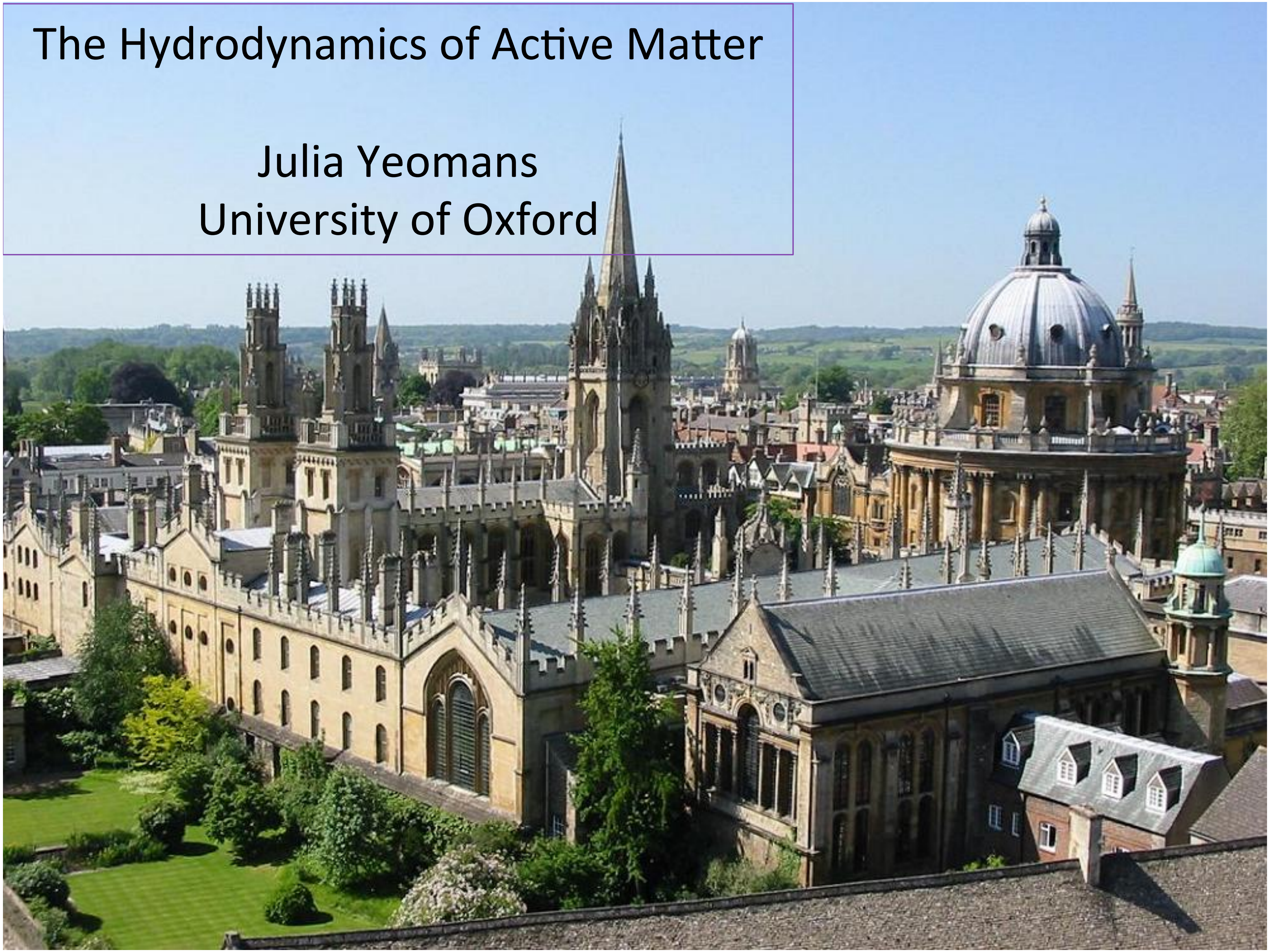
The Hydrodynamics of Active Matter

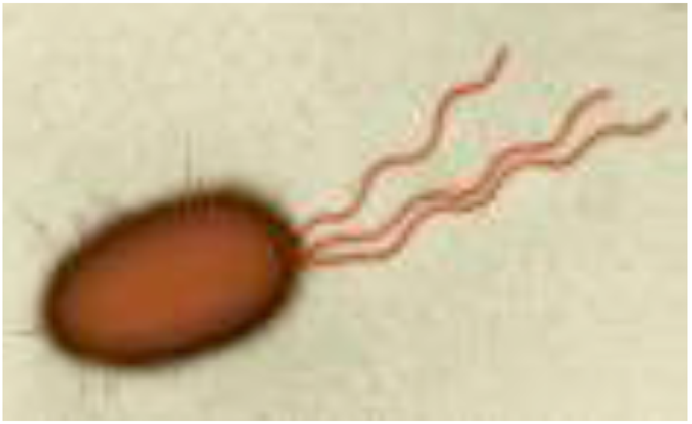
Julia Yeomans
University of Oxford

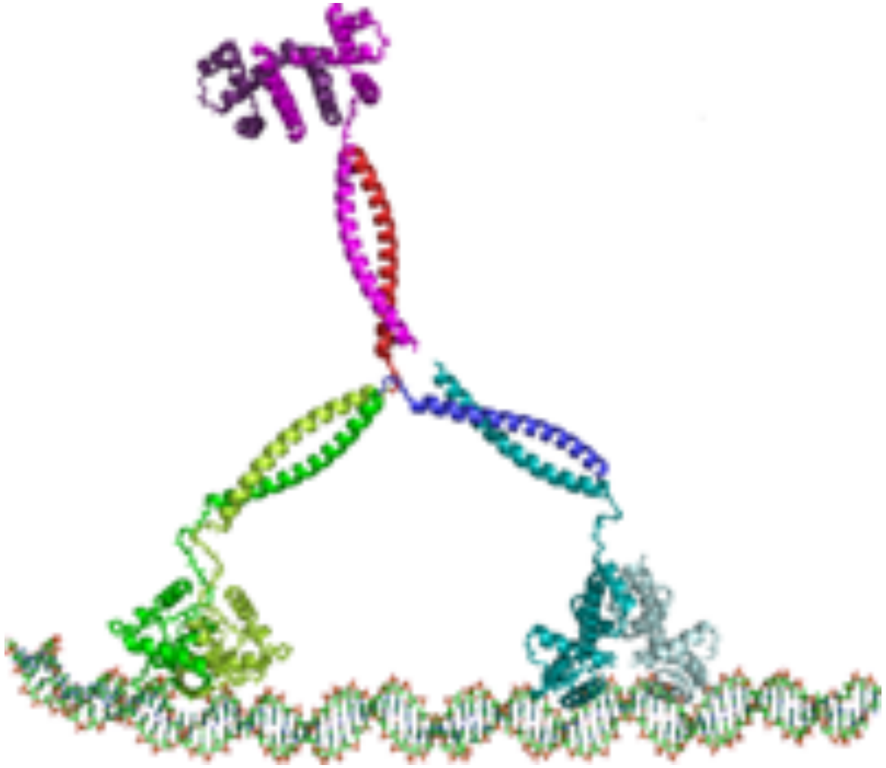


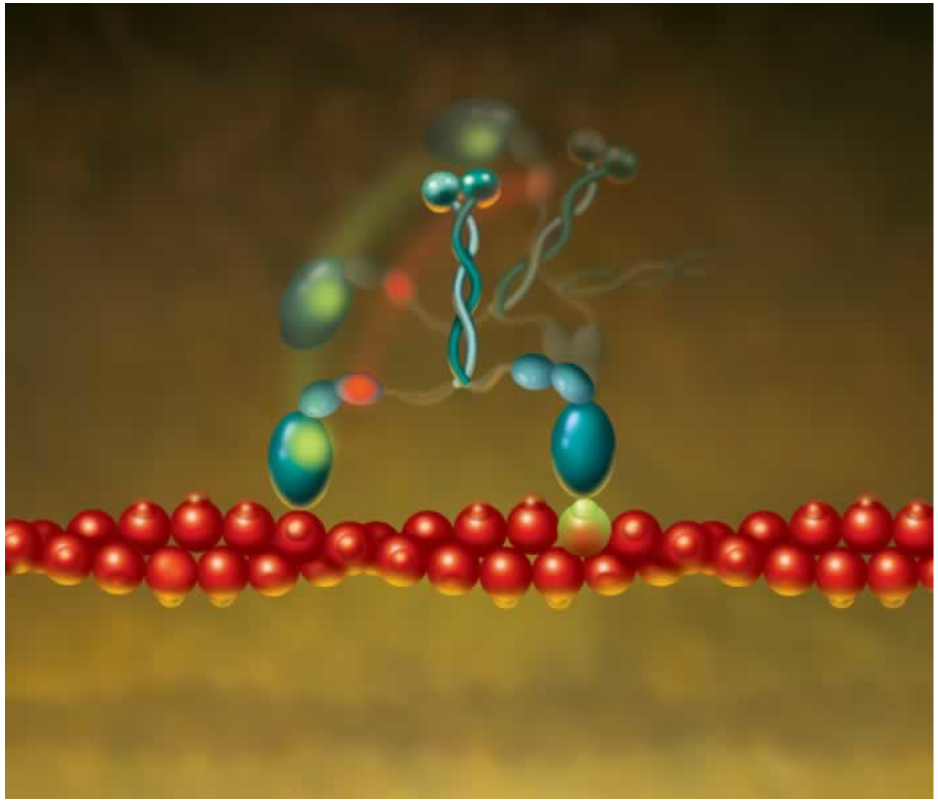
What is active matter?

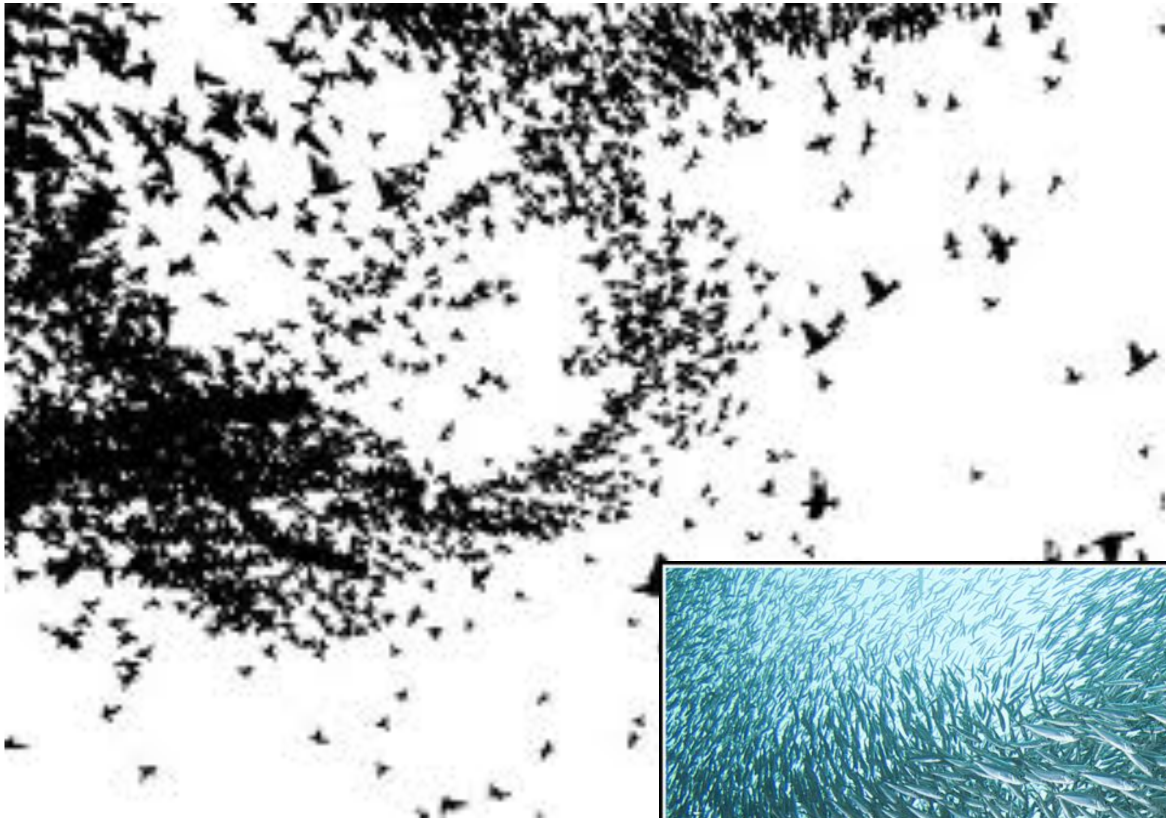
Why is it interesting?

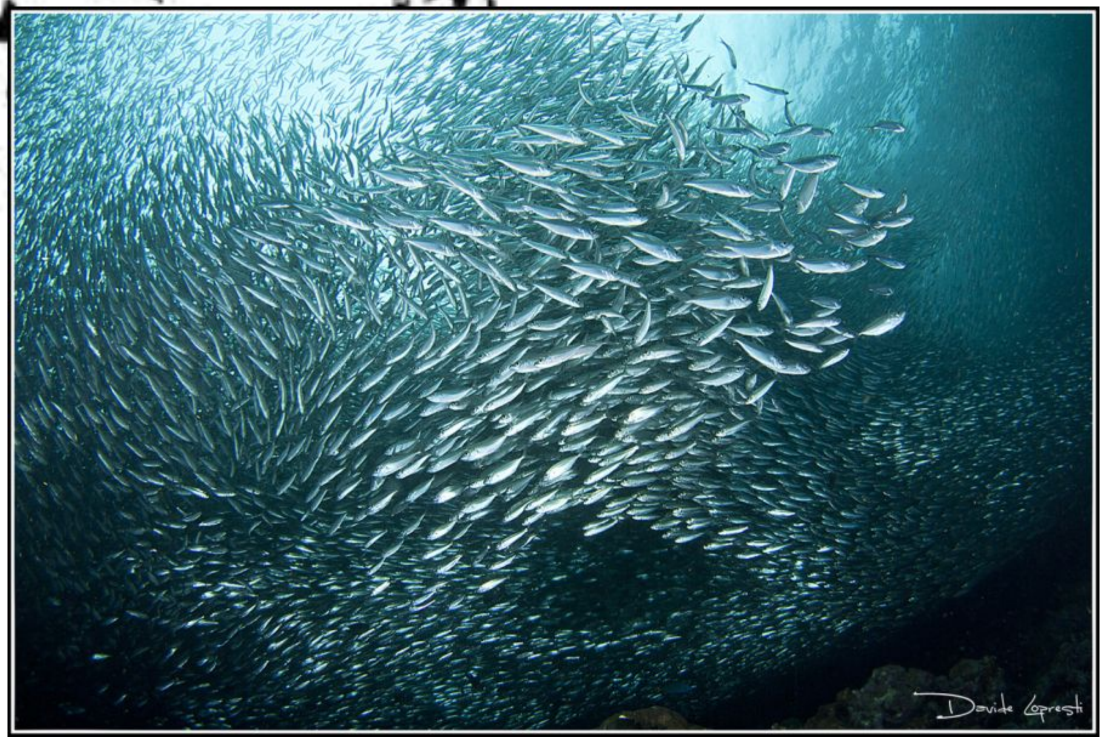
Why is it interesting now?

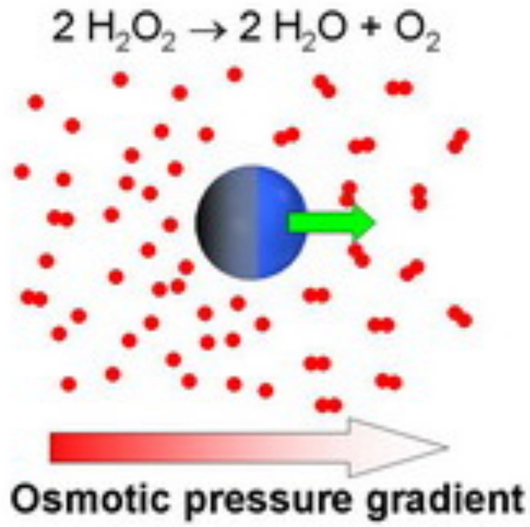


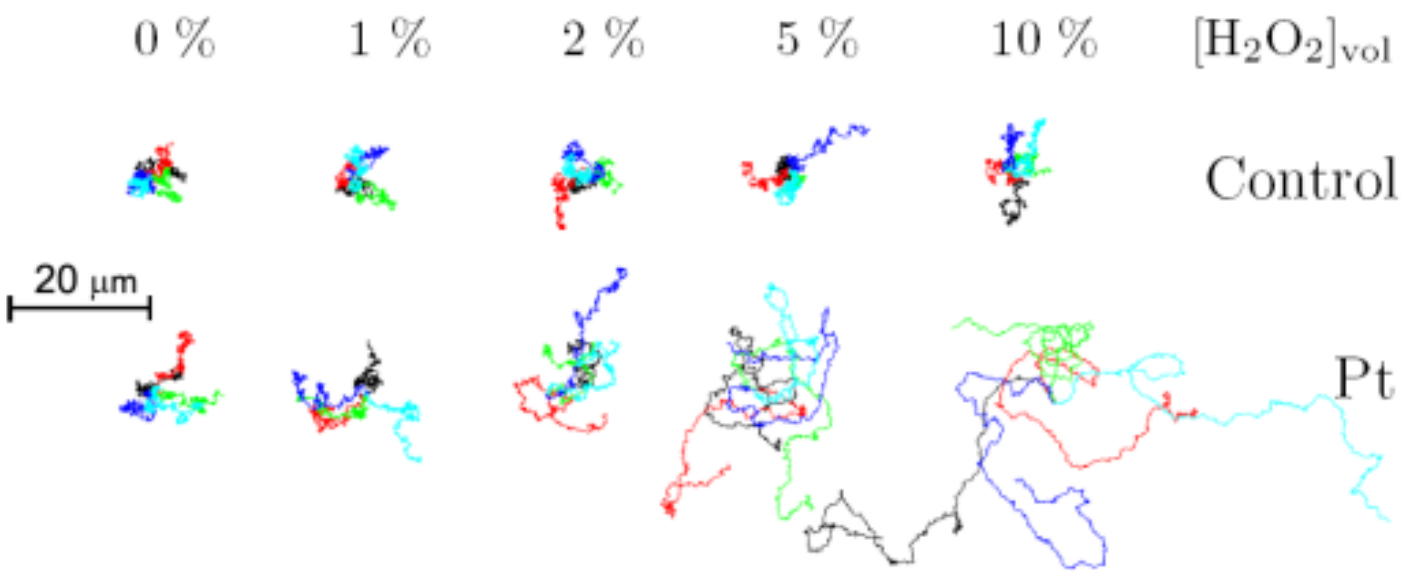







Davide Lopresti





Trajectories of catalytic microswimmers (Jon Howse, Univ of Sheffield*)*

Active particles convert chemical
energy to motion

Energy enters the system on a single particle level

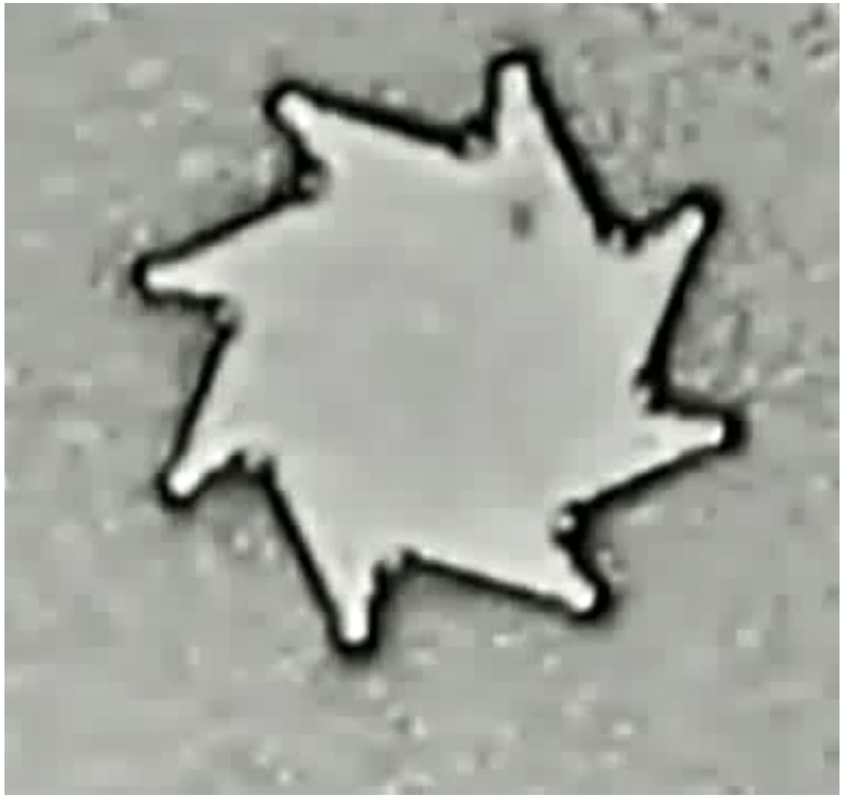
Meant to exist out of thermodynamic equilibrium

Dynamics of microswimmers + molecular motors

Collective behaviour – what is generic

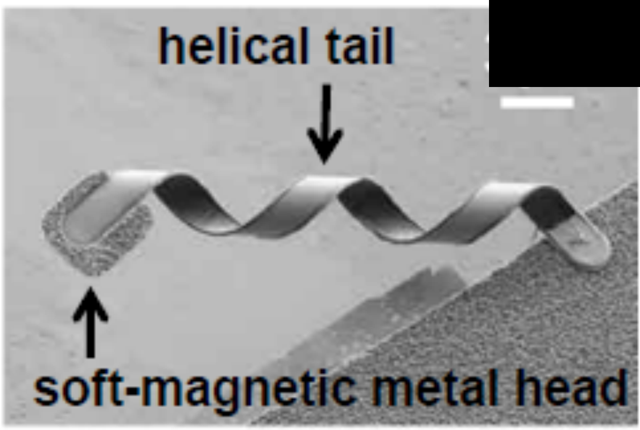
Testing ground for non-equilibrium statistical physics

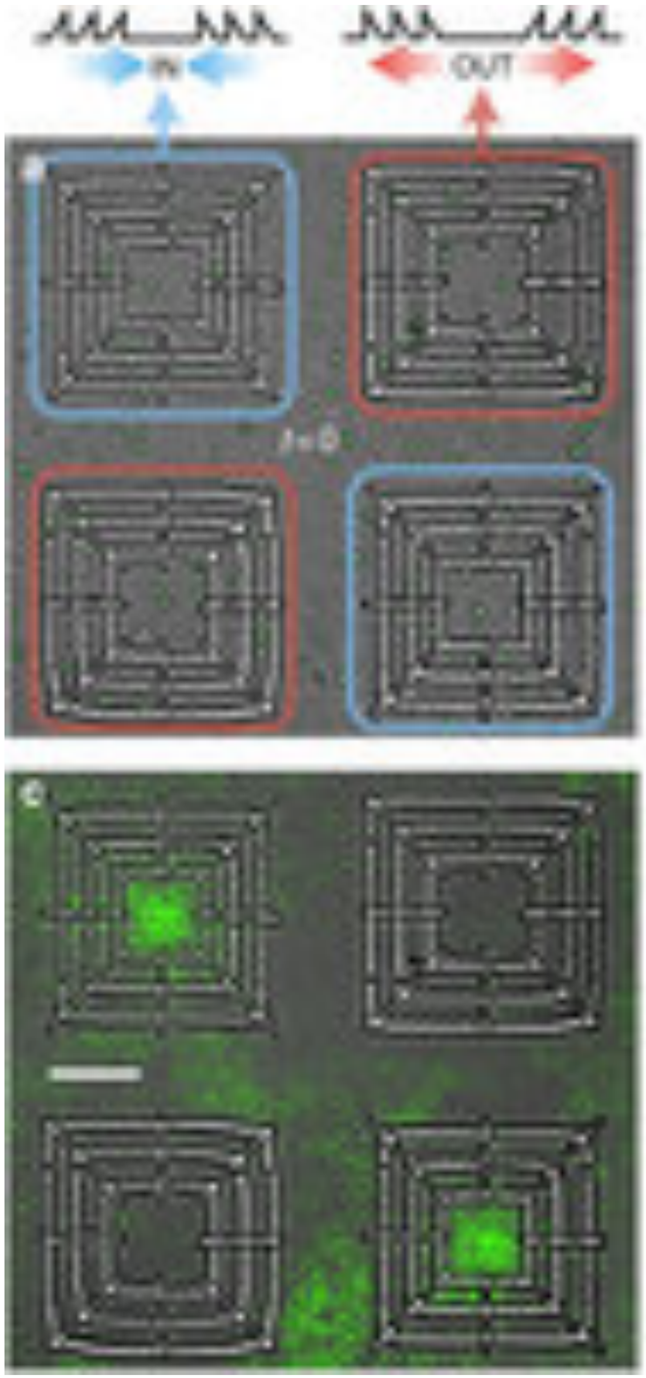
Applications as efficient machines



L. Angelani, R. Di Leonardo, Ruocco G.,
Phys. Rev. Lett., ***102***, *048104, (2009)*

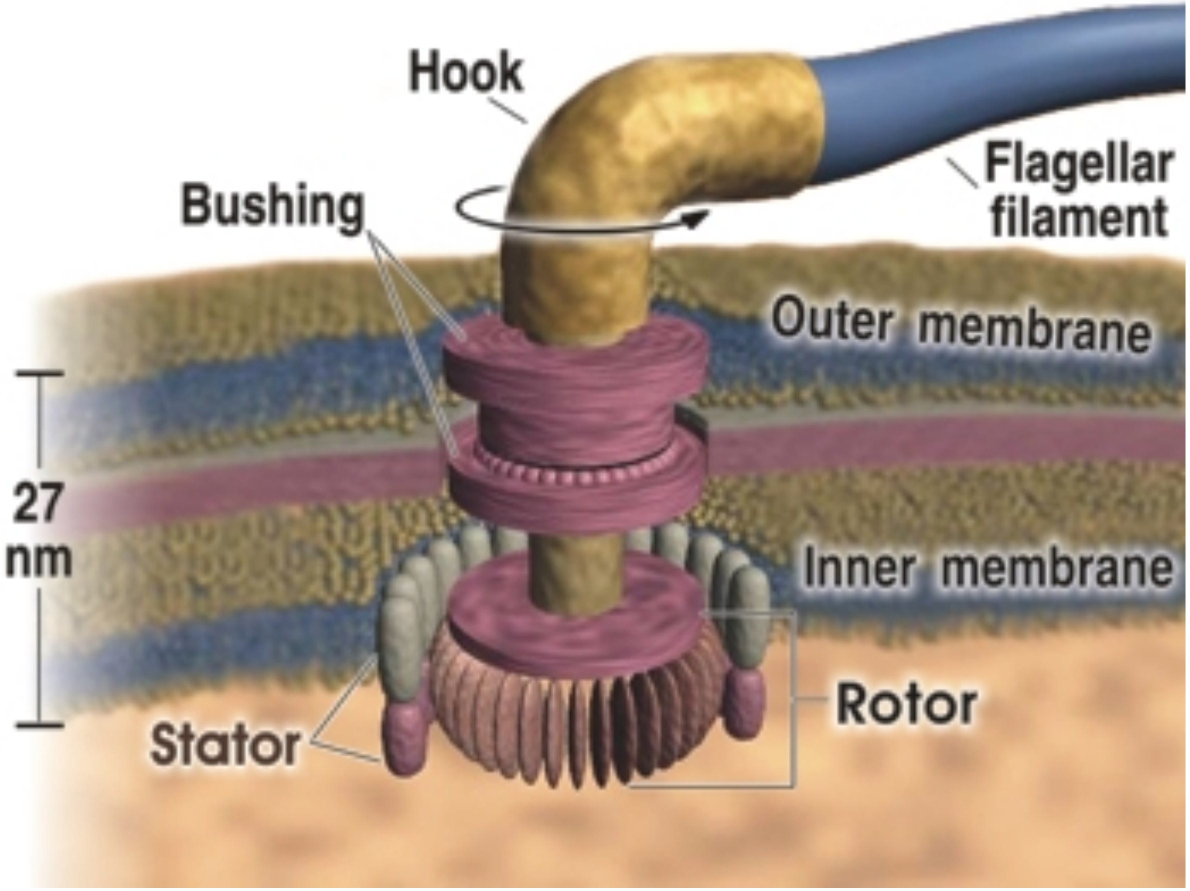

Video related to research article
appearing in *Lab on a Chip*
Bradley J Nelson et al. "Artificial Bacterial
Flagella for Micromanipulation"
Read the article at
http://xlink.rsc.org/?DOI=C004450B


helical tail
soft-magnetic metal head



active self assembly

Koumakis
Nature Comms. 2013



Bacterial flagellar motor

Lecture 1: The mathematics and physics of bacterial swimming

one active particle

Lecture 2: Applications

Lecture 3: Continuum models of dense active matter

lots of active particles

Lecture 4: Active turbulence and lyotropic active nematics

Lecture 1: The mathematics and physics of bacterial swimming

1. Low Re and the Stokes equations
2. The Scallop theorem
3. Dipolar flow fields
4. Multipole expansion for the Stokes equations

one active particle

Lecture 2: Applications

1. Swimming in Poiseuille flow

2. Stirring by microswimmers

Lecture 3: Continuum models of dense active matter

lots of active particles

Lecture 4: Active turbulence and lyotropic active nematics

$$\rho\left\{\frac{\partial \mathbf{v}}{\partial t}+(\mathbf{v}\cdot\nabla)\mathbf{v}\right\}=-\nabla p+\mu\nabla^2\mathbf{v}+\mathbf{f}$$

inertial terms

viscous terms

$$\tilde{v}=\frac{v}{V_0} \qquad \tilde{x}=\frac{x}{L_0}$$

$$\tilde{\nabla}=L_0\nabla \qquad \tilde{t}=\frac{V_0}{L_0}t \qquad \frac{\partial}{\partial\tilde{t}}=\frac{L_0}{V_0}\frac{\partial}{\partial t}$$

$$\frac{V_0^2}{L_0}\rho\left\{\frac{\partial\tilde{\mathbf{v}}}{\partial\tilde{t}}+(\tilde{\mathbf{v}}\cdot\tilde{\nabla})\tilde{\mathbf{v}}\right\}=-\nabla p+\frac{V_0}{L_0^2}\mu\tilde{\nabla}^2\tilde{\mathbf{v}}+\mathbf{f}$$

$$\left\{\frac{\partial\tilde{\mathbf{v}}}{\partial\tilde{t}}+(\tilde{\mathbf{v}}\cdot\tilde{\nabla})\tilde{\mathbf{v}}\right\}=-\frac{L_0}{V_0^2\rho}\nabla p+\frac{\mu}{L_0V_0\rho}\tilde{\nabla}^2\tilde{\mathbf{v}}+\frac{L_0}{V_0^2\rho}\mathbf{f}$$

$$\mathrm{Re}=\frac{\text{inertial response}}{\text{viscous response}}\sim\frac{\rho L_0V_0}{\mu}$$

$$\frac{V_0^2}{L_0}\rho\left\{\frac{\partial\tilde{\mathbf{v}}}{\partial\tilde{t}}+(\tilde{\mathbf{v}}\cdot\tilde{\nabla})\tilde{\mathbf{v}}\right\}=-\nabla p+\frac{V_0}{L_0^2}\mu\tilde{\nabla}^2\tilde{\mathbf{v}}+\mathbf{f}$$

$$\left\{\frac{\partial\tilde{\mathbf{v}}}{\partial\tilde{t}}+(\tilde{\mathbf{v}}\cdot\tilde{\nabla})\tilde{\mathbf{v}}\right\}=-\frac{L_0}{V_0^2\rho}\nabla p+\frac{\mu}{L_0V_0\rho}\tilde{\nabla}^2\tilde{\mathbf{v}}+\frac{L_0}{V_0^2\rho}\mathbf{f}$$

10^{-6}

10^{-6}

$$\mathrm{Re}=\frac{\text{inertial response}}{\text{viscous response}}\sim\frac{\rho L_0V_0}{\mu}$$

10^{6}

$$\frac{V_0^2}{L_0}\rho\left\{\frac{\partial\tilde{\mathbf{v}}}{\partial\tilde{t}}+(\tilde{\mathbf{v}}\cdot\tilde{\nabla})\tilde{\mathbf{v}}\right\}=-\nabla p+\frac{V_0}{L_0^2}\mu\tilde{\nabla}^2\tilde{\mathbf{v}}+\mathbf{f}$$

$$\left\{\frac{\partial\tilde{\mathbf{v}}}{\partial\tilde{t}}+(\tilde{\mathbf{v}}\cdot\tilde{\nabla})\tilde{\mathbf{v}}\right\}=-\frac{L_0}{V_0^2\rho}\nabla p+\frac{\mu}{L_0V_0\rho}\tilde{\nabla}^2\tilde{\mathbf{v}}+\frac{L_0}{V_0^2\rho}\mathbf{f}$$

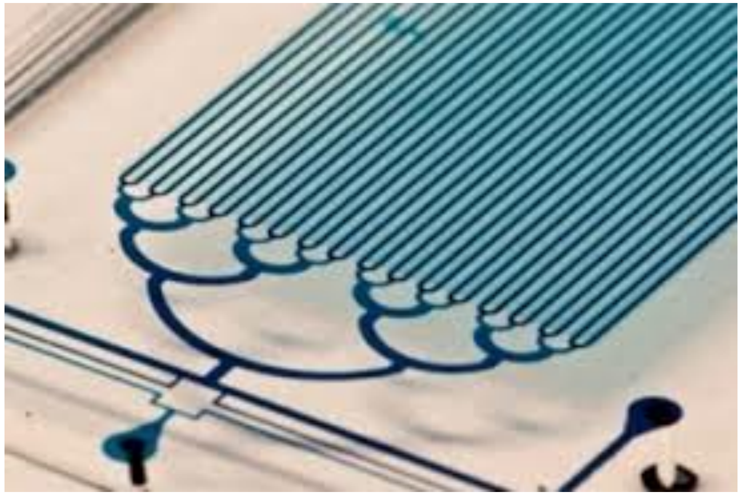
10^{-6}

10^{-6}

$$\mathrm{Re}=\frac{\text{inertial response}}{\text{viscous response}}\sim\frac{\rho L_0V_0}{\mu}$$

10^{6}

$$\boxed{\mathrm{Re}\sim 0}$$



Low Re

High Re

Stokes equations

$$\nabla p = \mu \nabla^2 \mathbf{v} + \mathbf{f} \qquad \nabla \cdot \mathbf{v} = 0$$

Why did the chicken cross the Mobius strip?

To get to the same side.

Purcell's Scallop Theorem

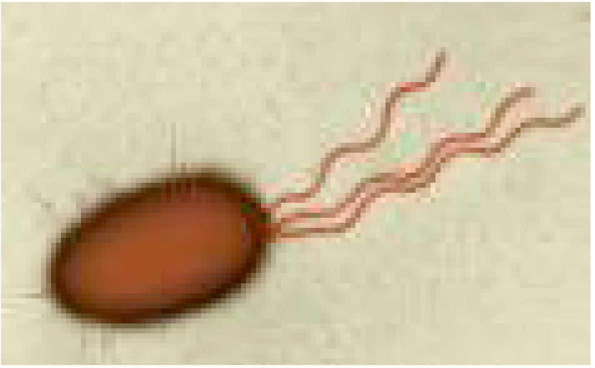
$$\nabla p = \mu \nabla^2 \mathbf{v} + \mathbf{f} \qquad\qquad \nabla \cdot \mathbf{v} = 0$$

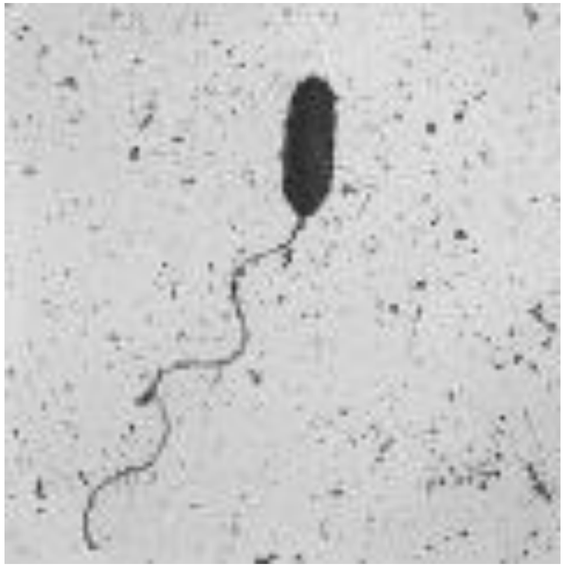
No time dependence implies the Scallop Theorem

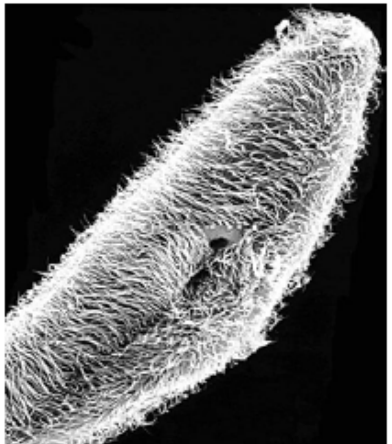
A swimming stroke must be non- invariant under time reversal

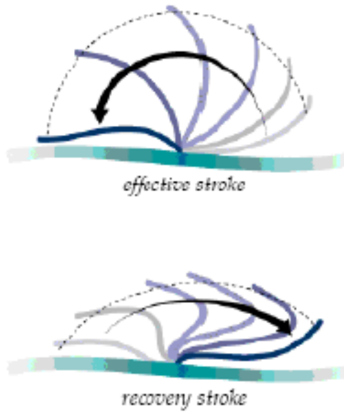
Purcell's Scallop Theorem

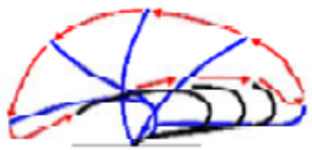
A swimmer strokes must be non-invariant under time reversal

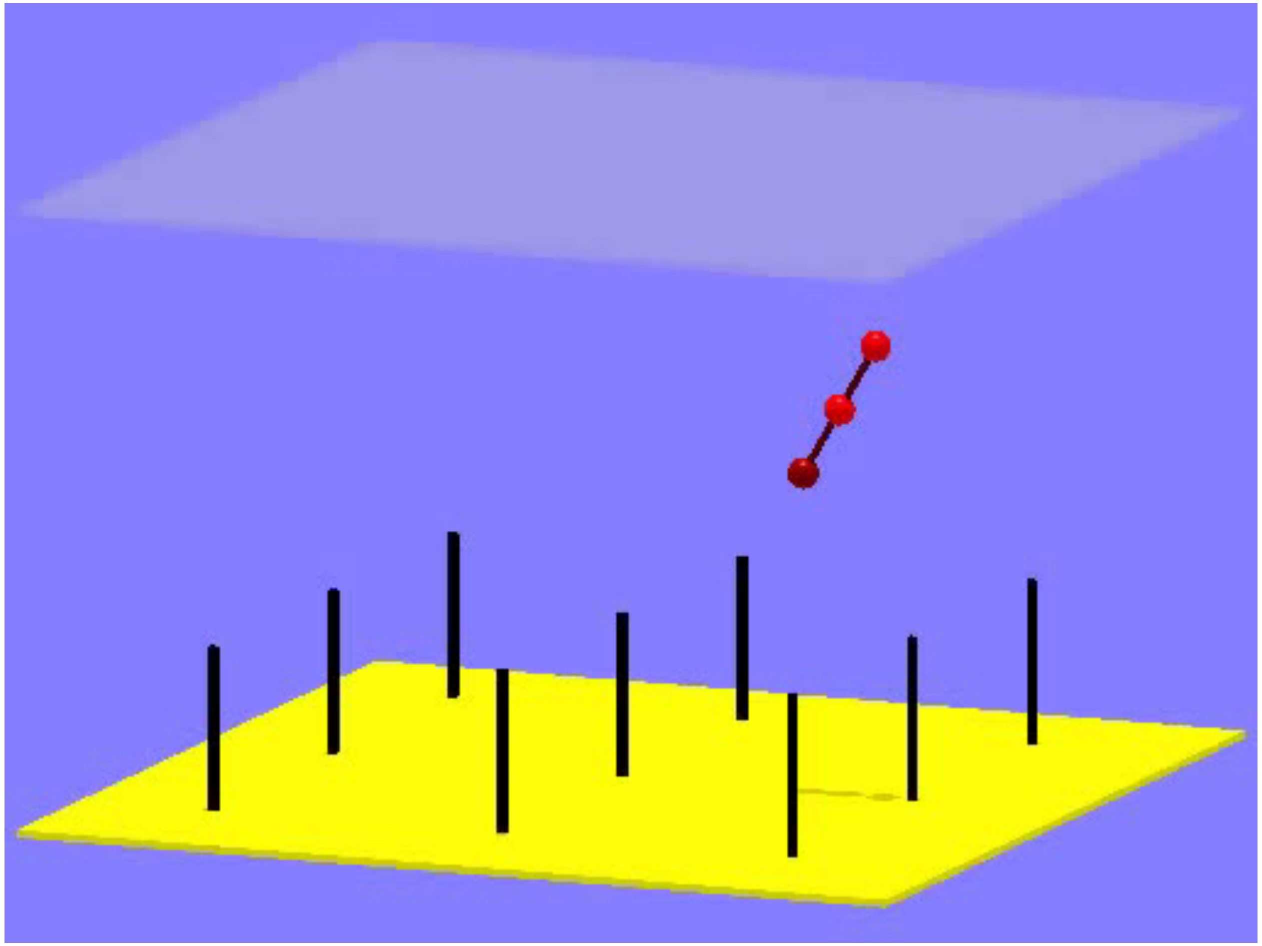


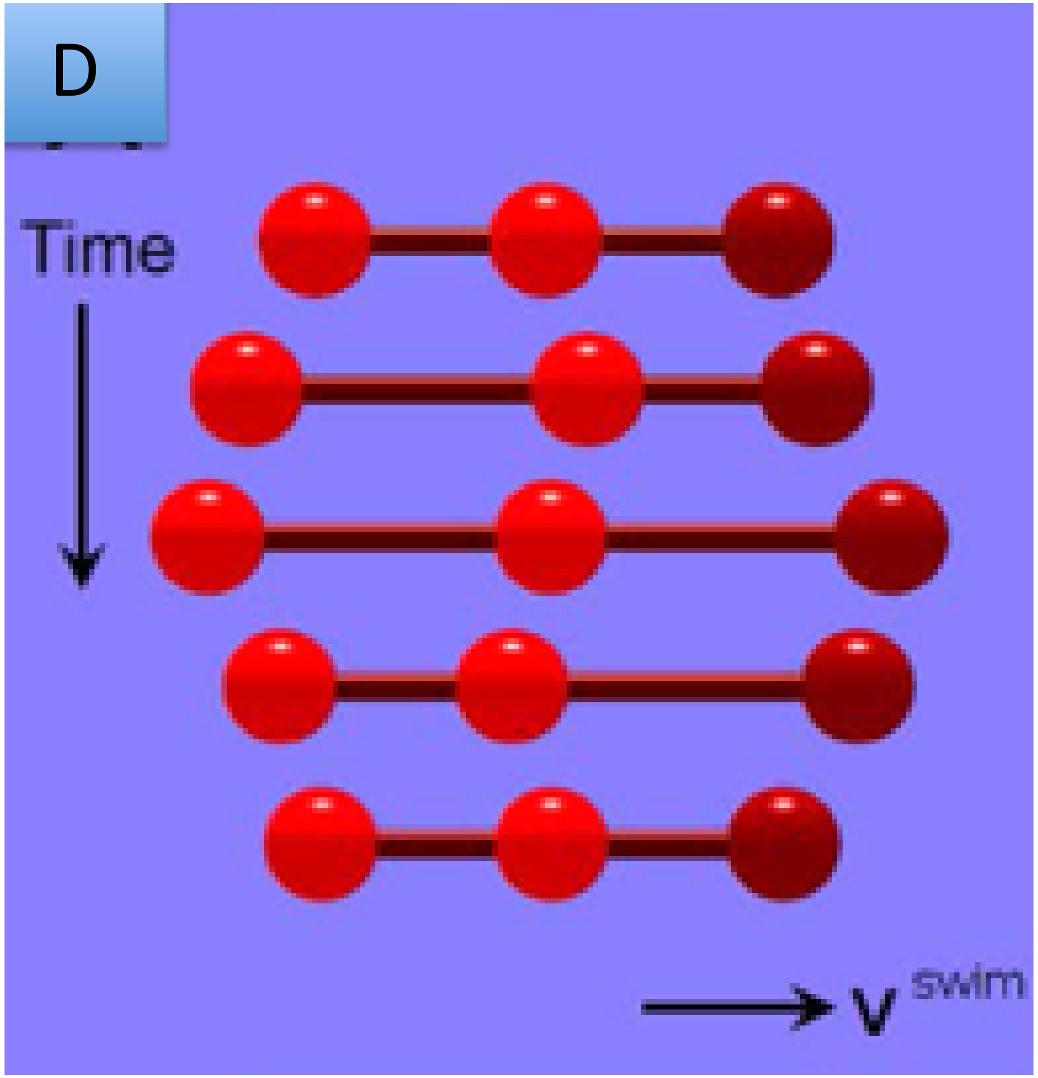






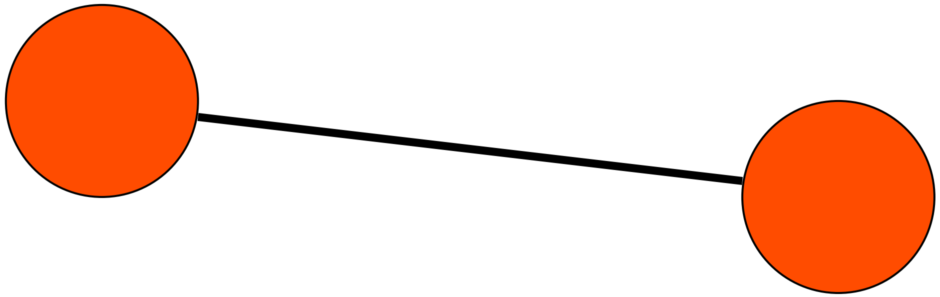


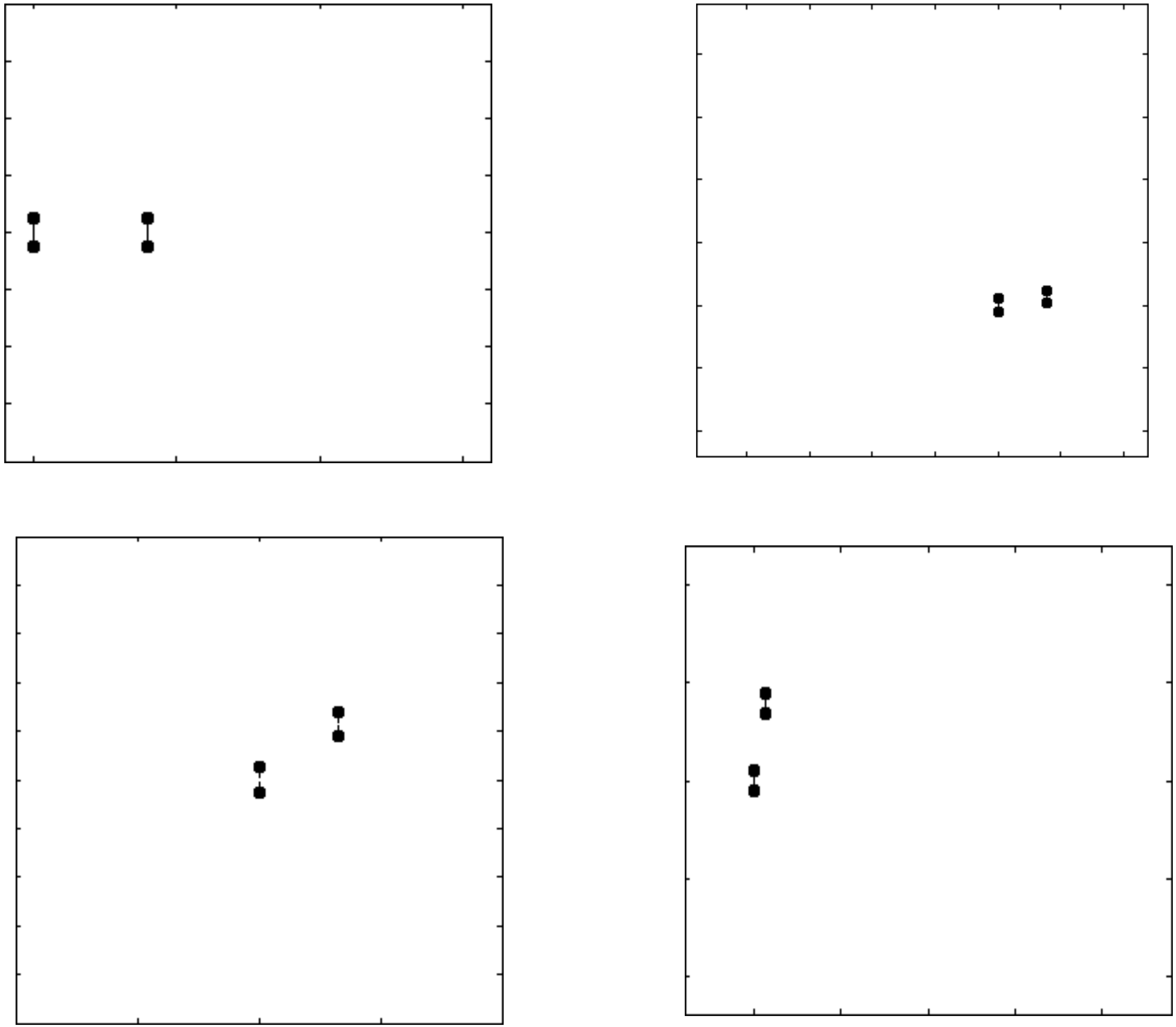




Najafi, Golestanian

Dumbbell swimmers





Lecture 1: The mathematics and physics of bacterial swimming

1. Low Re and the Stokes equations

2. The Scallop theorem

When can a creature swim at low Re?

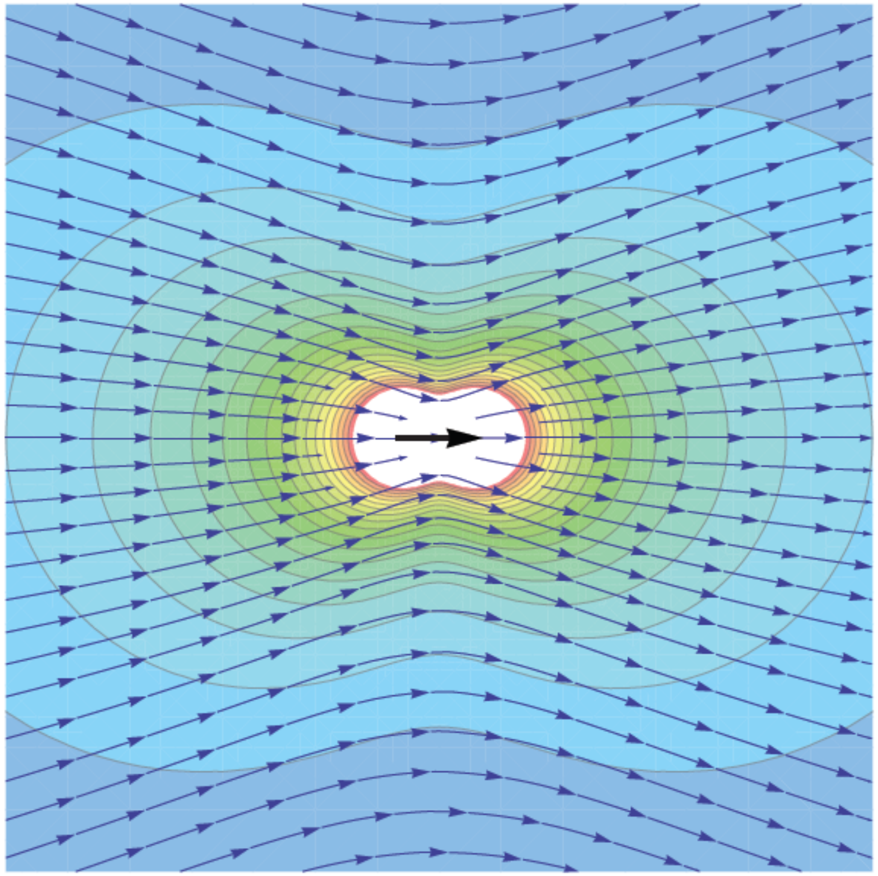
Swimming stroke must be different forwards and backwards in time

3. Dipolar flow fields

What does its flow field look like?

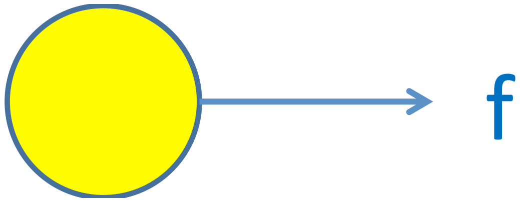
Green function of the Stokes equation (Stokeslet)

$$\nabla p = \mu \nabla^2 \mathbf{v} + \mathbf{f} \qquad \nabla \cdot \mathbf{v} = 0$$



$$\mathbf{v}(\mathbf{r}) = \frac{\mathbf{f}}{8\pi\mu} \cdot \left(\frac{\mathbf{I}}{r} + \frac{\mathbf{rr}}{r^3} \right)$$

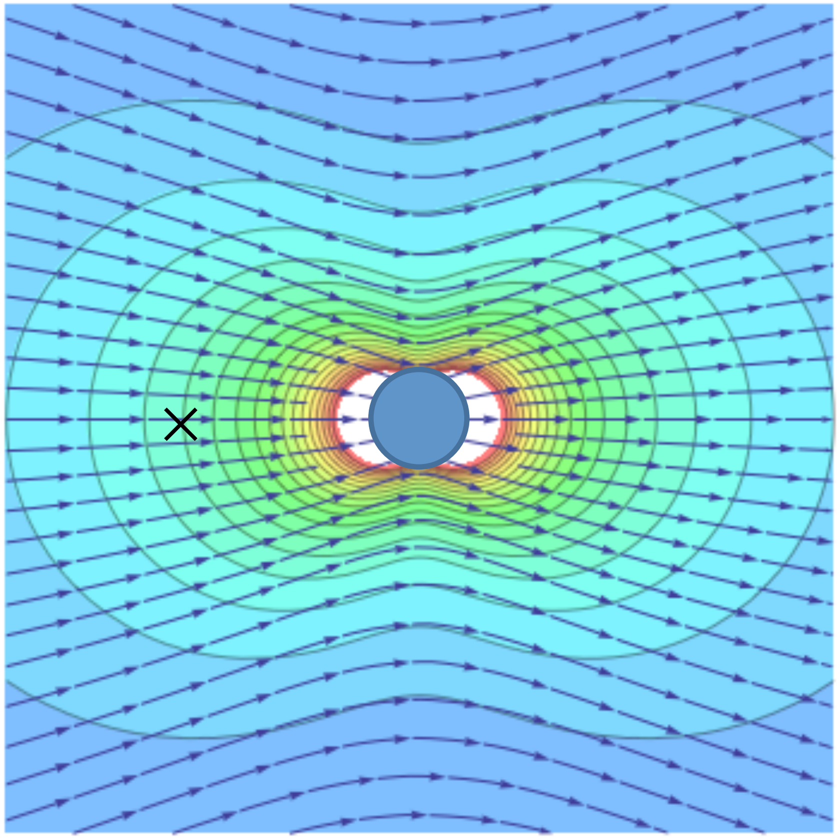
$$v_i(\mathbf{r}) = \frac{f_j}{8\pi\mu} \left(\frac{\delta_{ij}}{r} + \frac{r_i r_j}{r^3} \right) \equiv G_{ij}(\mathbf{r}) f_j$$

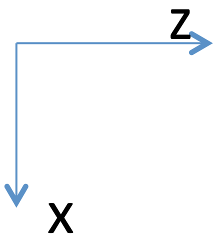


For derivation see Maciej Lisicki
http://www.fuw.edu.pl/~mklis/publications/Hydro/oseen.pdf

Green function of the Stokes equation

$$v_i(\mathbf{r}) = \frac{f_j}{8\pi\mu}\left(\frac{\delta_{ij}}{r} + \frac{r_i r_j}{r^3}\right) \equiv G_{ij}(\mathbf{r}) f_j$$





For a force in the z-direction:

$$v_x = \frac{f}{8\pi\mu}\left(\frac{zx}{r^3}\right)$$

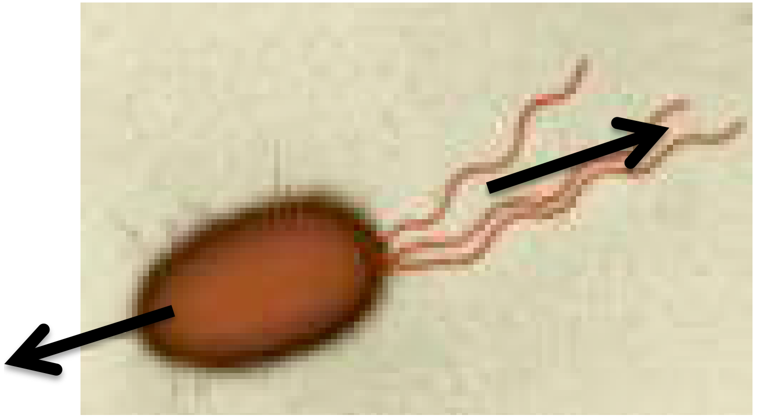
$$v_y = \frac{f}{8\pi\mu}\left(\frac{zy}{r^3}\right)$$

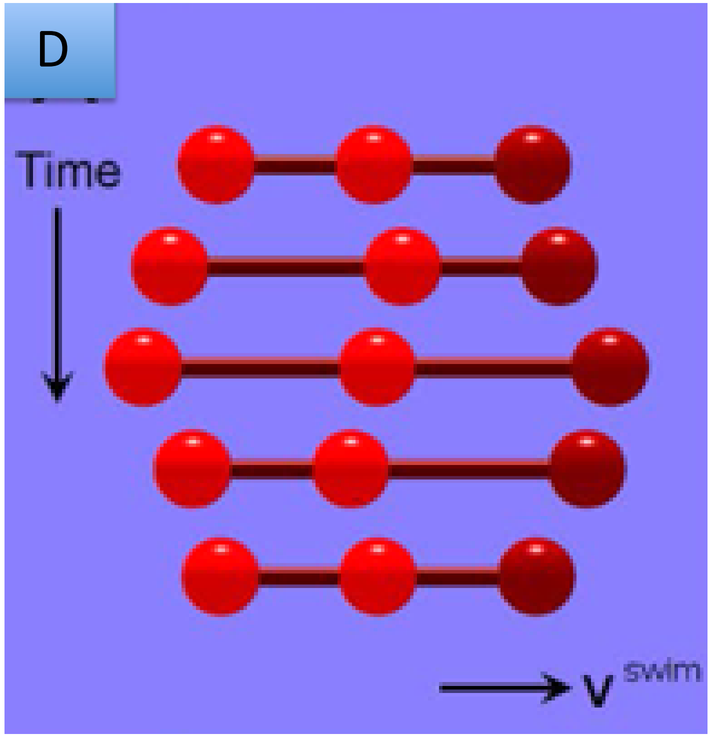
$$\boxed{v_z = \frac{f}{8\pi\mu}\left(\frac{1}{r} + \frac{z^2}{r^3}\right)}$$

Things expand when heated: that’s why days are longer in the summer

Swimmers have no external forces or torques acting on them.

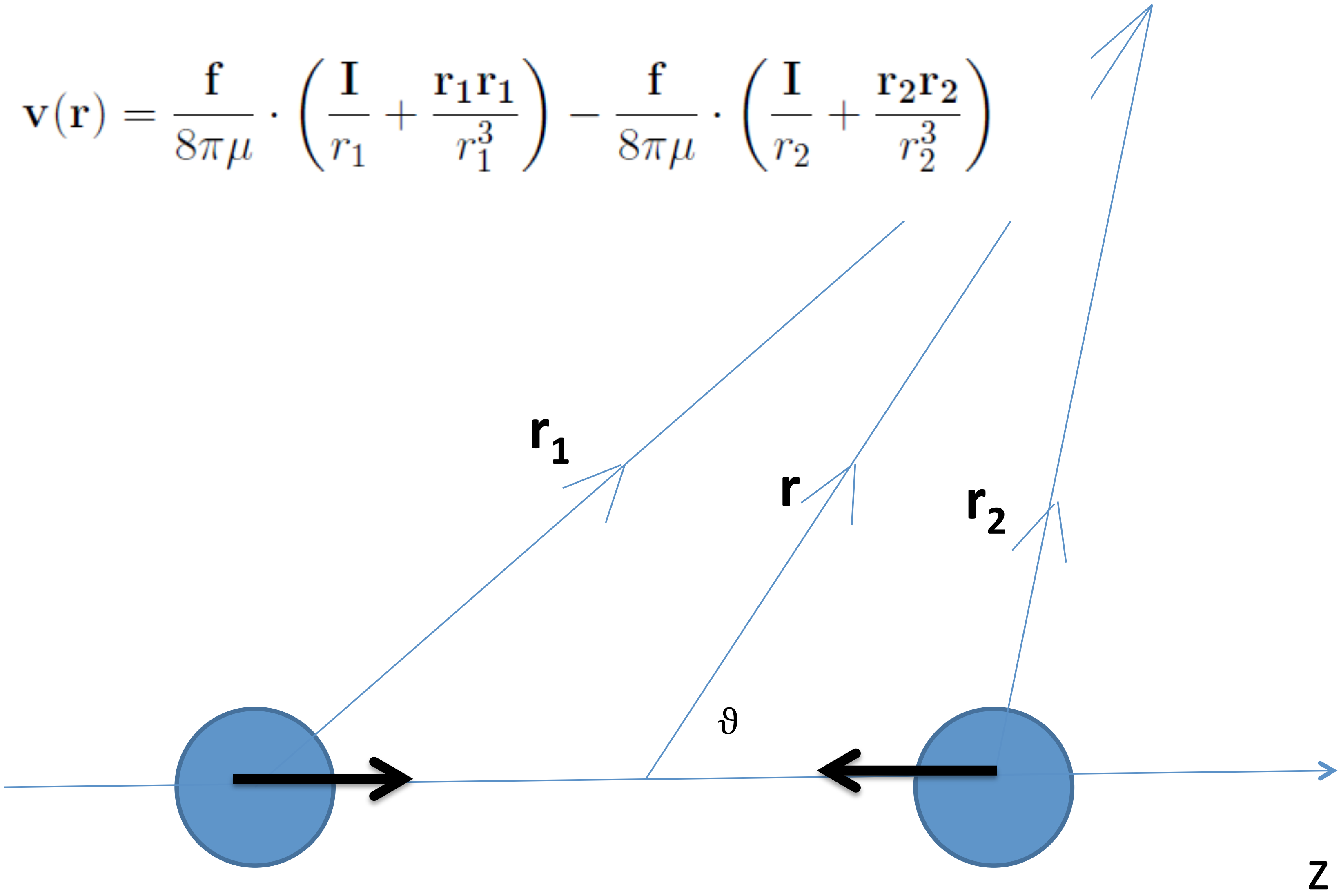
So all driving forces must act in equal and opposite pairs.





Far flow field of a swimmer

$$\mathbf{v}(\mathbf{r}) = \frac{\mathbf{f}}{8\pi\mu} \cdot \left(\frac{\mathbf{I}}{r_1} + \frac{\mathbf{r}_1\mathbf{r}_1}{r_1^3} \right) - \frac{\mathbf{f}}{8\pi\mu} \cdot \left(\frac{\mathbf{I}}{r_2} + \frac{\mathbf{r}_2\mathbf{r}_2}{r_2^3} \right)$$

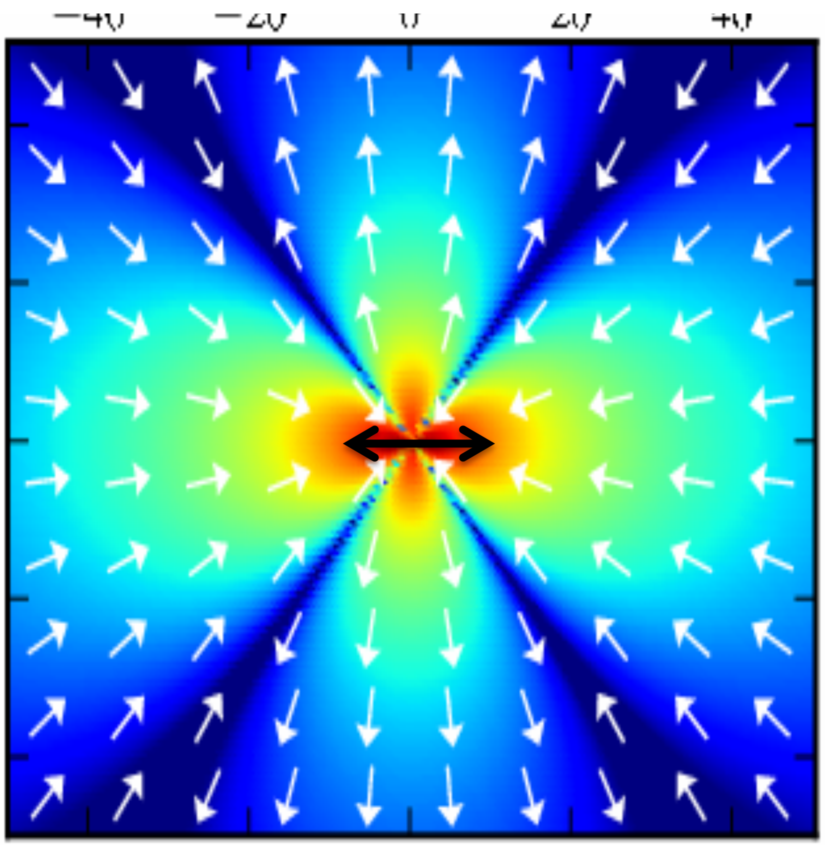


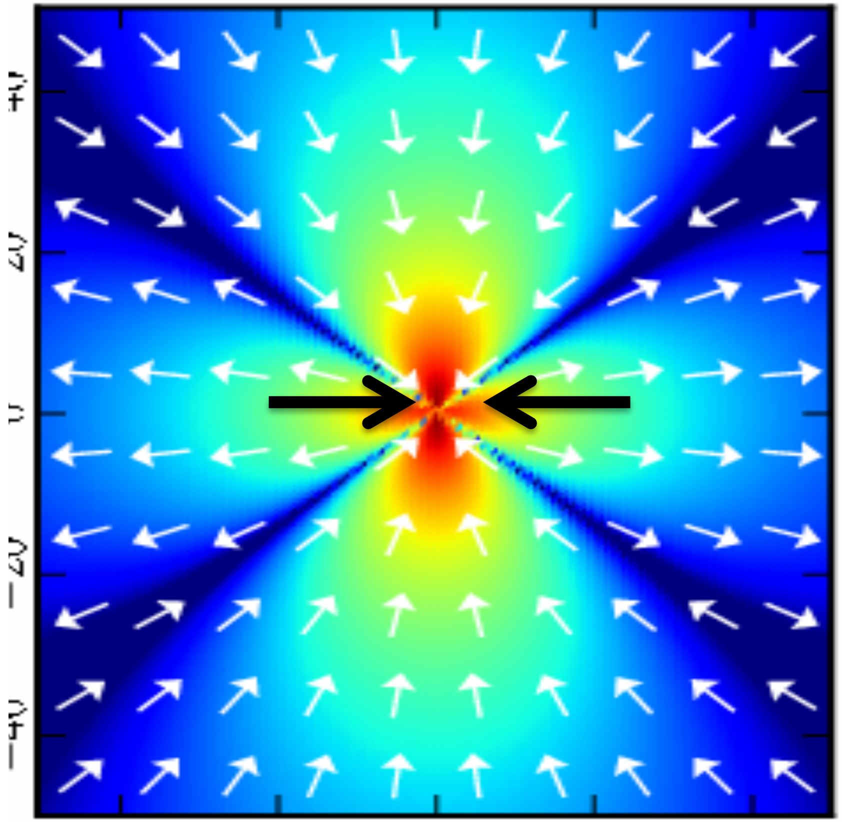
Far flow field of a swimmer

$$v_r = \frac{f}{4\pi\mu}\,\frac{L}{r^2}\,\left(3\cos^2\theta - 1\right)$$

Swimmers have dipolar far flow fields because they have no net force acting on them

Dipolar flow field



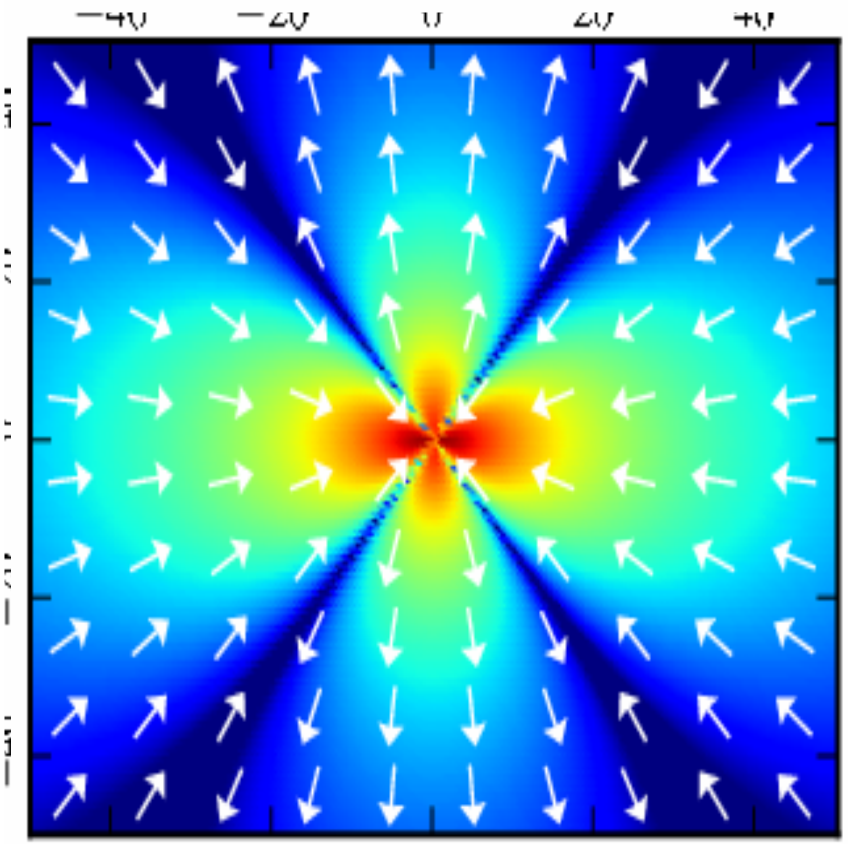


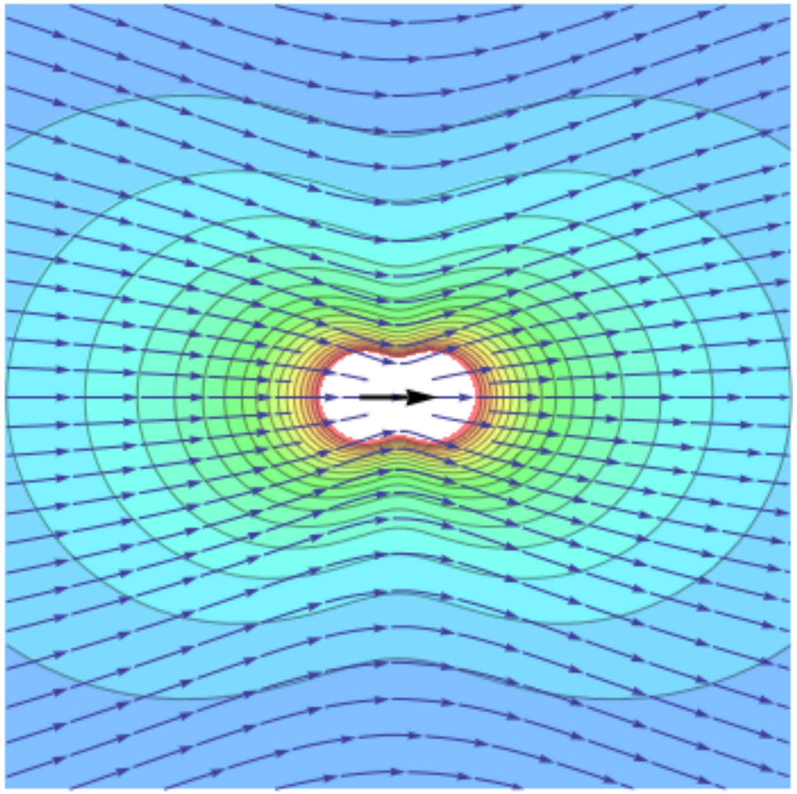
puller (contractile)

pusher (extensile)

$$v_r = \frac{f}{4\pi\mu} \; \frac{L}{r^2} \; (3\cos^2\theta - 1)$$

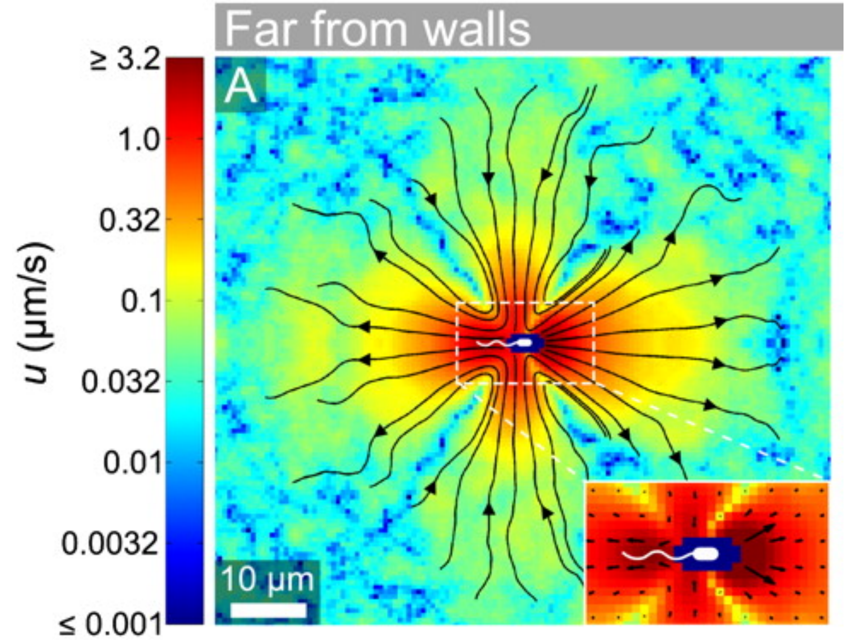
Swimmer and colloidal flow fields



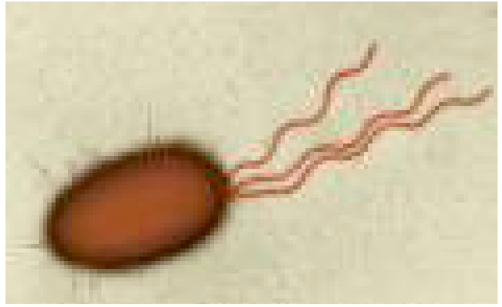


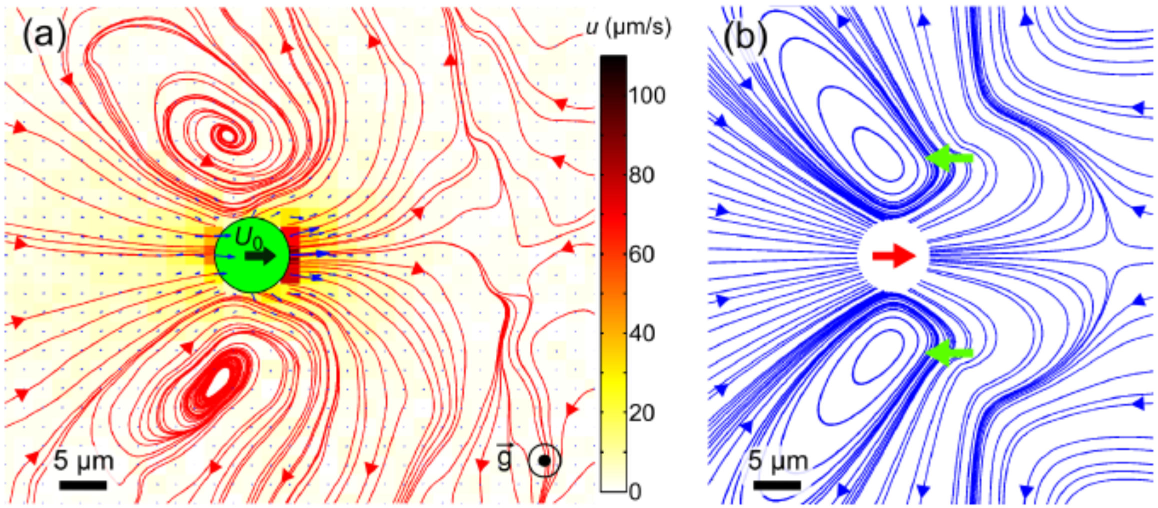
$$v \sim \frac{1}{r^2}$$

$$v \sim \frac{1}{r}$$



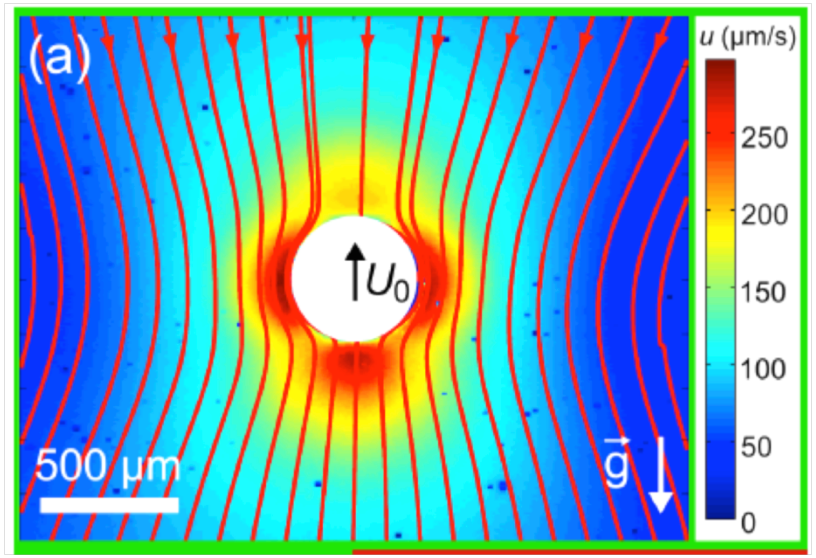
E-coli



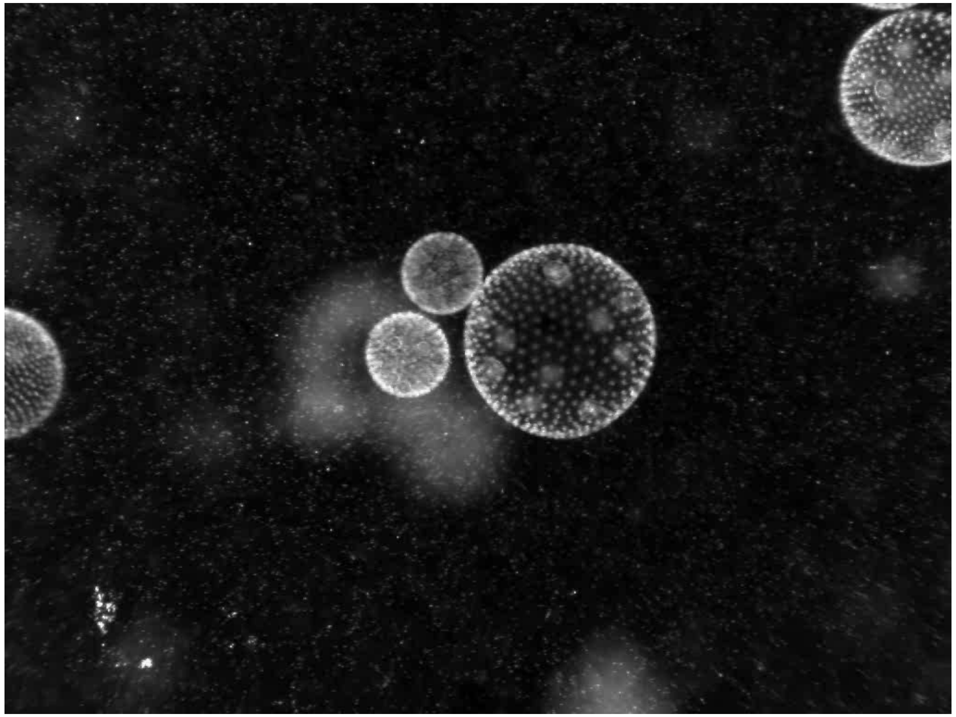


Chlamydomonas

Results from Goldstein group, University of Cambridge



Volvox



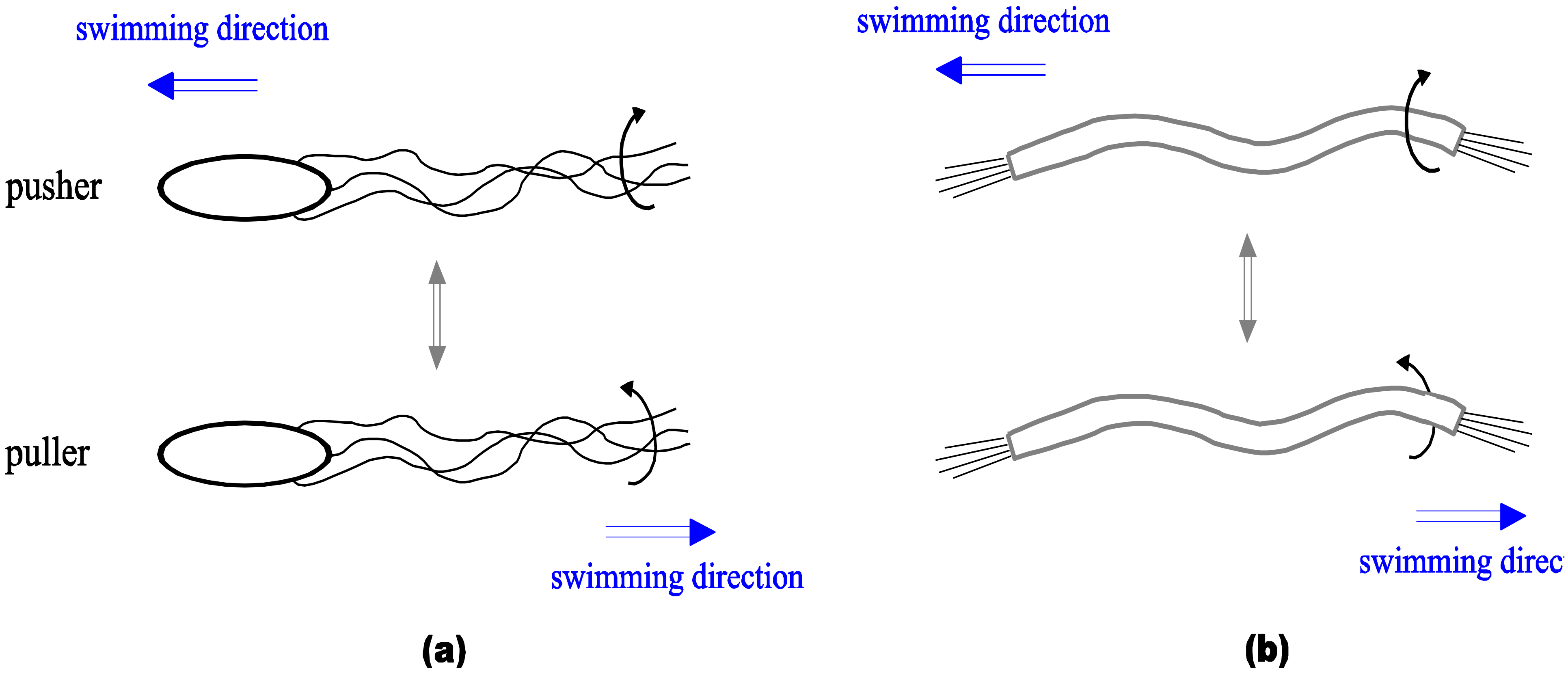
Dresher et al, PRL 105 (2010)
PNAS 108 (2011)

Quadrupolar swimmers

three sphere swimmer

Najafi and Golestanian,

quadrupolar symmetry

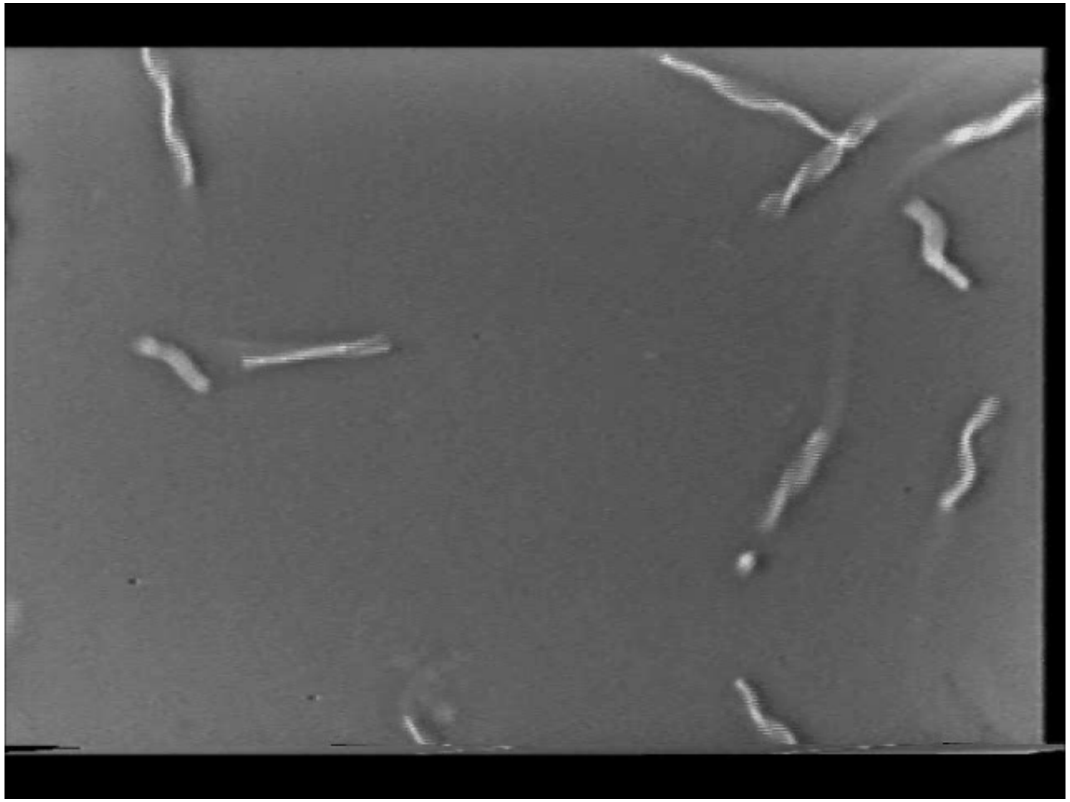


dipolar flow field

quadrupolar flow field

Spirillum volutans

Helical cell body with a short flagellar bundle at each end



From the Howard Berg Lab

http://www.rowland.harvard.edu/labs/bacteria/index_movies.html

Lecture 1: The mathematics and physics of bacterial swimming

1. Low Re and the Stokes equations
2. The Scallop theorem
3. Dipolar flow fields
4. Multipole expansion for the Stokes equations

one active particle

Lecture 2: Applications

1. Swimming in Poiseuille flow

2. Stirring by microswimmers

Lecture 3: Continuum models of dense active matter

lots of active particles

Lecture 4: Active turbulence and lyotropic active nematics

Lecture 1: The mathematics and physics of bacterial swimming

1. Low Re and the Stokes equations

2. The Scallop theorem

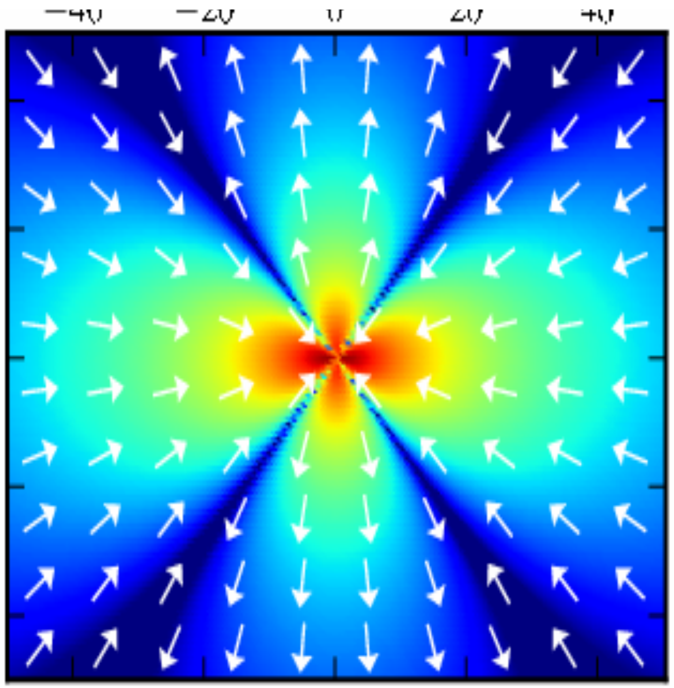
When can a creature swim at low Re?

Swimming stroke must be different forwards and backwards in time

3. Dipolar flow fields

What do the flow fields look like?

The far flow fields are generically dipolar



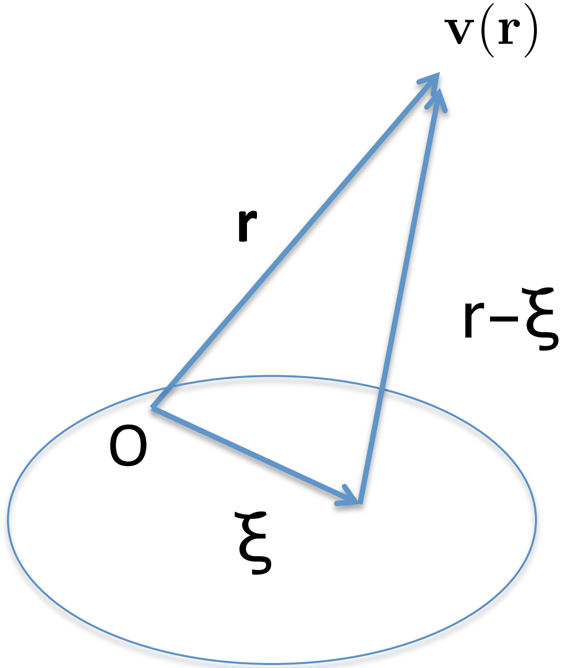
Index notation, etc

$$f_j$$

$$f_j r_j = f_x x + f_y y + f_z z = \mathbf{f} \cdot \mathbf{r}$$

$$G(\mathbf{r} - \xi) = G(\mathbf{r}) - \frac{\partial G}{\partial \xi_k}(\mathbf{r})\xi_k + \frac{1}{2}\frac{\partial^2 G}{\partial \xi_k \partial \xi_l}(\mathbf{r})\xi_k \xi_l \ldots$$

$$v_i(\mathbf{r}) = \frac{f_j}{8\pi\mu}\left(\frac{\delta_{ij}}{r} + \frac{r_i r_j}{r^3}\right) \equiv G_{ij}(\mathbf{r}) f_j$$



$$G_{ij}(\mathbf{r}) = \frac{1}{8\pi\mu}\left(\frac{\delta_{ij}}{r} + \frac{r_i r_j}{r^3}\right)$$

$$G_{ij}(\mathbf{r}-\xi) = \frac{1}{8\pi\mu}\left(\frac{\delta_{ij}}{|\ \mathbf{r}-\xi\ |} + \frac{(\mathbf{r}-\xi)_i(\mathbf{r}-\xi)_j}{|\ \mathbf{r}-\xi\ |^3}\right)$$

$$v_i(\mathbf{r}) = \int G_{ij}(\mathbf{r}-\xi) f_j(\xi)\ d\xi$$

$$v_i(\mathbf{r}) = \int G_{ij}(\mathbf{r}-\xi) f_j(\xi)\, d\xi$$

$$\begin{aligned} v_i(\mathbf{r}) &= \int \left\{ G_{ij}(\mathbf{r}) - \frac{\partial G_{ij}}{\partial \xi_k}(\mathbf{r})\xi_k + \frac{1}{2}\frac{\partial^2 G_{ij}}{\partial \xi_k \partial \xi_l}(\mathbf{r})\xi_k \xi_l \ldots \right\} f_j(\xi)\, d\xi \\ &= G_{ij}(\mathbf{r}) \int f_j(\xi)\, d\xi - \frac{\partial G_{ij}(\mathbf{r})}{\partial \xi_k} \int \xi_k f_j(\xi)\, d\xi + \ldots \end{aligned}$$

$$\boxed{\equiv \; G_{ij}(\mathbf{r}) F_j - \frac{\partial G_{ij}(\mathbf{r})}{\partial \xi_k} D_{jk} + \ldots}$$

$$G_{ij}(\mathbf{r}-\xi) = \frac{1}{8\pi\mu}\left(\frac{\delta_{ij}}{|\,\mathbf{r}-\xi\,|} + \frac{(\mathbf{r}-\xi)_i(\mathbf{r}-\xi)_j}{|\,\mathbf{r}-\xi\,|^3} \right)$$

Heisenberg might have lived here

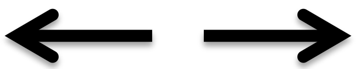
$$D_{jk} = \int \xi_k f_j \, d\xi.$$

stresslet

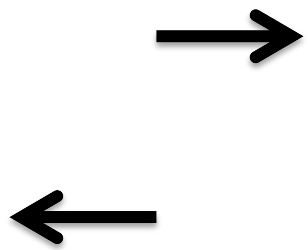
rotlet

$$D_{jk} - \frac{1}{3} D_{ii} \delta_{jk} = S_{jk} + T_{jk}$$

$$S_{jk} = \frac{1}{2} \int (\xi_k f_j + \xi_j f_k) \, d\xi - \frac{1}{3} \int \xi_i f_i \, \delta_{jk} \, d\xi$$

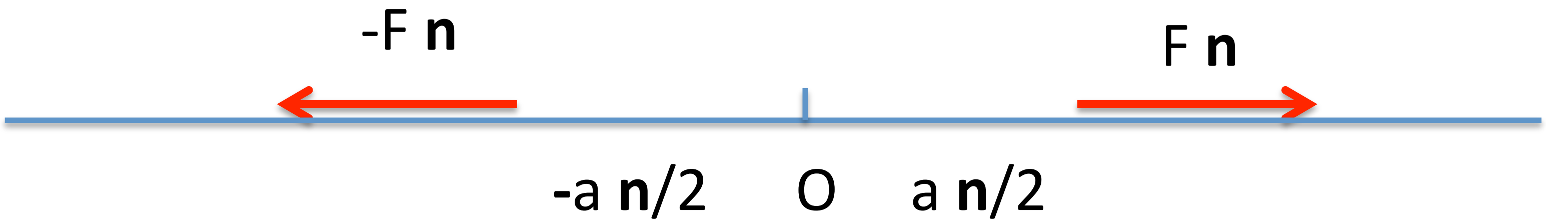


stresslet



rotlet

$$S_{jk} = \frac{1}{2}\int (\xi_k f_j + \xi_j f_k)\, d\xi - \frac{1}{3}\int \xi_i f_i\, \delta_{jk}\, d\xi$$



$$S_{jk} = \frac{1}{2}Fa(n_k n_j + n_j n_k) - \frac{1}{3}Fa n_i n_i\, \delta_{jk} = Fa(n_j n_k - \frac{\delta_{jk}}{3})$$

$$S_{xx} = -\frac{Fa}{3}, \qquad S_{yy} = -\frac{Fa}{3}, \qquad S_{zz} = \frac{2Fa}{3}$$

Lecture 1: The mathematics and physics of bacterial swimming

1. Low Re and the Stokes equations

2. The Scallop theorem

When can a creature swim at low Re?

Swimming stroke must be different forwards and backwards in time

3. Dipolar flow fields

What do the flow fields look like?

The far flow fields are generically dipolar