Strongly Interacting Fermi Gases

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(Dated: July 2014)

Abstract

These are notes on a series of lectures on strongly interacting Fermi gases given in Varenna, July 2014.

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Contents

I. FESHBACH RESONANCES

The very notion of a strongly interacting gas seems to be a contradiction in itself, so we start by discussing what is meant by this and why it is possible to realize strongly interacting Fermi gases not, however - at least in equilibrium or without an optical lattice - its counterpart with Bosons.

A gas requires densities n which are low enough that the average interparticle spacing $n^{-1/3} \gg b$ is much larger than the range b of interactions. In order to have nontrivial correlations in a gaseous state, the temperature has to be small enough that the thermal wavelength λ_T is of the order of or larger than the interparticle spacing. At these temperatures, the wave nature of the particles becomes relevant. In contrast to the non-degenerate limit, interactions can then no longer be described by point like collisions of particles which approach each other on distances of order b. Instead, at very low temperatures, the relevant scale which determines the strength of the interactions is the scattering length a, which may be much larger than the interaction range b.

The condition that the two-body interactions are due to s-wave scattering only, requires the thermal energy to be much less than the effective centrifugal barrier. In terms of the van der Waals length $l_{\rm vdw}$ defined in Eq. (3) below, the barrier is of order $\hbar^2/ml_{\rm vdw}^2$. Ultracold

gases are thus characterized by the following hierarchy of length scales

$$l_{\rm vdw} \ll n^{-1/3} \ll \lambda_T \,, \tag{1}$$

since the effective range $b \simeq l_{\rm vdw}$ of the interactions turns out to be essentially identical with the van der Waals length, see Eq. (7) below.

The distinction between weak and strong interactions now depends on whether the scattering length a is much smaller or much larger than the average interparticle spacing $n^{-1/3}$. The former limit is in fact the standard situation because, as follows from Eq. (5), generic values of the scattering length are of the order of the van der Waals length $l_{\rm vdw}$. Under special circumstances, however, the scattering length can become much larger than $l_{\rm vdw}$ and in fact even much larger than the interparticle spacing. In particular, Feshbach resonances allow to tune a from values of order $l_{\rm vdw}$ in the few nm range to values much larger than typical interparticle distances, which are about $0.5\,\mu$m. This allows to realize ultracold gases with strong interactions $n^{1/3}a \gg 1$.

A. Two-body scattering

We start by recalling some elementary facts about two-body scattering at low energies, using a simple toy model (Gribakin and Flambaum, 1993), where the van der Waals attraction at large distances is cutoff by a hard core at some distance r_c on the order of an atomic dimension. The resulting spherically symmetric potential

$$V(r) = \begin{cases} -C_6/r^6 & \text{if} \qquad r > r_c \\ \infty & \text{if} \qquad r \le r_c \end{cases} \tag{2}$$

is characterized by two quite different lenghts: Beyond r_c, the van der Waals coefficient C_6 which fixes the asymptotic behavior of the interaction potential defines a characteristic length

$$l_{\rm vdw} = \frac{1}{2}\left(\frac{mC_6}{\hbar^2}\right)^{1/4} \tag{3}$$

at which the kinetic energy of the relative motion of two atoms with mass m is of the order of their interaction energy. For alkali atoms, this length is typically on the order of several nano-meters. It is much larger than the atomic scale r_c because alkalis are strongly polarizable, resulting in a large C_6 coefficient. Since $l_{\rm vdW} \gg r_c$, the low energy scattering

properties are essentially determined by the van der Waals length $l_{\rm vdw}$, which is only sensitive to the asymptotic behavior of the potential.

The scattering length a and effective range r_e are defined by the low energy expansion

$$f(k) = \frac{1}{k\cot\delta_0(k) - ik} \;\to\; \frac{1}{-1/a + r_e k^2/2 + \cdots - ik} \tag{4}$$

of the s-wave scattering amplitude. For the simple model potential (2), the exact expression (Gribakin and Flambaum, 1993)

$$a = \bar{a}\,[1 - \tan(\Phi - 3\pi/8)] \tag{5}$$

for the scattering length shows that its characteristic magnitude is set by the mean scattering length $\bar{a} = 0.956\, l_{\rm vdw}$, which is basically identical with the van der Waals length. The short range part of the interaction only enters via the WKB-phase

$$\Phi = \int_{r_c}^{\infty} dr \sqrt{m|V(r)|}/\hbar = 2l_{\rm vdw}^2/r_c^2 \gg 1 \tag{6}$$

at zero energy, which is sensitive to the hard core scale r_c. Concerning the effective range r_e, the exact result for the toy-model potential is (Flambaum *et al.*, 1999)

$$r_e = 2.92\,\bar{a}\left(1 - 2\frac{\bar{a}}{a} + 2\left(\frac{\bar{a}}{a}\right)^2\right). \tag{7}$$

Its typical value $2 - 3\, l_{\rm vdw}$ is therefore again of the order of the van der Waals length, unless $a \ll \bar{a}$. In contrast to what might have been expected naively, the effective range in low energy scattering is thus much larger than the short range scale r_c. The property $r_e \gg r_c$ is in fact a generic result for low energy scattering in long range potentials, provided the number $N_b \simeq (l_{\rm vdw}/r_c)^2$ of bound states is much larger than one (Flambaum *et al.*, 1999).

B. Feshbach Resonances

The regime of strong interactions in dilute, ultracold gases can in practice be reached by Feshbach resonances, which allow to increase the scattering length to values not only beyond $l_{\rm vdw}$ but even far beyond the average interparticle spacing. In the following, we will briefly introduce the standard two-channel model for magnetically tunable Feshbach resonances. For a detailed exposition of the subject, see the review by Chin *et al.* (2010).

Quite generally, a Feshbach resonance in a two-particle collision appears whenever a bound state in a closed channel is coupled resonantly with the scattering continuum of an

open channel. The two channels may correspond, for example, to different spin configurations for the atoms. Taking the specific example of fermionic ^{6}Li atoms, which have electronic spin $S = 1/2$ and nuclear spin $I = 1$, for typical magnetic fields above 500 G, the electron spin is essentially fully polarized by the magnetic field, and aligned in the same direction for the three lowest hyperfine states. Thus, two lithium atoms collide with their electron spins aligned, hence in the triplet configuration. The incoming state or open channel is thus part of the triplet interatomic potential. In turn, the closed channel consists of states in the singlet potential, which has a much deeper attractive well. Due to the hyperfine interaction, that can trade electron spin for nuclear spin, the two atoms can resonantly tunnel from the triplet into bound states of the singlet potential and back.

What makes Feshbach resonances in the scattering of cold atoms particularly useful, is the ability to tune the scattering length simply by changing the magnetic field. This tunability relies on the finite difference $\Delta\mu$ in the magnetic moments of the closed and open channels, which allows to change the position of closed channel bound states relative to the open channel threshold by an external magnetic field. A standard parametrization for the magnetic field dependent scattering length in the open channel near a particular Feshbach resonance at $B = B_0$ is given by

$$a(B) = a_{\rm bg}\left(1 - \frac{\Delta B}{B - B_0}\right) = a_{\rm bg} - \frac{\hbar^2}{m r^\star \nu(B)}\,, \tag{8}$$

where $a_{\rm bg}$ is the off-resonant background scattering length in the absence of the coupling to the closed channel, while ΔB describes the width of the resonance expressed in magnetic field units. A physically more transparent expression for $a(B)$ is given in the second form of Eq. (8) in terms of the detuning $\nu(B) = \Delta\mu(B - B_0)$ away from the resonance and a characteristic length $r^\star > 0$, whose inverse $1/r^\star$ is a measure of how strongly the open and closed channels are coupled (see Eq. (13) below).

A microscopic understanding of the origin of the resonant contribution to the scattering length can be obtained most simply within a two-channel model. On the many-body level, this model describes the conversion of two atoms in an open channel into a closed channel

bound state. For a two-component Fermi gas, the effective Hamiltonian is

$$\hat{H} = \int d^3x \Bigg[\sum_\sigma \hat{\psi}^\dagger_\sigma \big(-\tfrac{\hbar^2}{2m}\nabla^2 \big) \hat{\psi}_\sigma \; + \hat{\Phi}^\dagger \big(-\tfrac{\hbar^2}{4m}\nabla^2 + \nu_{\rm res} \big) \hat{\Phi} \\ + \tilde{g} \int d^3x' \, \chi(|\boldsymbol{x} - \boldsymbol{x'}|) \Big(\hat{\Phi}^\dagger (\tfrac{\boldsymbol{x}+\boldsymbol{x'}}{2}) \, \hat{\psi}_\uparrow(\boldsymbol{x}) \hat{\psi}_\downarrow(\boldsymbol{x'}) + \text{h.c.} \Big) \Bigg] . \tag{9}$$

Here, the fermionic field operators $\hat{\psi}_\sigma(\boldsymbol{x})$ describe atoms in the open channel, with a formal spin variable $\sigma = \uparrow, \downarrow$ distinguishing two different hyperfine states. The bound state in the closed channel is denoted by the bosonic operator $\hat{\Phi}$, which is often called the dimer field. Its energy $\nu_{\rm res}$ measures the detuning of one bare dimer state with respect to two atoms at zero energy. The coupling is characterized by a strength $\tilde{g}$ and a dimensionless cutoff function $\chi(\boldsymbol{x})$ which only depends on the magnitude $r = |\boldsymbol{x} - \boldsymbol{x'}|$ of the two atoms in the open channel, consistent with the pure s-wave nature of scattering. As will be shown below, the characteristic range σ of the cutoff function is the van der Waals or mean scattering length. Physically, this means that the conversion between the open channel scattering states and the closed channel bound state takes place at a separation much larger than the short distance scale r_c. The absence of a term quartic in the fermionic fields in (9) implies that background scattering between fermions is neglected. This is justified close enough to resonance $|B - B_0| \ll |\Delta B|$, where the scattering length is dominated by its resonant contribution $a_{\rm res} \sim -(r^\star \nu)^{-1}$.

For just two atoms, the model in (9) is equivalent to an an off-diagonal coupling (Castin, 2007)

$$\hat{W} \, |\phi_{\rm res}\rangle = \tilde{g} \sum_k \chi(\boldsymbol{k}) \, ||\boldsymbol{k}\rangle \qquad\qquad \hat{W} \, ||\boldsymbol{k}\rangle = \tilde{g} \, \chi(\boldsymbol{k}) \, |\phi_{\rm res}\rangle \tag{10}$$

between a single bound state $|\phi_{\rm res}\rangle$ in the closed-channel and a pair of atoms with opposite momenta $\boldsymbol{k}, -\boldsymbol{k}$ in an open channel state $||\boldsymbol{k}\rangle$. The associated two-body problem can be reduced to a coupled eigenvalue equation in momentum space

$$\begin{aligned} \frac{\hbar^2 k^2}{m} \, \alpha(\boldsymbol{k}) + \tilde{g} \, \chi(\boldsymbol{k}) \, \sqrt{Z} &= E \, \alpha(\boldsymbol{k}) \\ \nu_{\rm res}(B) \, \sqrt{Z} + \tilde{g} \sum_k \chi(\boldsymbol{k}) \alpha(\boldsymbol{k}) &= E \, \sqrt{Z} \, . \end{aligned} \tag{11}$$

by decomposing an eigenstate with energy E in terms of a sum $\sqrt{Z} \, |\phi_{\rm res}\rangle + \sum_k \alpha(\boldsymbol{k}) \, ||\boldsymbol{k}\rangle$, with Z as a measure of the closed channel admixture. The set of Eqs. (11) can then be

solved exactly, giving rise to an open channel scattering amplitude (Castin, 2007)

$$f_k = \frac{m}{4\pi\hbar^2} \frac{\tilde{g}^2 \chi^2(k)}{\nu_{\rm res}(B) - \frac{\hbar^2 k^2}{m} + \frac{m\tilde{g}^2}{\hbar^2} \int_q \frac{\chi^2(q)}{k^2 - q^2 + i0}} \,. \tag{12}$$

Expanding this at low energies, the resulting expressions for the scattering length and the effective range defined in (4) are

$$\frac{1}{a} = -\frac{m r^\star}{\hbar^2} \nu_{\rm res}(B) + \frac{1}{2\sigma} \quad \text{and} \quad r_e = -2r^\star + 3\sigma \left(1 - \frac{4\sigma}{3a}\right) \tag{13}$$

where we have introduced $r^\star = 4\pi\hbar^4/(m^2\tilde{g}^2)$ as an intrinsic length scale for the strength of the Feshbach coupling. Specifically, the cutoff function has been choosen to be of a Lorentzian form $\chi(\boldsymbol{k}) = 1/\left(1 + (k\sigma)^2\right)$ in momentum space (Schmidt *et al.*, 2012). Expanding the detuning $\nu_{\rm res}(B) = \Delta\mu(B - B_{\rm res})$ of the closed channel molecular state to linear order around a bare resonance position $B_{\rm res}$ leads to a scattering length which is indeed of the form given in (8). The resonance position is shifted from its bare value by

$$\Delta\mu \left(B_0 - B_{\rm res}\right) = \frac{\hbar^2}{2m r^\star \sigma} \,. \tag{14}$$

The magnetic field B_0 where the scattering length diverges thus differs from the value at which the energy of a bare molecule crosses zero. This resonance shift has been calculated within in a fully microscopic description of the Feshbach coupling based on interaction potentials with a van der Waals tail (Góral *et al.*, 2004). Comparison with this result yields the identification $\sigma = \bar{a}$, with the mean scattering length $\bar{a} \simeq 0.956\, l_{\rm vdw}$. The ratio

$$s_{\rm res} = \frac{\bar{a}}{r^\star} \tag{15}$$

between the two characteristic lengths which characterize the scattering length and range in potentials with a van der Waals tail and the strength of the coupling to the closed channel is called the resonance strength (Chin *et al.*, 2010). It allows to classify Feshbach resonances into two limiting cases: when $s_{\rm res} \gg 1$, the resonance is called an open channel dominated one because the closed channel fraction Z remains small compared to one over the whole range of detunings $|B - B_0| < |\Delta B|$. Moreover, in the relevant regime where $a \gg \bar{a}$, their effective range $r_e \to 3\,\bar{a}$ obtained from Eq. (13) is essentially identical to the corresponding result (7) for a single-channel potential with a $1/r^6$ tail. Resonances with $s_{\rm res} \ll 1$, in turn, are called closed channel dominated. Here, the near-threshold scattering and bound states have an open channel character only very close to resonance, where the two-body bound

state energy follows the universal behavior $\epsilon_b = \hbar^2/(ma^2)$ determined by the scattering length only. The effective range $r_e \to -2r^\star$ is now negative and large compared to the characteristic scale set by $\bar{a} \simeq l_{\rm vdw}$.

C. Three-body interactions, Efimov-effect

While (s-wave) Feshbach resonances appear for both Bose or two component Fermi gases, the strong interaction limit $a \gg n^{-1/3} \gg \bar{a}$ is in practice only accessible for Fermions. This is a result of the fact that for Fermions the lifetime due to three-body collisions is sufficiently large near a Feshbach resonance, in contrast to bosons, where it goes to zero. The basic physics which underlies the stability of Fermions near a resonance of the scattering length is the fact that relaxation into deep bound states is strongly suppressed by the Pauli-principle. By energy and momentum conservation, a relaxation into one of the deeply bound states requires that at least three fermions are at a distance of order $b \simeq l_{\rm vdw} \ll a$. In a two-component gas two of them are necessarily equal [1]. A detailed analysis of this problem has been given by Petrov *et al.* (2004), who have shown that the dependence of the relaxation rate on the scattering length can be inferred from the behavior of the three-body wave function at short distances. Quite generally, for a system of N Fermions with short range interactions, one may define an exponent $\gamma(N_\uparrow, N_\downarrow)$ by the behavior (Tan, 2004)

$$\Psi(\boldsymbol{r}_1, \sigma_1\, \boldsymbol{r}_2, \sigma_2 \ldots \boldsymbol{r}_N, \sigma_N) \;\to\; r^{\gamma(N_\uparrow, N_\downarrow)} \;\text{ as }\; r \to 0 \tag{16}$$

of the many-body wave function as $N_\uparrow$ up-spin Fermions and $N_\downarrow$ down-spin Fermions are within a small radius $r \ll a$ which is, however, still much larger than the actual range of the interaction potential. All other particles remain at a finite distance. From the standard expression $\psi_0(r) = 1/r - 1/a$ for the corresponding solution of the two-particle problem with a scattering length a, one has $\gamma(1,1) = -1$ trivially. For three Fermions, the solution of the three-body Schrödinger equation (Petrov *et al.*, 2004) yields $\gamma(2,1) = -0.2273\ldots$, i.e. the wave function is less singular than for two particles. The physical origin of the non-integer power law is an effective $1/R^2$-potential which appears in the three-body Schrödinger equation expressed in terms of the hyperradius $R = \sqrt{r_{12}^2 + r_{13}^2 + r_{23}^2}$ (Petrov *et al.*, 2004).

[1] Note that this is no longer the case for three or more component Fermigases, which therefore do not exhbit an enhanced stability for large scattering lengths.

By dimensional analysis, the probability that three Fermions get close depends on a via the prefactor $A(a) \sim a^{-3/2-\gamma}$ of the three-body wave function $\Psi(r \to 0) = a^{-3/2}\,(r/a)^{\gamma}\,F(\Omega) = A(a)\,r^{\gamma}\,F(\Omega)$. Here, $F(\Omega)$ is a function which depends on the remaining angular degrees of freedom and is regular as $r \to 0$. The relaxation rate $\alpha_{\rm rel}$ into deep bound states will be proportional to $|A(a)|^2$. Expressed in physical units cm^3/sec, it thus follows a power law (Petrov *et al.*, 2004)

$$\alpha_{rel}(a) = \text{const}\,\frac{\hbar\, l_{\rm vdw}}{m}\cdot\Big(\frac{l_{\rm vdw}}{a}\Big)^{s} \tag{17}$$

with a positive exponent $s = 3 + 2\gamma \simeq 2.55$. The dimensionless prefactor depends on short range physics below the scale $l_{\rm vdw}$ and thus cannot be calculated within the zero range approximation. Experimental results for the lifetime of Fermionic ^{40}K or ^{6}Li near their respective Feshbach resonances at $B_0 \simeq 202\,$G (Regal *et al.*, 2004) and $B_0 \simeq 832\,$G (Bourdel *et al.*, 2004) are consistent with the dependence predicted by Eq. (17). In a system with finite density $n \sim k_F^3$, the power law dependence on a is cut off at values beyond $k_F a = \mathcal{O}(1)$. The rate $\Gamma_3 = -\dot{N}_3/N$ for three-body losses due to decay into deeply bound states is then expected to scale like $\Gamma_3 \sim n\,\alpha_{\rm rel}(1/a \to k_F) \simeq (k_F\, l_{\rm vdw})^{s+3}$ (Petrov *et al.*, 2004). The fact that this loss rate is negligible in the experimentally relevant limit $k_F\, l_{\rm vdw} \ll 1$ is essential for the stability of the unitary Fermi gas. Expressing $\Gamma_3 = L_3 \cdot n^2$ in terms of a three-body loss rate coefficient L_3, the fact that $s + 3 = 6 + 2\gamma$ implies that $L_3 \sim (k_F\, l_{\rm vdw})^{2\gamma}$ exhibits a dependence on density $n \sim k_F^3$ which directly reflects the presence of a nontrivial short range scaling exponent. The unexpected power law with an exponent γ which is not a simple fraction can be understood from a field-theoretic point of view due to the appearance of an anomalous dimension for operators whose matrix elements first appear at the three-body level. Specifically, the value $\gamma = -0.227..$ appears in the anomalous dimension

$$\Delta_{\mathcal{O}} = \Delta_{\phi} + \Delta_{\psi\uparrow} + 1 + \gamma = 2 + 3/2 + 1 + \gamma = 4.272\ldots \tag{18}$$

of the operator

$$\mathcal{O} = \mathcal{O}^{(l=1)}_{\uparrow\uparrow\downarrow}(\boldsymbol{x}) = Z^{-1}(\Lambda)\,[2\{\phi\,\partial_i\psi_{\uparrow} - (\partial_i\phi)\psi_{\uparrow}]\,(\boldsymbol{x})\,. \tag{19}$$

It contains the gradient ($l = 1$) of a renormalized diatom operator $\hat{\phi}$ introduced in Eq. (59) below, combined with one additional up-spin Fermion. Here $i = x, y, z$ and $Z \sim \Lambda^{-\gamma}$ is the renormalization factor which is necessary for giving finite matrix elements of the operator in the zero range limit $\Lambda \to \infty$. The value $\Delta_{\mathcal{O}}$ also determines the energy $E_0 = \Delta_{\mathcal{O}}\hbar\omega$

of the ground state of three Fermions in a harmonic trap with frequency ω precisely at infinite scattering length, which has $l = 1$. For a detailed discussion of these connections, see Nishida and Son (2012).

The issue of inelastic collisions has an additional aspect, which is is crucial for the eventual stability of a many-body system of Fermions for arbitrary large scattering lengths. On the three-body level, this is related to the *repulsive* nature of elastic atom-dimer scattering, which is described by a positive scattering length $a_{ad} \simeq 1.2\ a$ in the regime $a > 0$ where two of the Fermions form a bound state with wave function $\varphi_0(r) \sim \exp(-r/a)$ (Petrov *et al.*, 2005). This result had in fact been first derived by Skorniakov and Ter-Martirosian in connection with neutron-deutron scattering. The repulsive tendency due to Fermi statistics also shows up in the four-body problem. This can be derived from an exact solution of the four-particle Schrödinger equation with pseudopotential interactions in the limit where the distance R between the centers of mass of two bosonic dimers is much larger than the dimer size a and at collision energies much smaller than their respective binding energies $\hbar^2/ma^2$. The wave function has the asymptotic form (Petrov *et al.*, 2004)

$$\Psi(\boldsymbol{x_1}, \boldsymbol{x_2}, \boldsymbol{R}) = \varphi_0(r_1)\varphi_0(r_2)\big(1 - a_{dd}/R\big) \quad \text{with} \quad a_{dd} = 0.6\, a\,, \tag{20}$$

where $\varphi_0(r) \sim \exp(-r/a)$ is the bound state wave function of an individual dimer and $\boldsymbol{x}_{1,2}$ are the respective interparticle distances between the two distinguishable fermions which they are composed of. Eq. (20) implies that the effective dimer-dimer interaction at low energies is characterized by a positive scattering length proportional to the original scattering length between its fermionic constituents. This guarantees the stability of molecular condensates and implies that, at least for short range interactions, there are no four-particle bound states. More generally, the stability of a Fermi gas at the many-body level for arbitrary strong attractive interactions $a \to -\infty$ relies crucially on the assumption that the range of the interactions is negligible. In fact, it is easy to show that the Pauli principle alone is unable to stabilize a gas of particles with purely attractive interactions if they have a finite range (Castin and Werner, 2012).

The stability of Fermionic gases also shows up in in a virial expansion around the non-degenerate limit $n\lambda_T^3 \ll 1$. Introducing a degeneracy factor g ($g = 2$ for a two-component

Fermi gas while $g = 1$ for the standard Bose case), this expansion has the well known form

$$\frac{P}{k_B T} = \lambda_T^3 \cdot g \sum_{l=1}^{\infty} b_l z^l \quad \text{and} \quad n\lambda_T^3 = g \sum_{l=1}^{\infty} l b_l z^l \tag{21}$$

with $z = \exp(\beta\mu)$ the fugacity and dimensionless virial coefficients b_l. The resulting density equation of state then reads

$$\frac{P}{n k_B T} = \sum_{l=1}^{\infty} a_l \left(\frac{n\lambda_T^3}{g}\right)^{l-1} \tag{22}$$

with dimensionless coefficients a_l such that $a_1 = 1$; $a_2 = -b_2$; $a_3 = 4b_2^2 - 2b_3$; .... Using $b_l^{(0)} = (-1)^{l-1}/l^{5/2}$ for an ideal Fermi gas, these coefficients turn out to be quite small, with $a_2^{(0)} = 1/(4\sqrt{2}) \simeq 0.1768$ and $a_3^{(0)} \simeq -0.0033$. For unitary Fermions, the coefficient $a_2 = -3/(4\sqrt{2}) \simeq -0.5303$ associated with just two body physics has changed sign and is three times larger, suggesting that the strong attractive interaction gives a Bose like behavior. The coefficient $a_3 \simeq 1.71$, however, is positive and larger by a factor of about 517(!) compared to that of an ideal Fermi gas. The atom-dimer repulsion thus has a huge effect on the density equation of state.

A completely different behavior appears for Bosons near a Feshbach resonance which, unfortunately, form an unstable system with a rather short lifetime once the scattering length exceeds about ten times its characteristic value $\bar{a}$. More precisely, the lifetime of Bose-Einstein condensate due to three-body losses *de*creases very strongly with increasing scattering length according to

$$-\dot{N}_3/N = L_3 \cdot n^2 = \text{const.}\, \frac{\hbar}{m} \left(na^2\right)^2 \tag{23}$$

This result, however, only describes the dependence on a *on average*. In fact, a closer analysis shows that there are maxima in the three-body loss coefficient $L_3 \sim \hbar a^4/m$ on top of the a^4-law at a discrete number $a_-^{(n)}$ of negative scattering lengths and minima at a set of positive a's which are due to the formation of many-body bound states. For the special case of three particles, the associated trimer states have been predicted first by Efimov in the 70s. They were eventually observed in an ultracold gas of Caesium atoms by Kraemer *et al.* (2006). Very recently, it has even been possible to observe the formation of a second trimer state at the much larger scattering length $a_-^{(1)} \simeq 21.0(1.3)\, a_-$ (Huang *et al.*, 2014). Remarkably, for Bosons, many-body bound states at negative scattering lengths appear to

exist for an arbitrary large number of particles, at least up to $N = 13$ (Stecher. J, 2010)! A detailed analysis has so far only been given for tetramer states, which again form an infinite sequence, with two tetramer states per Efimov trimer (Schmidt and Moroz, 2010; Stecher. J *et al.*, 2009). The lowest one appears at $a_{-(4)} \simeq 0.43\, a_{-(3)}$. It is likely that there are N-body bound states for arbitrary large N, a conjecture which is consistent with a recent theorem (Seiringer, 2012) which states that any pairwise interaction potential with negative scattering length a has an N-body bound state for some value of N, no matter how small $|a|$ may be.

II. TAN-RELATIONS, SCALE INVARIANCE, RF-SPECTROSCOPY

A. Tan relations

This chapter provides an introduction to a series of exact relations due to Shina Tan, which hold for Fermions [2] with short range interactions (Tan, 2008a,b,c). The Tan relations connect the short distance behavior of one- and two-particle correlations with thermodynamic properties. They can be extended to time dependent correlations, giving rise to sum rules and power law tails at high frequency of RF-spectra (Braaten *et al.*, 2010; Punk and Zwerger, 2007; Schneider and Randeria, 2010) or of response functions like the dynamic shear viscosity (Enss *et al.*, 2011; Hofmann, 2011; Taylor and Randeria, 2010). For a more detailed discussion of the subject see Braaten (2012) and Castin and Werner (2012)

The study of a non-relativistic system of fermions with spin-independent two-body interactions appears as a generic many-body problem in different areas of physics. Except for the particular case of one-dimension, there are, unfortunately, very few exact results on this problem beyond the perturbative regime. It is therefore of considerable interest to derive relations for the many-body problem that hold independent of the interaction strength. As realized by Shina Tan, a whole new class of exact relations may be derived in the context of strongly interacting ultracold gases, where the range of the interactions can effectively be set to zero. In this special case, it turns out that the momentum distribution exhibits a universal C/k^4 decay in the regime where k is larger than other characteristic momentum scales in the problem. The constant C is the same for both particle species even in strongly imbalanced Fermi gases and is called the *contact*, because it is a measure of the probability that two fermions with opposite spin are close together. A crucial feature of the Tan relations is the fact that they apply to *any* state of the system, e.g. both to a Fermi-liquid or a superfluid state, at zero or at finite temperature and also in a few-body situation. The only change is the value of the contact C. The origin of this universality was elucidated by Braaten and Platter (Braaten and Platter, 2008) who have shown that the Tan relations are a consequence of operator identities that follow from a Wilson operator product expansion.

The contact which appears as the central quantity in the Tan relations turns out to be of

[2] For an extension of the Tan relations to interacting Bosegases, where the presence of the Efimov effect has to be accounted for, see (Braaten *et al.*, 2011)

significance both on the macroscopic and the microscopic level. To understand the nontrivial connection between these two different aspects, it is convenient to start by introducing the contact first in a purely thermodynamic fashion. The complete equilibrium thermodynamics of an interacting fluid in a micro-canonical situation, which is appropriate for ultracold gases, is fixed by its entropy $S(U, V, N)$ as a function of the conserved variables energy U, volume V and total particle number N. The condition $S(\lambda U, \lambda V, \lambda N) = \lambda\, S(U, V, N)$ that the gas is extensive implies the Gibbs-Duhem relation $G = \mu N$ for the free enthalpy $G = U - TS + pV$, or - equivalently - the relation $dp = n\, d\mu + s\, dT$, where $s = S/V$ is the entropy density. These relations are valid for abitrary phases of the many-body system. Of course, in order to calculate the equation of state say in the form $p(\mu, T)$ requires to explicitely determine e.g. the microcanonical partition function $S(U, V, N)$ for a given form of the interaction between the particles. In the context of ultracold gases, a new situation arises in this respect because

- the whole interaction is embodied in a *single* parameter, namely the scattering length
- the interaction can be changed externally via Feshbach resonances

It thus makes sense to consider the entropy of the gas not only as a function of the conserved and extensive variables U, V, N but also of the - for later convenience - inverse scattering length $1/a$. The associated complete differential

$$dS(U, V, N, 1/a) = \frac{1}{T}\, dU + \frac{p}{T}\, dV - \frac{\mu}{T}\, dN - \frac{X_{1/a}}{T}\, d\,(1/a) \tag{24}$$

then defines a new 'generalized force' $X_{1/a}$ (Balian, 1991). Its thermodynamic meaning becomes evident by rewriting (24) as the differential change in free energy

$$dF(T, V, N, 1/a) = -S\, dT - p\, dV + \mu\, dN + X_{1/a}\, d\,(1/a)\ . \tag{25}$$

Thus $X_{1/a}\, d\,(1/a)$ is the work done *on* the gas in an infinitesimal change $d(1/a)$ of the scattering length, keeping T, V and N fixed. Considering, for instance, a situation where a gas with a strongly repulsive interaction $l_{\rm vdw}/a \gtrsim 0$ is turned into a weakly interacting gas $l_{\rm vdw}/a \simeq 1$, it is evident that, similar to an expansion $dV > 0$, the gas does work on its environment. As a result, $X_{1/a} = -\hbar^2\, C/(4\pi m) < 0$ defines a *positive* quantity C, which has dimensions of an inverse length. For reasons that will become clear below, C will be called the 'contact'. In the case of a trapped gas with non-uniform density $n(\boldsymbol{R})$, the contact

$C = \int_{\boldsymbol{R}} \mathcal{C}$ can be written as an integral of an intensive contact density $\mathcal{C}(\boldsymbol{R})$. Fixing T, V and N, the relation

$$\frac{\partial F(T)}{\partial(1/a)} = \frac{\partial U(S)}{\partial(1/a)} = -\frac{\hbar^2}{4\pi m} \cdot \int_{\boldsymbol{R}} \mathcal{C}(\boldsymbol{R}) \tag{26}$$

which follows from (25) is called the Tan adiabatic theorem (Tan, 2008a). Using the fact that the entropy is extensive, this implies that the contact density $\mathcal{C}$ can be obtained from the change in pressure with inverse scattering length at fixed μ and T via

$$d\,p(\mu, T, 1/a) = n\,d\mu + s\,dT + \frac{\hbar^2}{4\pi m}\,\mathcal{C}\,d\,(1/a)\ . \tag{27}$$

The Tan adiabatic theorem states that the derivative of the total internal energy U or the free energy F with respect to the inverse scattering length at fixed values of the entropy S or temperature T, respectively, define a new and positive thermodynamic variable C. As a result, knowledge of the contact $C = \int_{\boldsymbol{R}} \mathcal{C}(\boldsymbol{R})$ as a function of the scattering length completely determines the thermodynamics by an integration which starts with the non-interacting system at $a = 0$.

An important exact relation for the total *energy* in an inhomogeneous situation is provided the Tan virial theorem (Tan, 2008c)

$$U = \langle \hat{H}_{\rm kin} + \hat{H}_{\rm int} + \hat{H}_{\rm ext} \rangle = 2 \int_{\boldsymbol{R}} V_{\rm ext}(\boldsymbol{R})\, n(\boldsymbol{R}) \ - \frac{\hbar^2}{8\pi m a} \int_R \mathcal{C}(\boldsymbol{R})\,, \tag{28}$$

which holds for harmonic trap potentials $V_{\rm ext}(\boldsymbol{R})$ even if they are anisotropic. Considering, in particular, the case of a unitary gas at infinite scattering length, where the last term vanishes since the contact density is finite at $a = \infty$, this relation implies that the total energy may be determined from in situ measurements of the density profile $n(\boldsymbol{R})$ (Thomas, 2008; Thomas *et al.*, 2005) [3]. The relation (28) is a simple consequence of dimensional analysis combined with the Tan adiabatic theorem. Consider an isotropic harmonic trap potential $V_{\rm ext}(\boldsymbol{R}) = m\omega^2 \boldsymbol{R}^2/2$. At a fixed number of particles, the free energy

$$F(T, \omega, 1/a) = \hbar\omega\, \tilde{F}\left(\frac{k_B T}{\hbar\omega}, \frac{\omega}{\hbar/ma^2}\right) \tag{29}$$

can be expressed in terms of a dimensionless function $\tilde{F}$, which depends only on dimensionless ratios. From (29), one can deduce the simple scaling law $F(\lambda T, \lambda\omega, \sqrt{\lambda}/a) =$

[3] This may be viewed as an utterly trivial example of density functional theory, where the non-trivial part of the functional $E[n]$ related to the kinetic and interaction energy is identically equal to $\langle \hat{H}_{\rm ext} \rangle$!

$\lambda F(T, \omega, 1/a)$ whose derivative with respect to λ at $\lambda = 1$ yields

$$\left(T\frac{\partial}{\partial T} + \omega\frac{\partial}{\partial \omega} + \frac{1}{2a}\frac{\partial}{\partial(1/a)} \right) F = F\,, \tag{30}$$

where all the partial derivatives are to be understood as leaving all other system variables constant. Since the free energy is just the Legendre transform of the energy, its partial derivatives at constant temperature T with respect to ω and $1/a$ are equal to those of the energy at the associated value of the entropy. Therefore, using $\partial F/\partial T = -S$, the energy turns out to obey the differential equation

$$\left(\omega\frac{\partial}{\partial \omega} + \frac{1}{2a}\frac{\partial}{\partial(1/a)} \right) U = U. \tag{31}$$

This leads immediately to the relation (28) by using the Tan adiabatic theorem (26) and $\omega\,\partial E/\partial\omega = 2\langle V_{\rm ext}\rangle$.

For a uniform gas, a further exact relation is the Tan pressure relation

$$p = \frac{2}{3}\,\epsilon + \frac{\hbar^2}{12\pi m a}\,\mathcal{C}\,, \tag{32}$$

which relates pressure p and energy density ϵ. Similar to the argument above, its proof relies on dimensional analysis. Indeed, again at a fixed number of particles, the entropy $S(U, V, 1/a) = \tilde{S}(u/(\hbar^2/ma^2), v/a^3)$ is only a function of dimensionless ratios that can be formed from the energy u and volume v per particle and the scattering length a. Taking the derivative of this relation with respect to $1/a$, the definition of the contact via (24) implies that

$$\frac{\hbar^2}{4\pi m}\,C = a\,[-2U + 3pV] \tag{33}$$

from which the pressure relation immediately follows. Anticipating again that $\mathcal{C}$ is finite at $a = \infty$, this implies that pressure and energy density are related in a trivial manner, identical to the relation which is known to hold for non-interacting gases, both Bose and Fermi. The deep underlying reason for this remarkable result is that at infinite scattering length, the gas is scale invariant, a property that will be discussed in more detail in section II.B below.

As noted by Werner *et al.* (2009) and Zhang and Leggett (2009), an observable which provides a direct measure of the contact is the number

$$N_b = \int d^3R\,\langle \hat{\Phi}^\dagger(\boldsymbol{R})\hat{\Phi}(\boldsymbol{R})\rangle \tag{34}$$

of closed channel molecules near a Feshbach resonance. This connection is a simple consequence of the observation above that the contact is a measure of the work done on a gas upon changing the (inverse) sattering length. For a magnetically tuned Feshbach resonance, this can be expressed in terms of the work needed to change the magnetic field by an infinitesimal amount dB in the presence of a finite magnetization $M = N_b\Delta\mu$, where $\Delta\mu$ is the difference in the magnetic moment between the molecule and the open-channel atoms. Indeed, using the two-channel description of Eq. (9), the only term in the Hamiltonian which depends on the magnetic field B is the bare detuning $\nu_{\rm res}(B) = \mu(B - B_{\rm res})$ of the closed channel bound state. Knowledge of the dependence $a(B)$ of the two-body scattering length on the external field B thus allows to extract the derivative of the many-body free energy F as a function of the inverse scattering length $1/a$ from the observed number of closed channel molecules via

$$N_b\Delta\mu = \frac{\partial F}{\partial B} = \frac{\partial F}{\partial(1/a)} \cdot \frac{d(1/a)}{dB} = -\frac{\hbar^2}{4\pi m} \cdot \int_{\boldsymbol{R}} \mathcal{C}(\boldsymbol{R}) \cdot \frac{d(1/a)}{dB}\,. \tag{35}$$

Here, in the second equality, we have used the Tan adiabatic theorem (26). The integrated contact density of a trapped gas can thus be obtained from the measured number of closed channel molecules, an experiment which had been done even before its relation with the Tan contact was realized. Specifically, Partridge et.al. (Partridge *et al.*, 2005) have determined the closed channel fraction $Z = N_b/(N/2)$ of a near resonant gas of ^{6}Li by exciting the closed channel molecules with a laser that transfers them to another, short lived molecular state. This results in a detectable loss of atoms from the trap. The observed very small value $Z \simeq 10^{-4}$ near the Feshbach resonance at $B_0 \simeq 832\,\mathrm{G}$ reflects its broad nature. Indeed, as defined in Eq. (8), the slope with which the inverse scattering length

$$\frac{1}{a(B)} = -\frac{\Delta\mu(B - B_0)\cdot m r^\star}{\hbar^2} + \ldots \tag{36}$$

vanishes near the resonance at $B = B_0$ as a function of the renormalized detuning $\Delta\mu(B-B_0)$ precisely defines the characteristic length $r^\star$. Using (35), the number of closed channel molecules near resonance

$$N_b(B \approx B_0) = \frac{r^\star}{4\pi} \int_{\boldsymbol{R}} \mathcal{C}(\boldsymbol{R}) \tag{37}$$

is therefore a direct measure of the many-body contact $C = \int_{\boldsymbol{R}} \mathcal{C}(\boldsymbol{R})$ multiplied with the two-body parameter $r^\star$ which characterizes the intrinsic width of the Feshbach resonance. In a spin-balanced Fermi gas, $\mathcal{C}(\boldsymbol{R}) = s \cdot k_F^4(\boldsymbol{R})$ scales with the fourth power of the local

Fermi wavevector $k_F(\boldsymbol{R})$, with s a dimensionless number, which is close to $s \approx 0.1$ for the zero temperature, unitary gas at $1/a = 0$ (Haussmann *et al.*, 2009). The resulting closed channel fraction $Z(B \approx B_0) \simeq k_F r^\star/2 \ll 1$ is therefore very small for 'broad' Freshbach resonances, an issue that will be discussed further in section II.B below.

For an understanding of the *microscopic* physics behind the Tan relations, it is useful to start from a Hamiltonian for a system of Fermions with two possible internal states $\sigma = \pm 1$ [4]. Its interaction part

$$\hat{H}_{\rm int} = \frac{1}{2}\sum_\sigma \int_x \int_{x'} V(\boldsymbol{x}-\boldsymbol{x}')\, \hat{\psi}^\dagger_\sigma(\boldsymbol{x})\hat{\psi}^\dagger_{-\sigma}(\boldsymbol{x}')\hat{\psi}_{-\sigma}(\boldsymbol{x}')\hat{\psi}_\sigma(\boldsymbol{x}) \tag{38}$$

contains a spherically symmetric two-body potential $V(\boldsymbol{x})$ which only acts between particles of opposite spin. Working with such a simplified single channel interaction is justified in the low density regime $n^{-1/3} \gg r_e$ that is relevant for cold atoms, where the average interparticle spacing is much larger than the range r_e of the interaction. In the following, we want to discuss the consequences and simplifications in the many-body problem which result from the fact that the effective interaction range is small compared with the average interparticle distance. Since the total density $n = n_\uparrow + n_\downarrow$ of the two-component Fermi gas defines a Fermi wavevector via $n = k_F^3/3\pi^2$ in the standard manner, the relevant limit is $k_F r_e \to 0$. [5] At finite temperature, the thermal wavelength λ_T appears as an additional length scale. Since the rescriction to s-wave scattering is justified only as long as λ_T is larger than the van der Waals length, the effective range remains the shortest length scale even for cold gases in the crossover regime $k_F\lambda_T \simeq 1$ to classical behavior. For short range interactions which only affect Fermions with opposite spin, the interaction energy

$$\langle \hat{H}_{\rm int}\rangle = \int_R \int_x \langle \hat{n}_\uparrow(\boldsymbol{R}+\frac{\boldsymbol{x}}{2})\, \hat{n}_\downarrow(\boldsymbol{R}-\frac{\boldsymbol{x}}{2})\rangle\, V(\boldsymbol{x}) \quad \stackrel{r_e\to 0}{\longrightarrow} \quad \int_R n_\uparrow(\boldsymbol{R}) n_\downarrow(\boldsymbol{R}) g^{(2)}_{\uparrow\downarrow}(0,\boldsymbol{R}) \int_x V(\boldsymbol{x}) \tag{39}$$

[4] in the context of ultracold atoms, one typically considers a mixture of two different hyperfine states of a Fermionic atom, and $\sigma = \pm 1$ may be formally thought of as labelling the z-component of an effective spin one-half. It is important to note, however, that the 'spin' polarization of both components is conserved *individually* and not only their sum because transitions between different hyperfine states are forbidden in most cases

[5] we assume r_e to be positive, as fulfilled e.g. by the range given in Eq. (7). Note, however, that the effective range defined by Eq. (4) is not necessarily positive, not even for single channel potentials. As evident from Eq. (13), closed channel dominated Feshbach resonances have negative $r_e \to -2r^\star$, in which case the diluteness condition is $k_F|r_e| \ll 1$.

in a situation with spatially varying densities $n_\sigma(\boldsymbol{R})$ due e.g. to an external trap potential, is expected to be determined by the local value $g^{(2)}_{\uparrow\downarrow}(0,\boldsymbol{R})$ of the dimensionless pair distribution function. All information about the many-body physics is then encoded in a single number that will have a nontrivial dependence on the interaction strength $k_F a$ and on temperature. By contrast, the detailed form of the two-body interaction is irrelevant since it only appears as an overall factor $\int V(\boldsymbol{x})$. As will be shown below, a modified form of this expected behavior is indeed valid for short range interacting Fermi gases, however with a quite important difference: the local value $g^{(2)}_{\uparrow\downarrow}(0,\boldsymbol{R})$ of the pair distribution at vanishing separation is ill defined and must be replaced by the Tan contact density $\mathcal{C}(\boldsymbol{R})$, defined properly in Eq. (45) below. For a derivation of the correct form of Eq. (39), we follow Zhang and Leggett (2009) and consider the spectral representation

$$\langle \boldsymbol{x}_1\uparrow, \boldsymbol{x}_2\downarrow|\hat{\rho}_2|\boldsymbol{x}_1\uparrow, \boldsymbol{x}_2\downarrow\rangle = \sum_n \lambda_n^{(2)}\,|\Phi_n(\boldsymbol{x},\boldsymbol{R})|^2 \tag{40}$$

of the diagonal elements of the exact two-particle density matrix $\hat{\rho}_2$ in the subspace of relative singlet states. Its eigenvalues $\lambda_n^{(2)}$ can be interpreted as the number of $\uparrow\downarrow$-pairs that are found in a two-particle wave function Φ_n, whose spatial part depends on the center-of-mass and relative coordinates $\boldsymbol{R} = (\boldsymbol{x}_1+\boldsymbol{x}_2)/2$ and $\boldsymbol{x} = \boldsymbol{x}_1 - \boldsymbol{x}_2$. Inserting the expansion (40) to calculate the interaction energy, one obtains

$$\langle \hat{H}_{\rm int}\rangle = \int_R \int_x \sum_n \lambda_n^{(2)}\,|\Phi_n(\boldsymbol{x},\boldsymbol{R})|^2\, V(\boldsymbol{x})\,. \tag{41}$$

Since the effective interaction potential vanishes beyond a scale $l_{\rm vdw}$ that is much shorter than the interparticle spacing, the integral over the relative coordinate $\boldsymbol{x}$ picks out the short distance regime $r = |\boldsymbol{x}| \le r_e$ of the exact eigenfunctions $\Phi_n(\boldsymbol{x},\boldsymbol{R})$ where their dependence on r is fixed by *two-body* physics. In the relevant regime that contributes to the integral in (41), the eigenfunctions are therefore expected to factorize in the form

$$\Phi_n(\boldsymbol{x},\boldsymbol{R}) \quad \stackrel{r\to 0}{\longrightarrow} \quad c_n(\boldsymbol{R})\,\psi_0(r) \tag{42}$$

where $\psi_0(r) = 1/r - 1/a$ is the zero energy solution of the two-particle problem. A formal proof of this important result is provided by the operator product expansion (OPE) (Braaten, 2012). It starts by considering the position space represemtation of the N-particle wavefunction Ψ in an arbitrary N-particle state $|\Psi_N\rangle$. This function can be expanded as a formal

power series

$$\Psi(-\frac{\boldsymbol{r}}{2},\uparrow;\frac{\boldsymbol{r}}{2},\downarrow;\boldsymbol{r}_3,\sigma_3;\ldots) = \frac{1}{\sqrt{N_\uparrow! N_\downarrow!}}\,\langle 0|\psi_\uparrow(-\frac{\boldsymbol{r}}{2})\psi_\downarrow(\frac{\boldsymbol{r}}{2})\prod_{l=3}^{N}\psi_{\sigma_l}(r_l)|\Psi_N\rangle$$
$$= \sum_n W_n(r)\,\frac{1}{\sqrt{N_\uparrow! N_\downarrow!}}\,\langle 0|\mathcal{O}_n(0)\prod_{l=3}^{N}\psi_{\sigma_l}(r_l)|\Psi_N\rangle, \qquad (43)$$

where $W_n(r)$ are called the Wilson coefficients in an OPE of the operator $\psi_\uparrow(x)\psi_\downarrow(y)$. The leading term can be obtained by calculating the expectation value of the spin singlet operator $(\psi_\uparrow(\boldsymbol{x})\psi_\downarrow(\boldsymbol{y}) - \psi_\downarrow(\boldsymbol{x})\psi_\uparrow(\boldsymbol{y}))/2$ between the vacuum and a *two-particle* state with (on-shell) energy p^2/m. Now, the exact eigenfunction $\psi_{\rm ex}(r;k)$ for relative motion in s-wave states - which always dominate at short distances - is determined by the effectively 1D Schrödinger equation

$$\left[\frac{d^2}{dr^2} - \frac{mV(r)}{\hbar^2}\right]\,\chi_{\rm ex}(r;k) = k^2\,\chi_{\rm ex}(r;k) \qquad (44)$$

for $\chi_{\rm ex}(r;k) = r\cdot\psi_{\rm ex}(r;k)$ with boundary condition $\chi_{\rm ex}(0;k) = 0$. The asymptotic behavior $\chi_{\rm ex}(r;0) \sim (1-r/a)$ of the associated zero energy solution at distances large compared to the range of the potential defines the exact scattering length. The assumption underlying (42) is now that the details of the short distance behavior of $V(r)$ - which in practice becomes infinitely repulsive and thus always dominates the kinetic energy $\sim k^2$ - is ignored and $\chi_{\rm ex}(r;k)$ is replaced by its zero energy asymptotic form $1-r/a$ at *all* distances. This leaves the scattering length a as the single variable characterizing the two-body interaction. As a result, $W_{\psi\psi}(r) \to \psi_0(r) = 1/r\, - 1/a$ solves the two-particle s-wave Schrödinger equation at zero energy. A similar result can in fact be derived for repulsive two-body potentials $V(r\to 0) \sim 1/r^s$ which have the form of a power law at short distances e.g. Coulomb - or dipolar interactions with $s = 1$ or $s = 3$ respectively. In the latter case, one finds $W_{\psi\psi}(r) = 1 + r/2a_B + \ldots$ up to linear order in r, giving rise to a universal power law $n(k\to\infty) \sim 1/k^8$ in the momentum distribution of systems with Coulomb interactions. This applies both at the few-body level as in the hydrogen atom or in the well known jellium problem at finite density (Hofmann *et al.*, 2013).

The many-body nature of the problem is now hidden in the complex numbers $c_n(\boldsymbol{R})$, which fix the amplitudes with which $\psi_0(r)$ is contained in the exact eigenfunctions of $\hat{\rho}_2$ at short distances. Defining a contact density

$$\mathcal{C}(\boldsymbol{R}) = 16\pi^2\sum_n \lambda_n^{(2)}\,|c_n(\boldsymbol{R})|^2\,, \qquad (45)$$

the expectation value of the interaction energy (or of any short range function) can be written as (Zhang and Leggett, 2009)

$$\langle \hat{H}_{\rm int} \rangle = \frac{1}{4\pi} \int_R \mathcal{C}(\boldsymbol{R}) \int_0^\infty r^2 dr\, V(r)\, \psi_0^2(r)\,. \tag{46}$$

Comparing this result with the exact expression (39), the short distance behavior

$$\langle \hat{n}_\uparrow(\boldsymbol{R} + \frac{\boldsymbol{x}}{2})\, \hat{n}_\downarrow(\boldsymbol{R} - \frac{\boldsymbol{x}}{2}) \rangle = \frac{\mathcal{C}(\boldsymbol{R})}{16\pi^2} \left(\frac{1}{r^2} - \frac{2}{ar} + \dots \right) \tag{47}$$

of the density correlation for opposite spins is therefore singular as $r \to 0$, in contrast to the result $n_\uparrow(\boldsymbol{R}) n_\downarrow(\boldsymbol{R}) g^{(2)}_{\uparrow\downarrow}(0, \boldsymbol{R})$ which is expected naively. For a simple physical interpretation of this anomalous behavior, it is useful to recall the standard definition of a pair distribution function and ask how many $\uparrow$-Fermions will - on average - be found in a sphere of radius b around a $\downarrow$-Fermion fixed at some position $\boldsymbol{R}$. At large distances $b \gg n^{-1/3}$, the presence of a $\downarrow$-Fermion at $\boldsymbol{R}$ becomes irrelevant and $N_\uparrow(b, \boldsymbol{R}) = n_\uparrow(\boldsymbol{R}) \cdot 4\pi b^3/3$ scales linearly with the volume of the sphere (we assume b to be less than the scale over which the average density of $\uparrow$-Fermions might change). By contrast, in the limit where $r_e < b \ll n^{-1/3}$ this number can be calculated using the short distance behavior (47), which gives

$$N_\uparrow(b \to 0, \boldsymbol{R}) = \int_{|\boldsymbol{x}|<b} d^3x\, \frac{\langle \hat{n}_\uparrow(\boldsymbol{R} + \boldsymbol{x})\, \hat{n}_\downarrow(\boldsymbol{R}) \rangle}{n_\downarrow(\boldsymbol{R})} = \frac{\mathcal{C}(\boldsymbol{R})}{4\pi n_\downarrow(\boldsymbol{R})} \cdot b\,. \tag{48}$$

For b much smaller than the interparticle distance, therefore, the number of $\uparrow$-Fermions scales linearly with the *radius* of the sphere instead of linearly with its volume! This anomalous behavior is a result of the $1/r^2$-dependence of the probability density $|\psi_0(r)|^2$ at short distances, which cancels the factor $4\pi r^2$ from the volume element. As a result, the number of $\uparrow$-Fermions in the range $r_e < b \ll n^{-1/3}$ vanishes much more slowly than the naively expected b^3-law, which would apply if $g^{(2)}_{\uparrow\downarrow}(0, \boldsymbol{R})$ were finite. Of course, for distances $b < r_e$, the details of the short range repulsion matter, typically leading to an exponentially small $N_\uparrow(b \ll r_e) \sim \exp\left(-(r_e/b)^\alpha\right)$. [6] For an ultracold, dilute gas, however, the energies involved are too small to see the behavior at distances less than r_e. Indeed, it is precisely the separation of length scales

$$n^{-1/3} \simeq k_F,\ a,\ \lambda_T \ \gg\ r_e \tag{49}$$

[6] This is a result of the fact that the exact two-body wave function $\psi_{\rm ex}(r)$ vanishes very quickly at short distances. Taking a Lenard-Jones potential for instance, the $1/r^{12}$ repulsion leads to an exponent $\alpha = 5$.

which is the origin of the 'universality' which characterizes these systems: none of the details of the interaction matter except for the scattering length a. This holds, in particular, in the unitary regime of strong interactions $k_F a \gg 1$ where even the scattering length a drops out of the problem, as discussed in more detail below.

For a formal derivation of the Tan relations, which connect the short distance physics encoded in the definition of the contact density via Eq. (47) with thermodynamic properties, it is convenient to use a strict zero range model with a two-body interaction $V(\boldsymbol{x}) = \bar{g}(\Lambda)\,\delta(\boldsymbol{x})$. The cutoff dependent coupling constant $\bar{g}(\Lambda) = 4\pi\hbar^2\bar{a}(\Lambda)/m$ must be adjusted such that the delta-function potential - which would lead to a *vanishing* scattering amplitude in $3d$ for any finite $\bar{g}$ - gives rise to a finite low energy scattering length a. Introducing $\tilde{f} = -4\pi f \to gm/\hbar^2$, the relation between the bare coupling constant $\bar{g}$ and its renormalized value $g = 4\pi\hbar^2 a/m$ at low energies follows from the solution of the Lippman-Schwinger equation

$$\tilde{f}(\boldsymbol{k} \to \boldsymbol{k}') = v(\boldsymbol{k}' - \boldsymbol{k}) + \int_q \frac{v(\boldsymbol{k}' - \boldsymbol{q})\tilde{f}(\boldsymbol{k} \to \boldsymbol{q})}{k^2 - q^2 + i0}\,, \tag{50}$$

where $v(\boldsymbol{k}) = 4\pi\bar{a}$ is the Fouriertransform of $mV(\boldsymbol{x})/\hbar^2$. Taking the limit $\boldsymbol{k} \to 0$, this gives

$$g = \bar{g}(\Lambda) - g\,\bar{g}(\Lambda)\int_q \frac{1}{2\epsilon_q} \quad \text{or} \quad \frac{1}{\bar{g}} = \frac{1}{g} - \int_q \frac{1}{2\epsilon_q}\,, \tag{51}$$

where $\epsilon_q = \hbar^2 q^2/2m$ is the energy of a free particle. Using a sharp cutoff Λ for the divergent integral in (51), the bare value of the scattering length is given by $\bar{a}(\Lambda) = a/(1 - 2a\Lambda/\pi)$, which implies that $\bar{a}(\Lambda)$ vanishes as the cutoff Λ is taken to infinity at fixed a. The somewhat counterintuitive result that finite values of a require the strength $\bar{g}(\Lambda)$ of the delta function to be negative and, moreover, vanish inversely with the cutoff can be understood by noting that two particles with mass m which interact via an attractive square well potential with range b and depth $V_0 = \hbar^2 k_0^2/m$, a potential which - in contrast to its repulsive counterpart - gives rise to scattering lengths in the full range $-\infty < a < \infty$, experience a non-vanishing scattering amplitude in the zero range limit $b \sim 1/\Lambda \to 0$, if and only if, the depth parameter $k_0 \sim \Lambda$ diverges linearly with $1/b$. As a result, the integral $\int V(\boldsymbol{x}) = \bar{g}(\Lambda) \sim -V_0 b^3$ vanishes like $1/\Lambda$.

In order to derive the Tan relations within the zero range model, we consider the formal expression for the operator of the interaction energy density

$$\hat{\epsilon}_{\rm int}(\boldsymbol{R}) = \bar{g}(\Lambda)\,\hat{\psi}^\dagger_\uparrow(\boldsymbol{R})\hat{\psi}^\dagger_\downarrow(\boldsymbol{R})\hat{\psi}_\downarrow(\boldsymbol{R})\hat{\psi}_\uparrow(\boldsymbol{R}) \equiv \bar{g}(\Lambda)\,\hat{\mathcal{O}}_c(\boldsymbol{R}) \tag{52}$$

which contains the bare coupling constant $\bar{g}$. Its expectation value $\epsilon_{\rm int} = \bar{g}(\Lambda)\,\langle\hat{\mathcal{O}}_c\rangle \sim \Lambda$ diverges linearly with the cutoff, as is evident from the divergence of the integral in Eq. (46) if $V(\boldsymbol{x})$ is taken to be proportional to a delta-function. The expectation value of the interaction energy alone is therefore ill-defined, as is the product $\hat{\mathcal{O}}_c$ of the four field operators at the same point in space, which scales like Λ^2. By contrast, the expectation value of the *total* energy density $\epsilon = \epsilon_{\rm kin} + \epsilon_{\rm int}$ should have a well defined zero range limit. Using the Hellman-Feynman theorem and Eq. (51)

$$\frac{\partial\epsilon}{\partial a} = \frac{\partial\bar{g}}{\partial a}\cdot\langle\hat{\mathcal{O}}_c\rangle = \frac{\bar{g}^2}{ga}\cdot\langle\hat{\mathcal{O}}_c\rangle\,, \tag{53}$$

the requirement of a finite energy density which depends in a continuous manner on the scattering length a, apparently implies that the combination

$$\bar{g}^2\cdot\langle\hat{\mathcal{O}}_c(\boldsymbol{R})\rangle = \frac{\hbar^4}{m^2}\cdot\mathcal{C}(\boldsymbol{R}) \tag{54}$$

remains finite as the cutoff is taken to infinity. The relation (54) defines the contact density $\mathcal{C}$ in the zero range limit, a definition which - as will be shown below - is in fact equivalent to the earlier one in (45). The definition leads immediately to the local form of the Tan adiabatic theorem

$$\frac{\partial\epsilon}{\partial a}(\boldsymbol{R}) = \frac{\hbar^2}{4\pi m a^2}\cdot\mathcal{C}(\boldsymbol{R}) \quad \text{or} \quad \frac{\partial\epsilon}{\partial(1/a)}(\boldsymbol{R}) = -\frac{\hbar^2}{4\pi m}\cdot\mathcal{C}(\boldsymbol{R})\,, \tag{55}$$

whose integral form has been derived in Eq. (26) within a purely thermodynamic reasoning. The Tan adiabatic theorem may be viewed as a special case of the Hellman-Feynman theorem for systems with zero range interactions. The total energy density in the translation invariant case

$$\epsilon = \sum_\sigma\int_q \epsilon_q\, n_\sigma(q) + \frac{1}{\bar{g}}\cdot\frac{\hbar^4\mathcal{C}}{m^2} \tag{56}$$

is a sum of the kinetic and the interaction contribution, which involve the momentum distribution $n_\sigma(q)$ and and the contact density, according to its definition in (54). Now, as noted above, both terms diverge individually. Their sum, however, is finite. To see this, the interaction energy contribution is rewritten by using Eq. (51). This leads to a sum of two finite contributions for the total energy density

$$\epsilon = \sum_\sigma\int_q \epsilon_q\left[n_\sigma(q) - \frac{\mathcal{C}}{q^4}\right] + \frac{\hbar^2\mathcal{C}}{4\pi m a}\,, \tag{57}$$

which is the Tan energy theorem (Tan, 2008a). The finiteness of the momentum integral implies that the contact density, which was defined via the *two*-body density matrix in Eq. (45), also determines the weight of the tail

$$\lim_{q\to\infty} n_\sigma(q) = \frac{\mathcal{C}}{q^4} \tag{58}$$

of $n_\sigma(q)$ at large momentum. It is a remarkable fact that the total energy of Fermions with zero range interactions can be expressed completely in terms of the momentum distribution, i.e. the Fourier transform of the *one*-particle density matrix [7]. A direct proof of the asymptotic behavior (58) of the momentum distribution has been given by Braaten and Platter (Braaten and Platter, 2008), who have shown that the singular operator $\hat{\mathcal{O}}_c$ which arises in the interaction energy and the density correlation $\hat{n}_\uparrow\hat{n}_\downarrow$ in the limit of small separation also appears as a non-analytic term $\sim |\boldsymbol{x}|\,\bar{g}^2\hat{\mathcal{O}}_c$ in the short-distance expansion of the one-particle density matrix $\hat{\psi}^\dagger_\sigma(\boldsymbol{R}+\boldsymbol{x}/2)\,\hat{\psi}_\sigma(\boldsymbol{R}-\boldsymbol{x}/2)$ as $|\boldsymbol{x}|\to 0$.

What still remains to be done, is to show the identity of the two - so far unrelated - definitions (45) and (54) of the contact density. To prove this, we introduce the renormalized diatom operator $\hat{\phi}(\boldsymbol{R}) = 4\pi\bar{a}(\Lambda)\,\hat{\psi}_\downarrow(\boldsymbol{R})\hat{\psi}_\uparrow(\boldsymbol{R})$, which turns out to have a finite limit as $\Lambda\to\infty$. In terms of this operator, the definition (54) of the contact density may be rewritten in the form

$$\mathcal{C}(\boldsymbol{R}) = \lim_{\Lambda\to\infty} \left(4\pi\bar{a}(\Lambda)\right)^2 \langle\hat{\mathcal{O}}_c(\boldsymbol{R})\rangle = \langle\hat{\phi}^\dagger(\boldsymbol{R})\hat{\phi}(\boldsymbol{R})\rangle\,. \tag{59}$$

Now, in fact, this relation is identical with our original definition (45) of the contact via the density correlation function at short distances. To see this, we note that Eq. (47) implies that the singular behavior of the operator product as $r\to 0$

$$\lim_{r\to 0} \hat{\psi}_\downarrow(\boldsymbol{R}-\boldsymbol{x}/2)\,\hat{\psi}_\uparrow(\boldsymbol{R}+\boldsymbol{x}/2) \;=\; \bar{a}(\Lambda)\,\left(\frac{1}{r}-\frac{1}{a}\right)\,\hat{\psi}_\downarrow(\boldsymbol{R})\hat{\psi}_\uparrow(\boldsymbol{R}) = \frac{\psi_0(r)}{4\pi}\,\hat{\phi}(\boldsymbol{R}) \tag{60}$$

is determined by the two-body wavefunction $\psi_0(r)$ and the diatom operator $\hat{\phi}(\boldsymbol{R})$ since

$$\langle\hat{n}_\uparrow(\boldsymbol{R}+\boldsymbol{x}/2)\,\hat{n}_\downarrow(\boldsymbol{R}-\boldsymbol{x}/2)\rangle = \langle\hat{\psi}^\dagger_\uparrow(\boldsymbol{R}+\boldsymbol{x}/2)\,\hat{\psi}^\dagger_\downarrow(\boldsymbol{R}-\boldsymbol{x}/2)\hat{\psi}_\downarrow(\boldsymbol{R}-\boldsymbol{x}/2)\hat{\psi}_\uparrow(\boldsymbol{R}+\boldsymbol{x}/2)\rangle \to \frac{\psi_0^2(r)}{16\pi^2}\mathcal{C}(\boldsymbol{R}) \tag{61}$$

[7] For interactions which are not of zero range, the total energy is still determined by the *one*-particle Green function, however it requires knowledge of its full momentum and frequency dependence (Fetter and Walecka, 1971). Note that the weight $\mathcal{C}$ of the tail in (58) is identical for both spin polarizations $\sigma=\pm 1$, even if the gas is not balanced.

This is identical with (47) and thus shows the equivalence with the alternative definition (54) of the contact density. As a final comment, we note that the derivation of the Tan relations in the zero range model provides a simple example of the field theoretic notion of an anomalous dimension. Indeed, the naive scaling dimension of the diatom operator $\hat{\phi}(\boldsymbol{R})$, which consists of a product of two Fermionic field operators $\hat{\psi}_\downarrow(\boldsymbol{R})\hat{\psi}_\uparrow(\boldsymbol{R})$ is three. The assumption, however, that the form of two-body wavefunction at short distances is $\psi_0(r) \sim (1/r - 1a)$, which ignores the true short distance regime where $\psi_0(r)$ eventually vanishes, makes the product of two field operators at the same point in space ill-defined. A finite result is thus obtained only by multiplying this product with $\bar{a}(\Lambda) \sim 1/\Lambda$, which gives an additional factor of length. As a result, the diatom operator has scaling dimension $\Delta_\phi = 2\Delta_\psi - 1 = 2$, as used in Eq. (18).

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Figures

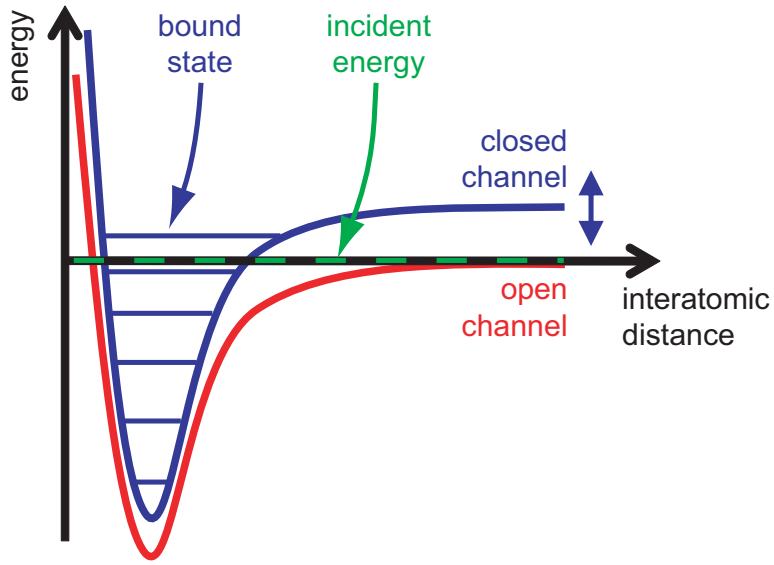

FIG. 1 Atoms prepared in the open channel, corresponding to the interaction potential (in red), undergo a collision at low incident energy. In the course of the collision the open channel is coupled to the closed channel (in blue). When a bound state of the closed channel has an energy close to zero, a scattering resonance occurs. The position of the closed channel can be tuned with respect to the open one, e.g., by varying the magnetic field B.

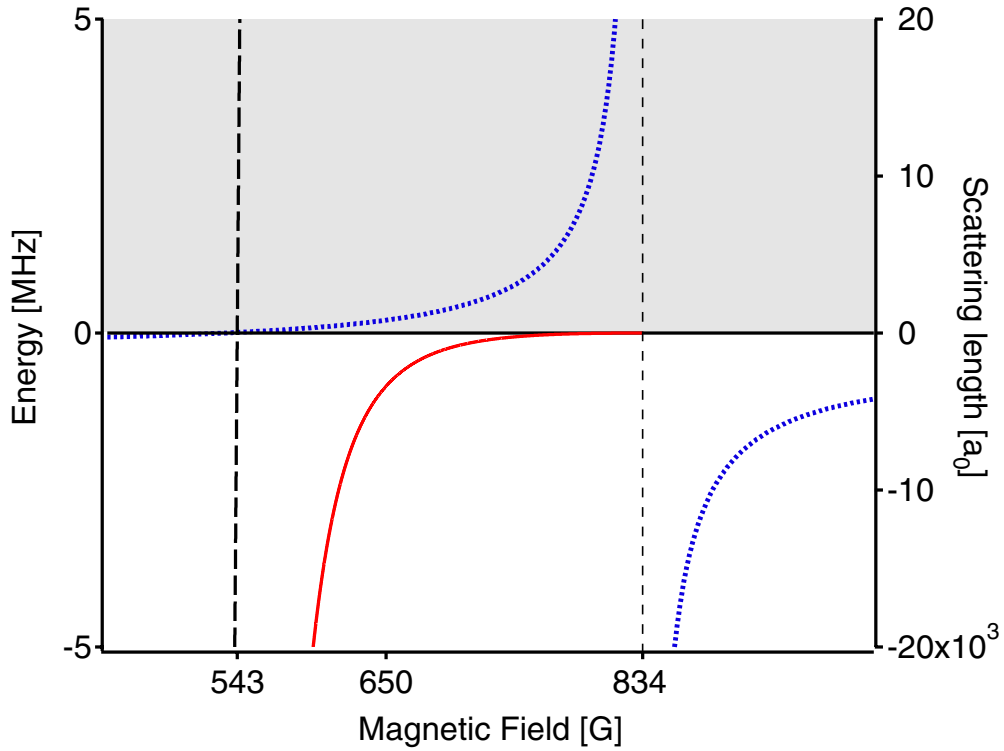

FIG. 2 Magnetic field dependence of the scattering length (blue curve) between the two lowest magnetic sub-states of ^{6}Li with a Feshbach resonance at $B_0 = 834\,\mathrm{G}$ and a zero crossing at $B_0 + \Delta B = 534\,\mathrm{G}$. The background scattering length $a_{\mathrm{bg}} = -1405\,a_0$ is exceptionally large in this case (where a_0 is the Bohr radius). The energy of the bound state causing the Feshbach resonance is shown in red.