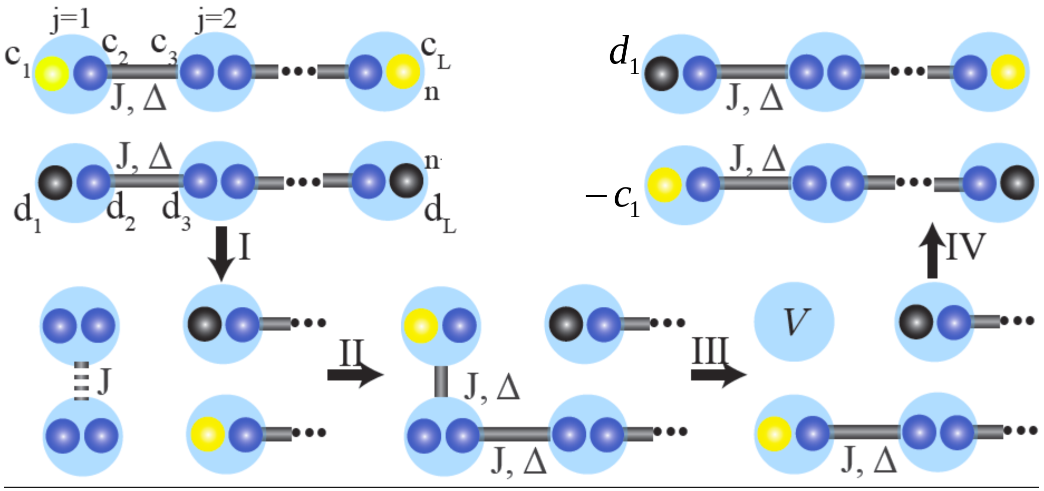
$$a_{u,j} = \frac{1}{2}(c_{2j-1} + i c_{2j})$$

$$a_{l,j} = \frac{1}{2}(d_{2j-1} + i d_{2j})$$

Four Majorana fermions $\gamma_1, \gamma_2, \gamma_3, \gamma_4$, we braid $\gamma_1 = c_1$ and $\gamma_3 = d_1$

Braiding protocol:

C.V. Kraus, P. Zoller, and M.A. Baranov, PRL 111, 203001 (2013)



Features: - small number of steps

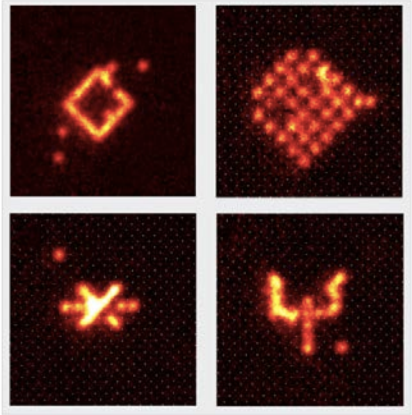
- only four sites and links between them are involved (local)

Requirement: - local cite/link addressing

J. Simon, et. al, Nature (London) 473:307-312, 2011

C. Weitenberg, et. al, Nature (London) 471:319-324, 2011

T. Fukuhara, et. al, Nat. Phys. 9:235, 2011



Needed local operations:

Single-link: switching on/off adiabatically

hopping $H_{jl}^{(J)} = -Ja_j^+ a_l - \mathrm{h.c.}$ and pairing $H_{jl}^{(p)} = \Delta a_j^+ a_l^+ + \mathrm{h.c.}$

between nearest sites j and l

Together give "Kitaev coupling" $H_{jl}^{(K)} = H_{jl}^{(J)} + H_{jl}^{(p)}$

Single-site: switching on/off adiabatically

on-site potential $H_j^{(loc)} = Va_j^+ a_j$

Braiding protocol: Step I

ϕ_t changes adiabatically from 0 to $\pi/2$

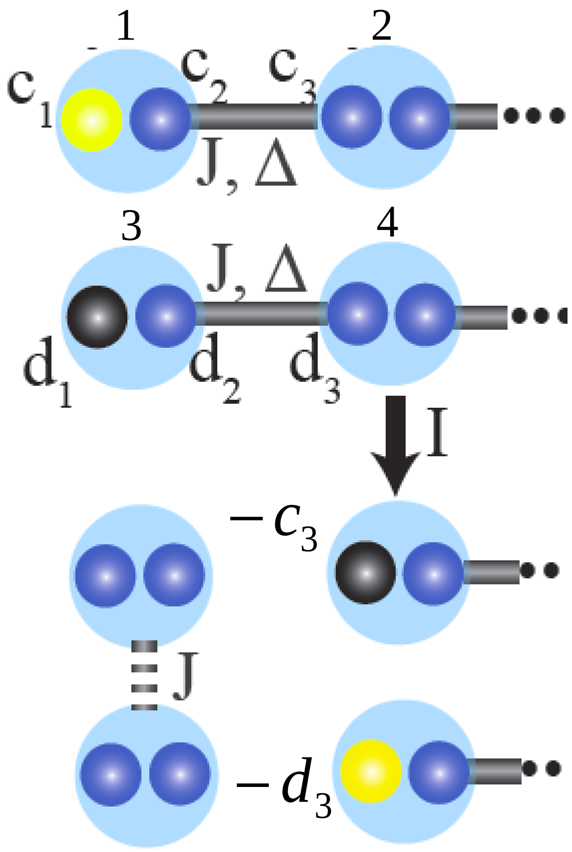
Turn off the couplings between sites 1-2 and 3-4;
turn on hopping between sites 1-3

$$H_I = \left(H_{12}^{(K)} + H_{34}^{(K)}\right)\cos\phi_t + H_{13}^{(J)}\sin\phi_t$$

$$\gamma_1(\phi_t) = (2c_1\cos\phi_t - d_3\sin\phi_t)/\sqrt{1+3\cos^2\phi_t}$$

$$\gamma_3(\phi_t) = (2d_1\cos\phi_t - c_3\sin\phi_t)/\sqrt{1+3\cos^2\phi_t}$$

$$\gamma_1 = c_1 \to -d_3$$
$$\gamma_3 = d_1 \to -c_3$$



Braiding protocol: Step II

ϕ_t changes adiabatically from 0 to $\pi / 2$

Turn on the couplings between sites 3-4;
turn on pairing between sites 1-3

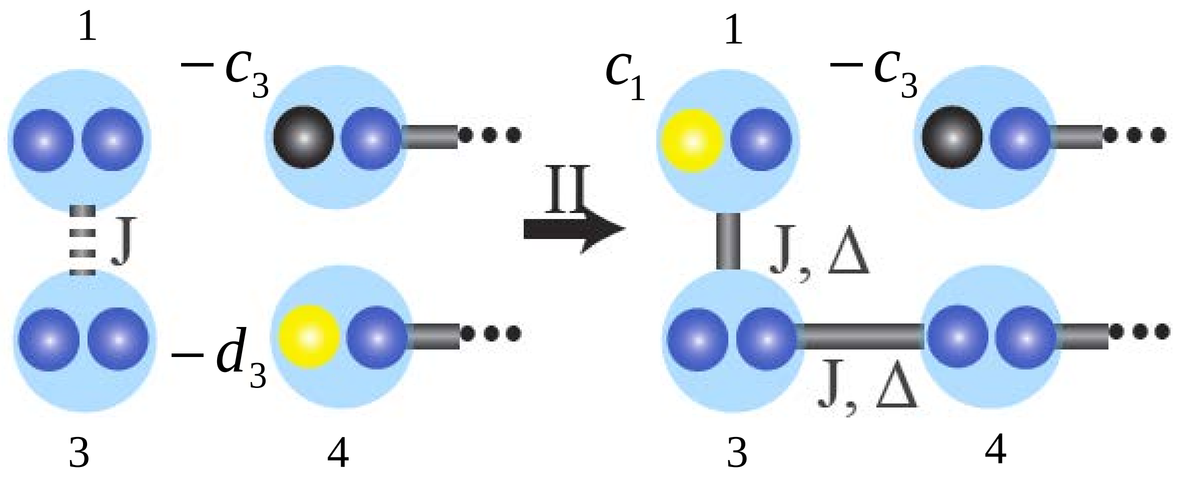
$$H_{II} = H_{13}^{(J)} + \left(H_{13}^{(p)} + H_{34}^{(K)}\right)\sin\phi_t$$

$$\gamma_1(\phi_t) = [2c_1 \sin\phi_t - d_3(1-\sin\phi_t)] / \sqrt{4\sin^2\phi_t + (1-\sin\phi_t)^2}$$

$$\gamma_3(\phi_t) = -c_3$$

$$\gamma_1 \rightarrow c_1$$

$$\gamma_3 \rightarrow -c_3$$



Braiding protocol: Step III

ϕ_t changes adiabatically from 0 to $\pi/2$

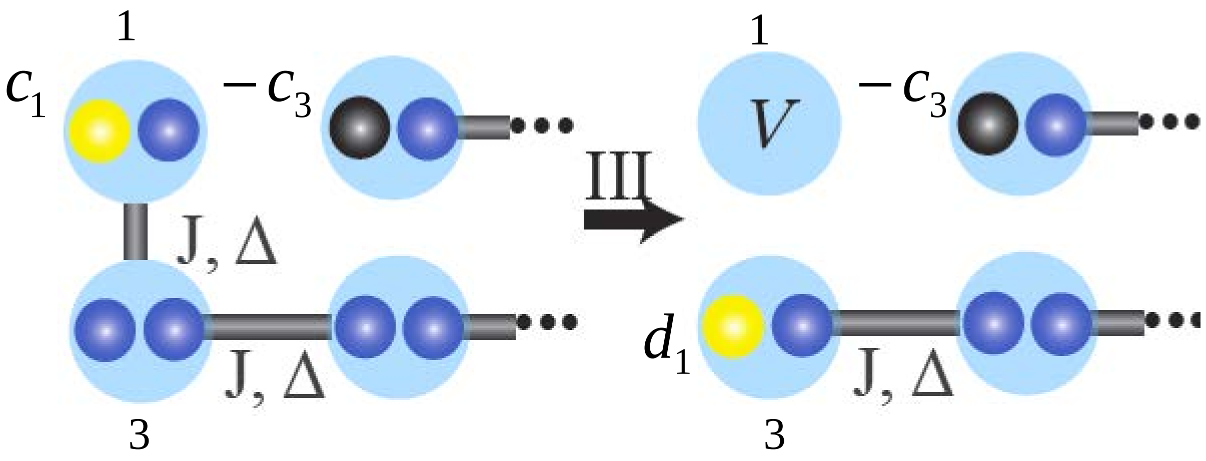
Ramp up local potential on site 1;
turn off couplings between sites1-3

$$H_{III} = H_1^{(loc)} \sin\phi_t + H_{13}^{(K)} \cos\phi_t + H_{34}^{(K)}$$

$$\gamma_1(\phi_t) = (Jc_1 \cos\phi_t + Vd_1 \sin\phi_t) / \sqrt{(J\cos\phi_t)^2 + (V\sin\phi_t)^2}$$

$$\gamma_3(\phi_t) = -c_3$$

$$\gamma_1 \to d_1$$
$$\gamma_3 \to -c_3$$



Braiding protocol: Step IV

ϕ_t changes adiabatically from 0 to $\pi/2$

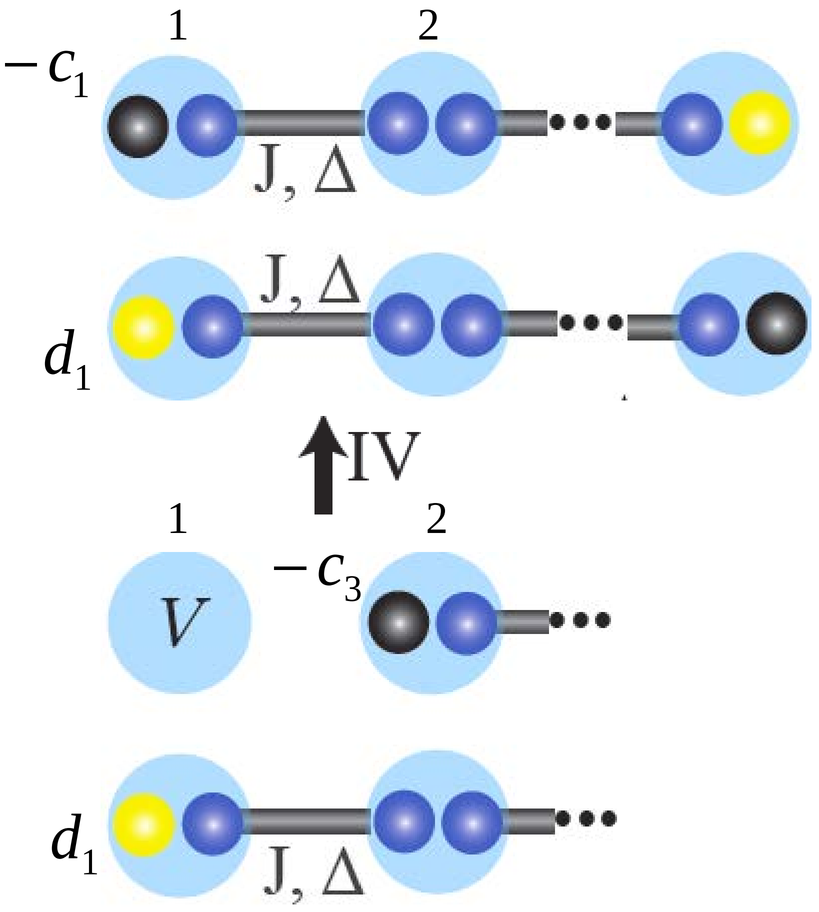
Ramp down local potential on site1;
turn on couplings between sites 1-2

$$H_{IV} = H_{12}^{(K)} \sin\phi_t + H_1^{(loc)} \cos\phi_t + H_{34}^{(K)}$$

$$\gamma_1(\phi_t) = d_1$$

$$\gamma_3(\phi_t) = -(Jc_1 \sin\phi_t + Vc_3 \cos\phi_t) / \sqrt{(J\sin\phi_t)^2 + (V\cos\phi_t)^2}$$

$$\gamma_1 \to d_1$$
$$\gamma_3 \to -c_1$$



Result of the braiding protocol:

$$\gamma_1 \to -\gamma_3$$
$$\gamma_3 \to \gamma_1$$

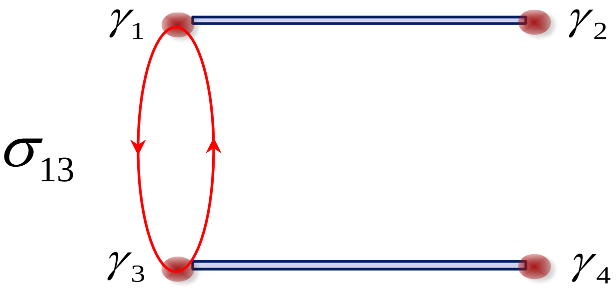
generated by

$$U_{13} = e^{i\frac{\pi}{8}} \exp\left(-\frac{\pi}{4}\gamma_1\gamma_3\right) = e^{i\frac{\pi}{8}} \frac{1}{\sqrt{2}}\left(1-\gamma_1\gamma_3\right)$$

Physics behind

one fermion is taken from the system
(either from the lower or from the upper wire)
and inserted into the lower wire

Physical consequences:



$$U_{13} = e^{i\frac{\pi}{8}} \frac{1}{\sqrt{2}} (1 - \gamma_1 \gamma_3)$$

In the basis $\{|++\rangle, |--\rangle\}$ of eigenfunctions of $-i\gamma_1\gamma_2$ and $-i\gamma_3\gamma_4$

$$\left.\begin{matrix} -i\gamma_1\gamma_2 \\ -i\gamma_3\gamma_4 \end{matrix}\right\} | p_1, p_2 \rangle = \left.\begin{matrix} p_1 \\ p_2 \end{matrix}\right\} | p_1, p_2 \rangle$$

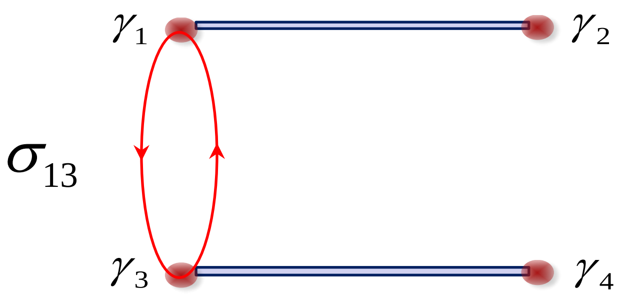
p_1: parity of the upper wire

p_2: parity of the lower wire

we have

$$U_{13} = \frac{1}{\sqrt{2}} e^{i\frac{\pi}{8}} \begin{pmatrix} 1 & -i \\ -i & 1 \end{pmatrix} \quad \text{and} \quad U_{13}^2 = e^{-i\frac{\pi}{4}} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

Physical consequences:



$$U_{13} = e^{i\frac{\pi}{8}} \frac{1}{\sqrt{2}} \left(1 - \gamma_1 \gamma_3\right)$$

Starting from $\left|++\right\rangle$

$$\left|++\right\rangle \xrightarrow{\sigma_{13}} U_{13}\left|++\right\rangle = e^{i\frac{\pi}{8}} \frac{1}{\sqrt{2}} \left[\left|++\right\rangle - i\left|--\right\rangle\right]$$

even-even parity

superposition of even-even and odd-odd

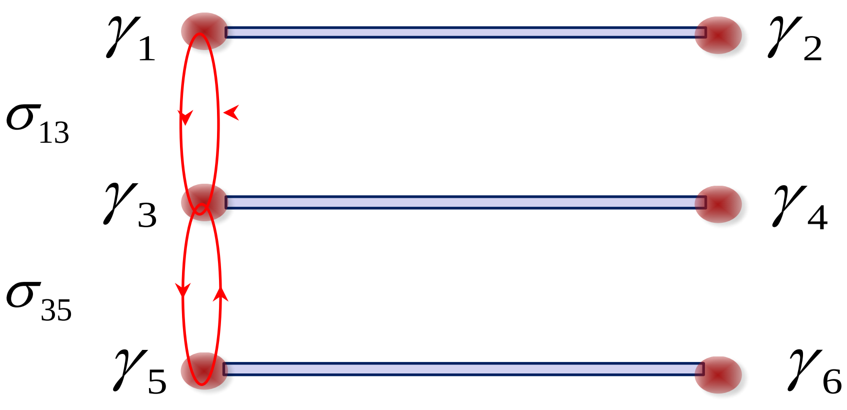
$$\left|++\right\rangle \xrightarrow{\sigma_{13}^2} U_{13}^2\left|++\right\rangle = e^{-i\frac{\pi}{4}} \left|--\right\rangle$$

even-even parity

odd-odd parity

Demonstration of non-Abelian character

Three wires



$$U_{13} == e^{i\frac{\pi}{8}} \frac{1}{\sqrt{2}}(1-\gamma_1\gamma_3)$$

$$U_{35} == e^{i\frac{\pi}{8}} \frac{1}{\sqrt{2}}(1-\gamma_3\gamma_5)$$

Starting from $|+++\rangle$ - eigenstate of $-i\gamma_1\gamma_2, -i\gamma_3\gamma_4, -i\gamma_5\gamma_6$

$$|+++\rangle \xrightarrow{\sigma_{35}\sigma_{13}} U_{35}U_{13}|+++\rangle = e^{i\frac{\pi}{4}} \frac{1}{2}[|+++\rangle - i|+--\rangle - i|--+\rangle + |-+-\rangle]$$

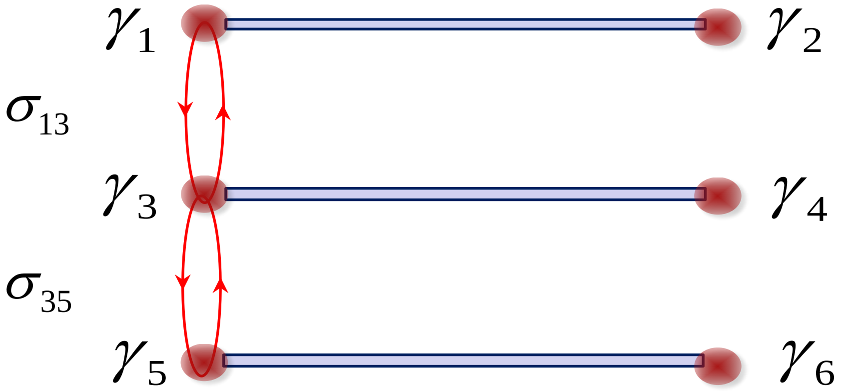
different

$$|+++\rangle \xrightarrow{\sigma_{13}\sigma_{35}} U_{13}U_{35}|+++\rangle = e^{i\frac{\pi}{4}} \frac{1}{2}[|+++\rangle - i|+--\rangle - i|--+\rangle - |-+-\rangle]$$

$$\sigma_{13}\sigma_{35} \neq \sigma_{35}\sigma_{13}$$

do not commute!

Another possibility: $\sigma_{13}\sigma_{35}$ and $\sigma_{35}\sigma_{13}$



$$U_{13} == e^{i\frac{\pi}{8}} \frac{1}{\sqrt{2}} (1 - \gamma_1\gamma_3)$$

$$U_{35} == e^{i\frac{\pi}{8}} \frac{1}{\sqrt{2}} (1 - \gamma_3\gamma_5)$$

Starting from $|+++\rangle$

$$|+++\rangle \xrightarrow{(\sigma_{35}\sigma_{13})(\sigma_{13}\sigma_{35})} U_{35}U_{13}^2U_{35}|+++\rangle = i|--+\rangle$$

different

$$|+++\rangle \xrightarrow{(\sigma_{13}\sigma_{35})(\sigma_{35}\sigma_{13})} U_{13}U_{35}^2U_{13}|+++\rangle = i|+--\rangle$$

$$(\sigma_{13}\sigma_{35})(\sigma_{35}\sigma_{13}) \neq (\sigma_{35}\sigma_{13})(\sigma_{13}\sigma_{35})$$

do not commute!

Using Majorana fermions for QC

Implementation of the Deutsch-Jozsa algorithm for two qubits

Although braiding does not provide a tool to build a universal set of gates, it still can be used for QC.

Example: Deutsch-Jozsa algorithm

Deutsch-Jozsa algorithm (2 qubits)

Function $g: \quad \{|0\rangle, |1\rangle\} \otimes \{|0\rangle, |1\rangle\} \mapsto \{0,1\}$ (oracle)

can be either constant or balanced

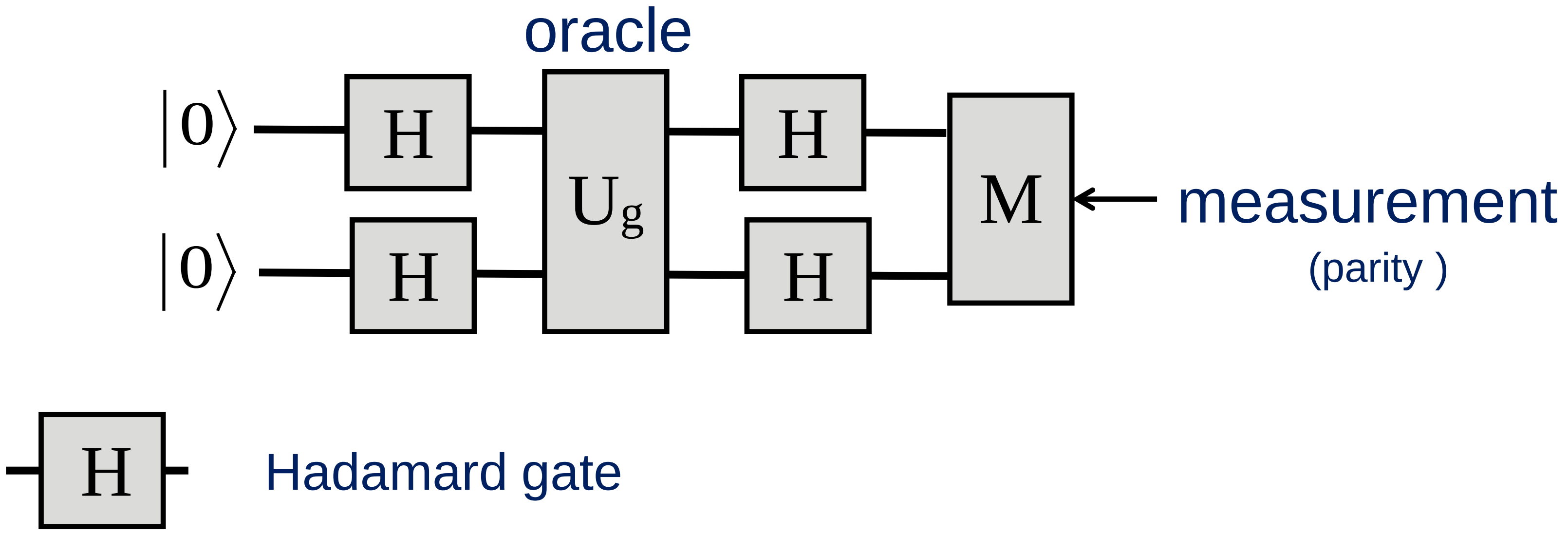
	$\|00\rangle$	$\|01\rangle$	$\|10\rangle$	$\|11\rangle$	
g_0	0	0	0	0	constant
g_1	0	0	1	1	balanced
g_2	0	1	1	0	balanced
g_3	0	1	0	1	balanced

Question: is a given unknown g constant or balanced?

Naïve way: three measurements (in the worst case)

Deutsch-Jozsa algorithm for two qubits: only one measurements!

When oracle is realized as a unitary $U_g|x\rangle = (-1)^{g(x)}|x\rangle$



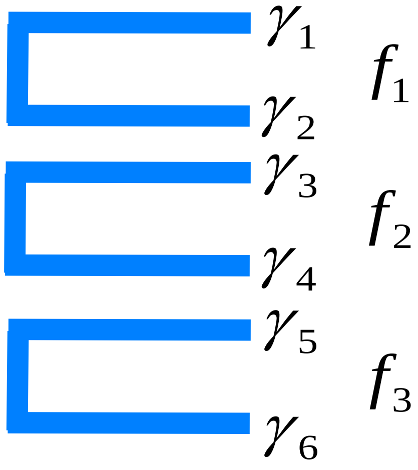
$g(x)$ constant: probability to measure $|00\rangle$ is 1

$g(x)$ balanced: probability to measure $|00\rangle$ is 0

Realization of the algorithm via braiding

C.V. Kraus, P. Zoller, and M.A. Baranov, PRL 111, 203001 (2013)

Setup: 3 Kitaev wires



controls parity

$$f_1 = (\gamma_1 + i\gamma_2)/2$$

$$f_2 = (\gamma_3 + i\gamma_4)/2$$

$$f_3 = (\gamma_5 + i\gamma_6)/2$$

encode 2 qubits

$$|00\rangle = f_2^+ |0\rangle_f$$

$$|01\rangle = f_3^+ |0\rangle_f$$

$$|10\rangle = f_1^+ |0\rangle_f$$

$$|11\rangle = f_1^+ f_2^+ f_3^+ |0\rangle_f$$

L. Georgiev, Phys. Rev. B 74 (2006)

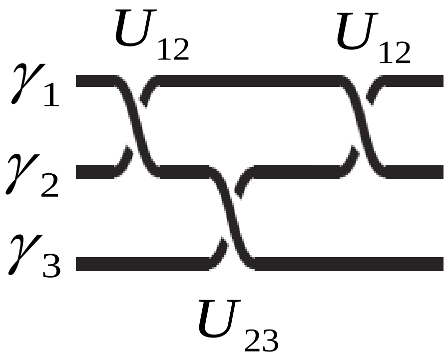
Hadamard gate

$$H \otimes H$$

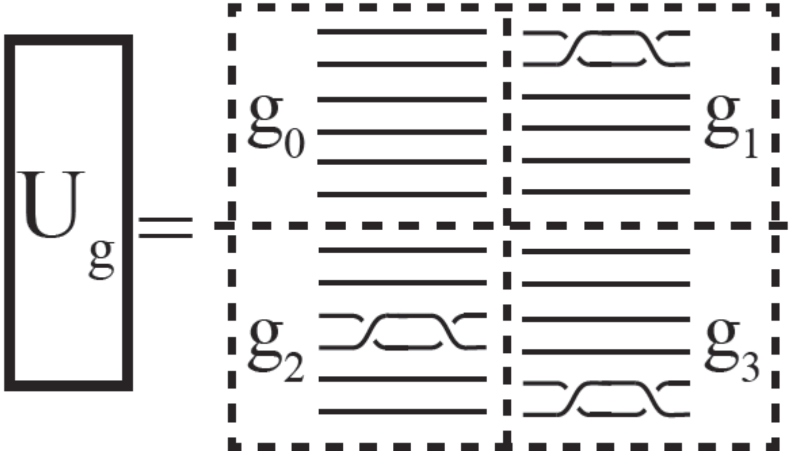
$$H \otimes 1 = U_{12} U_{23} U_{12}$$

$$1 \otimes H = U_{56} U_{45} U_{56}$$

$H \otimes 1 =$



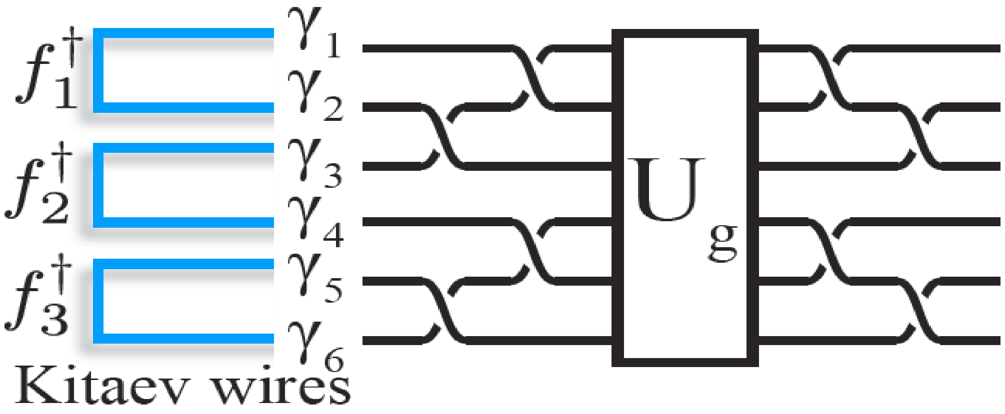
Unitary for an oracle



$$U_{g_0} = 1 \qquad U_{g_1} = U_{12}^{\;2}$$

$$U_{g_2} = U_{34}^{\;2} \qquad U_{g_3} = U_{56}^{\;2}$$

Realization of the algorithm via braiding (optimum sequence)



$$U_{D-J}(g_i) = U_{45}U_{56}\; U_{23}U_{12}\; U_{g_i}\; U_{56}U_{45}\; U_{12}U_{23}$$

can be done in parallel

can be done in parallel

Realization of the algorithm in five steps!

Results:

$$U_{D-J}(g_0)\left|00\right\rangle = \left|00\right\rangle \qquad U_{D-J}(g_0)\left|00\right\rangle = \left|11\right\rangle$$

$$U_{D-J}(g_1)\left|00\right\rangle = i\left|10\right\rangle \qquad U_{D-J}(g_1)\left|00\right\rangle = i\left|01\right\rangle$$

Read out: measuring parities (particle numbers) in the wires

Summary

Majorana fermions provide an example of non-Abelian anyons

- fundamental physical interest
- applications for quantum computation

Cold atomic/molecular systems provides a possibility
to implement and to manipulate Majorana fermions

Thank you for your attention!