Principles of gravitational wave detection

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Outline

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1. Effect of gravitational waves
2. Signal and noise
3. Position noise
4. Measurement noise
5. The Null instrument

Gravitational waves

Gravitational waves

The field equations of General Relativity

$$G^{\alpha\beta} = \frac{8\pi G}{c^4} T^{\alpha\beta}$$

are highly non linear, as the resulting metric enters in the equations themselves.
For small perturbations to a flat space

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}, \quad |h_{\mu\nu}| \ll 1$$

Einstein's equations can be expanded to the first order in h.
Calling the trace of $h^{\alpha\beta}$

$$h = h^{\alpha}{}_{\alpha}$$

and defining

$$\bar{h}^{\alpha\beta} = h^{\alpha\beta} - \frac{1}{2}\eta^{\alpha\beta} h$$

one arrives (see for example Schutz2009) at

$$\bar{h}^{\mu\nu}{}_{,\alpha\alpha} = \Box \bar{h}^{\mu\nu} = -\frac{16\pi G}{c^4} T^{\mu\nu},$$

that is a wave equation for the metric perturbation $\bar{h}_{\mu\nu}$ with source $T^{\mu\nu}$, that propagates at the speed of light c.

Expected amplitude

The amplitude of h in the static case of one point mass is

$$h = \frac{2\mathrm{G}}{\mathrm{c}^2}\frac{M_\odot}{R}$$

or about 4×10^{-6} at the surface of the Sun. For gravitational wave emission of non relativistic systems a rough estimate is given (Schutz 2009) :

$$\begin{aligned} h_{jk} &= \frac{2\mathrm{G}}{\mathrm{c}^4 R}\frac{\mathrm{d}^2 Q_{jk}}{\mathrm{d}t^2} \\ &\lesssim \frac{2\mathrm{G}}{\mathrm{c}^2 R}\frac{2Mv^2}{\mathrm{c}^2 r} \\ &\sim \frac{2\mathrm{G}M}{\mathrm{c}^2 R}\frac{2\mathrm{G}M}{\mathrm{c}^2 2r} \\ &= \frac{r_S}{R}\frac{r_S}{d} \\ &= \phi_{ext}\phi_{int} \end{aligned}$$

r_S is a typical Schwarzschild radius of the system. ϕ_{int} and ϕ_{ext} are the gravitational potential of one mass within the system and at the observer respectively.
For binary black holes of 30 $M_\odot$ ($r_S = 3$ km) at 400 Mpc, forming a system of size 600 km

$$h \sim 10 \times 10^{-21}$$

Distance ladder

For reference the main scales of distance are

Quantity	Value	Note
1 light-year	0.946×10^{16} m	
1 parsec (pc)	3.086×10^{16} m	3.262 light-year: observed parallax error of one arcsecond when the Earth moves transversally by 1 AU
Galactic center	10 kpc	High stellar density, massive black hole
Galaxy diameter	$\sim$ 30 kpc	There are of the order of 10^9 neutron stars in our Galaxy
Local group Galaxies	50 kpc - 1.4 Mpc	50 galaxies
Virgo galaxy cluster	$\sim$ 20 Mpc	2500 galaxies
Horizon	5 Gpc	15 billion light-year

Compact objects

In order to obtain a maximal quadrupole moment derivative one needs coherent motion at high speed.
Motion is driven mostly by gravitation
Neutron stars and black holes are the heaviest and most compact astrophysical objects known
For neutron stars the typical mass is $1.4M_{\odot}$ with a radius of 10 km
For black holes dimensions are given by their Schwarzschild radius $r_S = 2GM/\mathrm{c}^2$
For a 30 $M_{\odot}$ black hole r_S= 90 km

Neutron stars are formed when a massive star undergoes core collapse when a supernova event occurs.
Matter gets compressed under gravitational pressure and the inverse β decay reaction is favoured energetically

$$e^- + p \rightarrow n + \nu_e$$

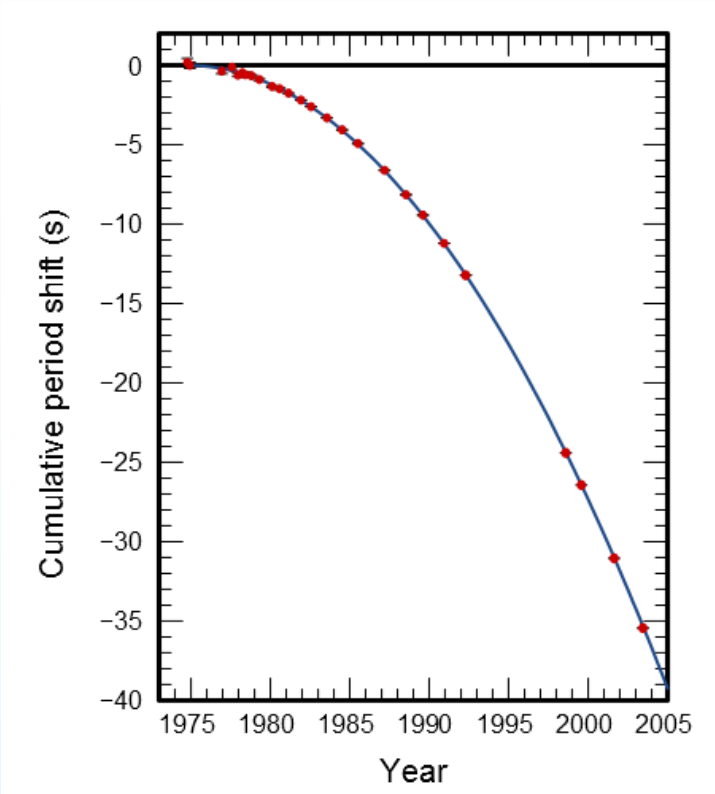
leading to a compact body made of neutrons, with a solid crust
Current understanding is that if the progenitor star has a mass between $\sim 8 - 25M\odot$, it is likely that out of its remnants a neutron star will come out while the rest of the mass will be dispersed in space
If the start has a mass greater than $\sim 25M_{\odot}$, neutron matter compresses until a black hole forms

Single compact objects

Core collapse and rotation of non axisymmetric neutron stars are sources of gravitational waves.
Due to angular momentum conservation the neutron star will be rotating quite fast.
The presence of rotating asymmetries either on the crust or of hydrodynamical origin lead to gravitional wave emission
The expected amplitude is very low: $h_{psr} \lesssim 10^{-26}$, currently only upper limits have been placed
The fascinating point about these sources is that they are continuous and can be observed in principle for years (careful of Earth motion)

During stellar core collapse a large mass undergoes high acceleration. However most of this motion is axisymmetric so that emission of radiation low
With more and more accurate simulations the emitted energy has gone down from $0.01 M_\odot$ to $10^{-8} M_\odot$ which makes them observable with current detectors in our Galaxy. But the rate is very low (1 every 40 years)
In the last years simulations of stellar core collapse have shown a wealth of physics that can studied: from relativistic magnetohydrodynamics to nuclear physics to probing neutrino structure.
Signal shape was first thought to last a few milliseconds, having a characteristic frequency of 1 kHz.
Now simulations show a signal lasting for 1 second, with high and low frequency components
CAN'T MISS IT

Indirect evidence: PSR 1913+16



Cumulative of the periastron epoch in seconds for the binary system PSR B1913+16 due to energy loss through gravitational radiation per perdita di energia per irraggiamento di onde gravitazionali.
Data from J. M. Weisberg and J. H. Taylor, Relativistic Binary Pulsar B1913+16: Thirty Years of Observations and Analysis, July 2004, http://arxiv.org/abs/astro-ph/0407149.

Parameters of PSR1913+16

Parameter	Symbol (units)	Value[1]
(i) **"Physical" Parameters**		
Right Ascension	α	$19^h15^m28.^s00018(15)$
Declination	δ	$16°06'27.''4043(3)$
Pulsar Period	P_p (ms)	59.029997929613(7)
Derivative of Period	$\dot{P}_p$	$8.62713(8) \times 10^{-18}$
(ii) **"Keplerian" Parameters**		
Projected semimajor axis	$a_p \sin i$ (s)	2.3417592(19)
Eccentricity	e	0.6171308(4)
Orbital Period	P_b (day)	0.322997462736(7)
Longitude of periastron	ω_0 (°)	226.57528(6)
Julian date of periastron	T_0 (MJD)	46443.99588319(3)
(iii) **"Post-Keplerian" Parameters**		
Mean rate of periastron advance	$\langle\dot{\omega}\rangle$ (° yr^{-1})	4.226621(11)
Redshift/time dilation	γ' (ms)	4.295(2)
Orbital period derivative	$\dot{P}_b$ (10^{-12})	−2.422(6)

[1]Numbers in parentheses denote errors in last digit.
Data from http://puppsr8.princeton.edu/psrcat.html

Table 6: Parameters of the Binary Pulsar PSR 1913+16

Gravitational waves II

Gauge freedom allows to write a simple wave equation.
In addition one can impose $\bar{h}^{\mu\nu}$ to be traceless and transverse with respect to the direction of propagation.
The general solution for a plane wave propagating in the z direction is:

$$\bar{h}_{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & h_+ & h_\times & 0 \\ 0 & h_\times & -h_+ & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} e^{i(\Omega t - kz)}$$

There are two independent solutions corresponding to the "+" e "×" polarization.

The symmetry properties attribute to the graviton a spin of two and 2 helicity states, as for a massless particle.
For a free mass and reference system in free fall with TT gauge, one applies the geodetic equation

$$\frac{\mathrm{d}U^\alpha}{\mathrm{d}\tau} + \Gamma^\alpha{}_{\mu\nu} U^\mu U^\nu = 0$$

For a mass initially at rest ($U^\alpha = (1,0,0,0)$) the equation at that moment becomes

$$\left.\frac{\mathrm{d}U^\alpha}{\mathrm{d}\tau}\right|_0 = -\Gamma^\alpha{}_{00} = -\frac{1}{2}\eta^{\alpha\beta}(h_{\beta 0,0} + h_{0\beta,0} - h_{00,\beta}) = 0$$

The coordinate remains constant if the mass is initially at rest.

Effect on a pair of masses

Consider the $+$ polarization. The metric element is

$$\mathrm{d}s^2 = \mathrm{d}t^2 - (1+h_+)\mathrm{d}x^2 - (1-h_+)\mathrm{d}y^2 - \mathrm{d}z^2$$

Consider a light ray ($\mathrm{d}s^2 = 0$). If travelling in the transverse direction x

$$\mathrm{d}s^2 = 0 = \mathrm{c}\mathrm{d}t^2 - (1+h_+)\mathrm{d}x^2$$

What can be measured by a single clock is the round trip time between two free falling test mass.
In the TT gauge coordinates don't change.
Travel time is given by:

$$\int_{t_0}^{t_1} \mathrm{d}t = \int_{\mathrm{path}} \sqrt{1+h_+}\mathrm{d}x$$

which is function of $h_+(t)$.
For $h_+(t)$ approximately constant during the round trip

$$t_1 - t_0 = \frac{2L}{\mathrm{c}}\left[1 + \frac{h_+(t)}{2}\right].$$

Using time measurements

Consider a periodic wave $h_+ = h\cos(\Omega t + \phi)$ and light travelling between two masses at coordinates $x = 0$ and $x = L$.
Integration along that path gives

$$t_1 - t_0 = \int_{x=0}^{x=L} (1 + \frac{h_+}{2})\mathrm{d}x = \frac{2L}{c}\left[1 - \frac{h}{2}\frac{\sin\frac{\Omega L}{c}}{\frac{\Omega L}{c}}\cos\left(\Omega\left(t_1 - \frac{L}{c}\right) + \phi\right)\right]$$

In the lab frame of the first mass (not in TT gauge) this corresponds to a displacement of the second mass

$$\delta x = L\frac{h}{2}\frac{\sin\frac{\Omega L}{c}}{\frac{\Omega L}{c}}\cos\left(\Omega\left(t_1 - \frac{L}{c}\right) + \phi\right)$$

If $\Omega L/c \ll 1$, or $2\pi\frac{L}{\lambda_{GW}} \ll 1$ the round trip time reproduces the signal shape.

Using time measurements II

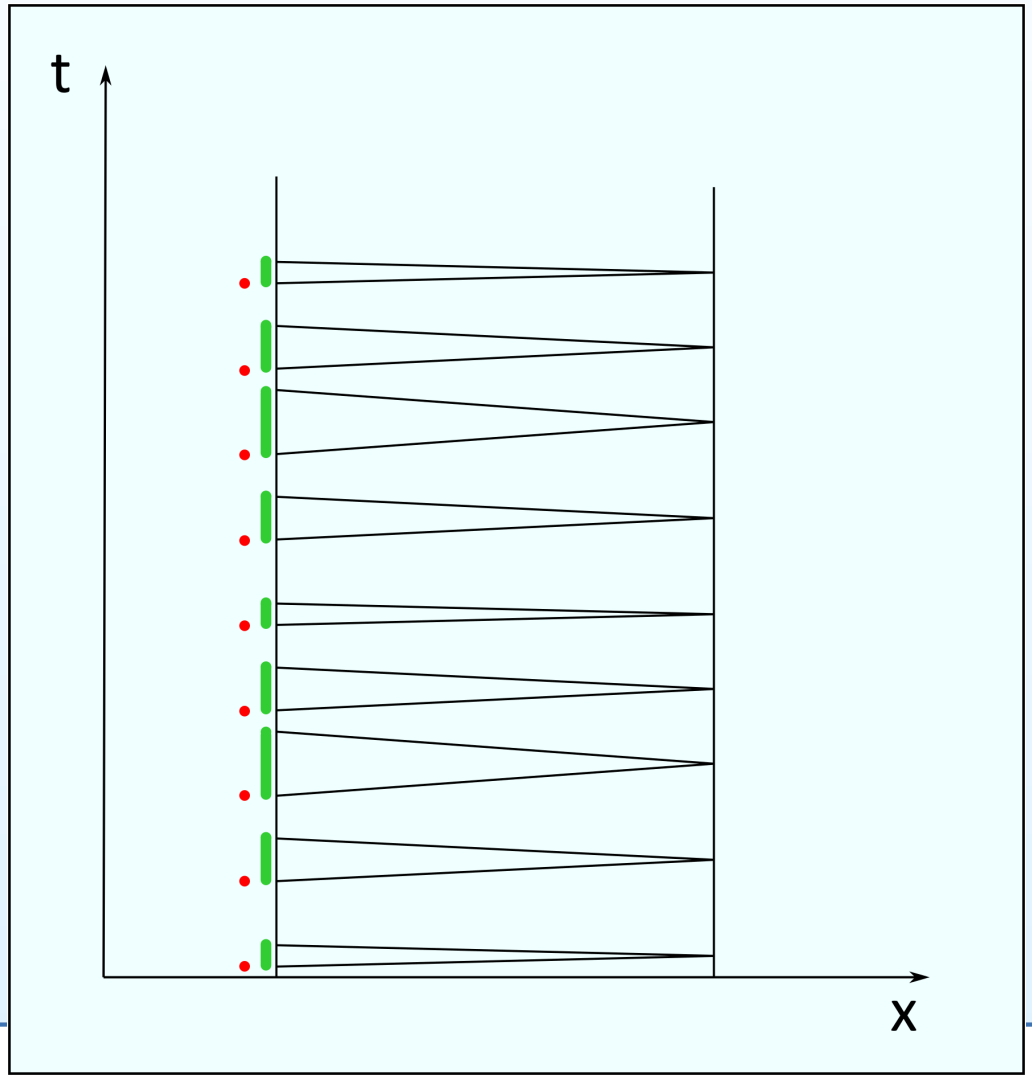
It is possible to increase the response to h by increasing L

$$\delta x = L\frac{h}{2}\frac{\sin\frac{\Omega L}{c}}{\frac{\Omega L}{c}}\cos\left(\Omega\left(t_1 - \frac{L}{c}\right) + \phi\right)$$

In a multiple delay line one can have N round trips, leading to substituting L with NL.
If $\Omega L/c = k\pi$, that is $2L = n\lambda_{GW}$, the response is zero. The signal gained going from one mass to the other is lost on the return trip.
For $\Omega_1\ \pi c/L$ the response is in an envelope proportional to $1/(\Omega L/c)$.
This procedure averages the measurement of h(t) over the time it takes to go from the light source to the detector.
The light path acts as a low pass filter with pole at $f_{LP} = c/NL$.
Sources recently detected on Earth generate an $h \sim 10^{-21}$ on Earth for $L = 3 - 4\,\mathrm{km}$, the distance variation is

$$\delta x = L\frac{h}{2} = 1.5 - 2 \times 10^{-18}\,\mathrm{m}$$

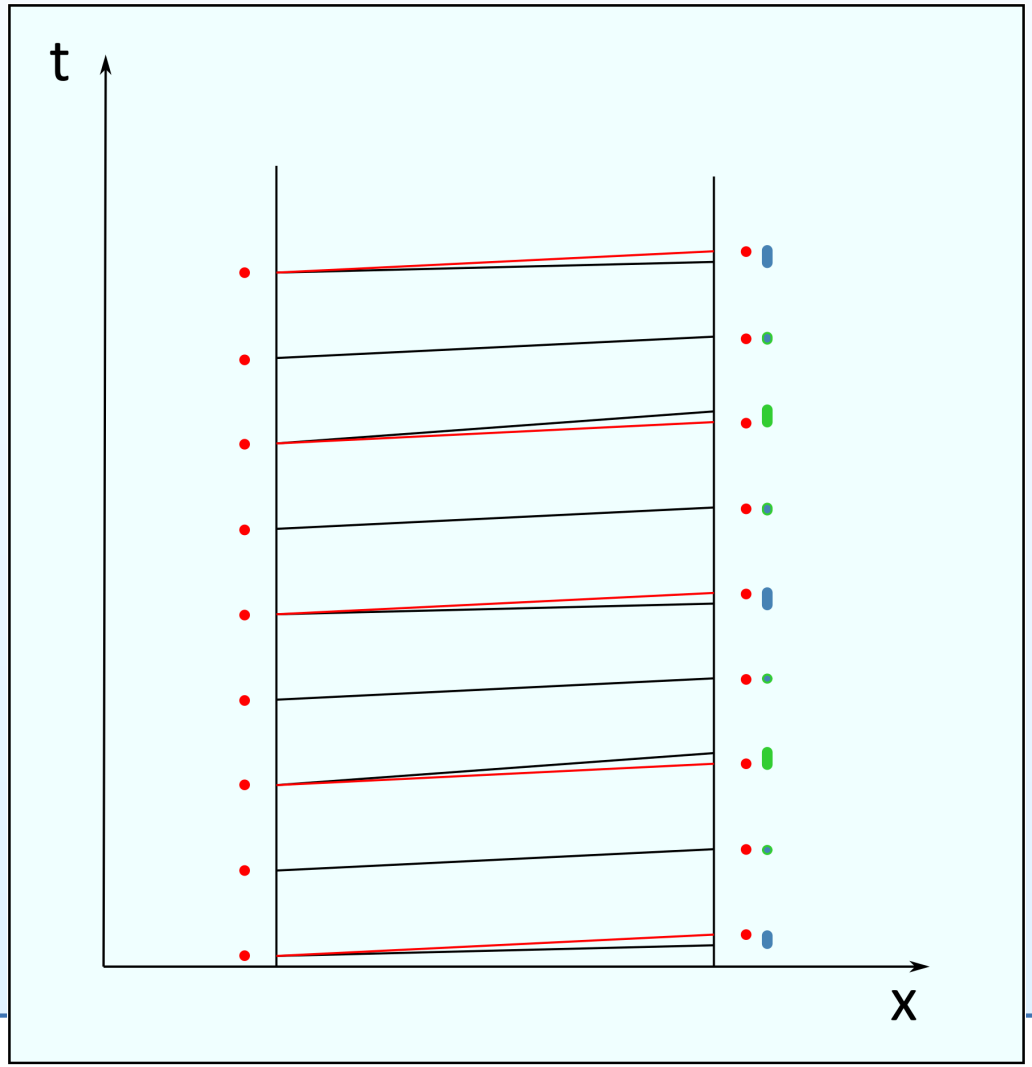
Using time measurements III



By having light bouncing back from the second mass one can compare timings in a single place, as is done with interferometry.

Using time measurements IV

Having two stable clocks allows to use one way light transmission, timing relies on the stability of the remote clock.
This can be done using the emission from highly stable pulsars. See 2016MNRAS.458.1267V and references therein.



Signal and noise

Noise power spectrum

Consider a stochastic process $X(t, K)$ representing the output of a continuous measurement from an experiment K
The autocorrelation is defined by

$$R(t, \tau) =< n(t, K)n(t + \tau, K) >$$

where the average $<>$ is computed on the various instances of the experiment K.
For stationary noise $n(t)$ (statistical properties constant in time) one can define

$$R_n(\tau) =< n(t)n(t + \tau) >$$

The power spectrum $S_n(f)$ of $n(t)$ is

$$S_n(f) = \int_{-\infty}^{+\infty} R_n(\tau) \exp(i2\pi f\tau)\mathrm{d}\tau$$

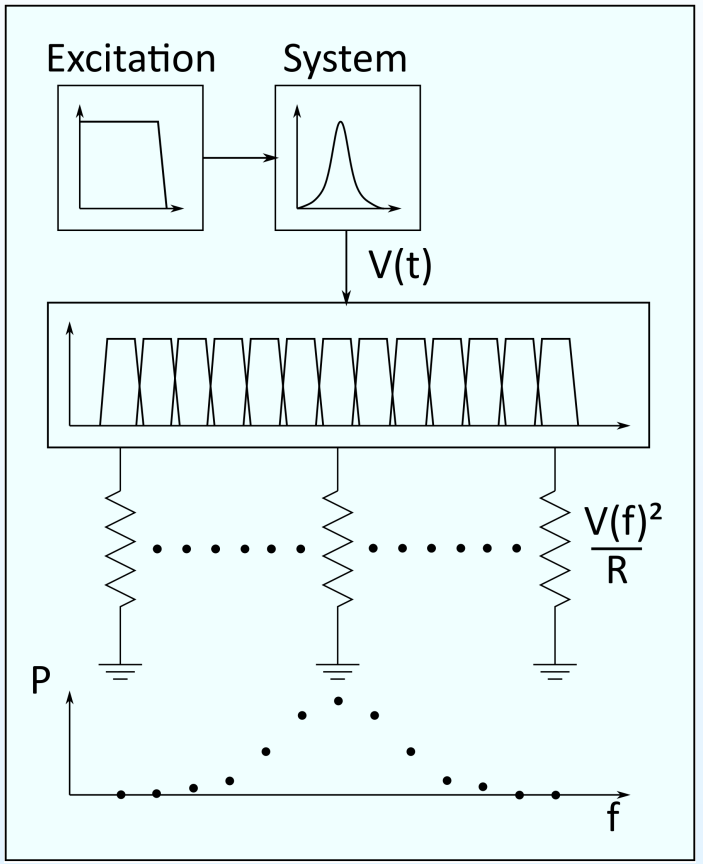
From Parseval's theorem one has that the noise over the full frequency band

$$\begin{aligned} \int_{-\infty}^{+\infty} S_n(f)\mathrm{d}f &= \int_{-\infty}^{+\infty} R_n(\tau) \int_{-\infty}^{+\infty} \exp(i2\pi f\tau)\mathrm{d}f\mathrm{d}\tau \\ &= R_n(0) \\ &= \mathrm{var}(n) \end{aligned}$$

This allows to explain the name given to $S_n(f)$.

Power spectrum

Consider the output $n(t)$ of a noisy channel and distribute it to a battery of bandpass filters 1 Hz wide.
Measure for each filter the power dissipated on a 1 Ω resistor.
It is



$$\int_{-\infty}^{+\infty} S_n(f)\mathrm{d}f = 2\int_{f}^{f+1\mathrm{Hz}} S_n(f)\mathrm{d}f \cong S_n(f)$$

$S_n(f)$ is the variance of noise per unit frequency.

Power spectrum II

As in the case of single measurements one prefers to use the standard deviation $\sigma = \sqrt{\mathrm{var}(n)}$ rather than the variance.
One finds often

$$\tilde{n}(t) = \sqrt{S_n(f)}$$

called the linear power spectrum (LPS).
$S_n(f)$ has units of $n(t)^2/\,\mathrm{Hz}$
$\tilde{n}(f)$ has units of $n(t)/\sqrt{\mathrm{Hz}}$

The power spectrum for the noise from several uncorrelated sources is the sum of the power spectra.
The linear power spectrum is obtained summing in quadrature the various LPS.
For Gaussian noise, that is assuming that the real and the imaginary components of the Fourier transform of the signal are independent and Gaussian distributed with mean 0 and variance σ^2, the probability density distribution for $r = S_n(f)$ is a Rayleigh distribution

$$p(r) = \frac{r}{\sigma^2} \exp\left(-\frac{r^2}{2\sigma^2}\right)$$

which is asymmetric with a tail toward higher values

Power spectrum III

Important caveat: check whether the power spectrum is defined for $0 \leq f < +\infty$ or $-\infty < f < +\infty$.
Physically one measures only $f \geq 0$ but the definition is for $-\infty < f < +\infty$
The power spectrum defined considering $f > 0$ is twice the one defined over the full real axis.
The linear power spectrum defined considering $f > 0$ is $\sqrt{2}$ times the one defined for $-\infty < f < +\infty$.
Experimentally noise is measured making many averages over time:

$$S_n(f) = E\left[\frac{1}{2T}\left|\int_{-T}^{+T} n(t)\exp(i2\pi ft)\mathrm{d}t\right|^2\right]$$

T will determine the frequency resolution. The number of averages will reduce the fluctuations of the measured $S_n(f)$.
This usually works well, a typical pathology is when one is in presence of a monochromatic signal.

Power spectrum IV

Nowadays signal spectrum analyzers are digital. The signal is sampled at some frequency f_s. The Nyquist frequency $f_N = f_s/2$ is the maximum frequency that can be analyzed. Higher frequency signals appear as lower frequency one, like when one sees a car wheel accelerating in a movie. When the rotation is too fast it seems that the wheel is going backward, more and more slowly

This is called aliasing and high frequency components must be filtered BEFORE sampling.

Using sampled signals the Fourier transform becomes a series that will provide Fourier coefficients a_k, b_k for the fundamental frequency $f_1 = 1/T$ and its harmonics $f_k = kf_1$ up to f_N. The time average is obtained repeating the computation of the coefficients and averaging $S_(f_k) = E[a_k^2 + b_k^2]$ for a given number of times n.

The Fourier analysis is made in terms of functions that have period T, and trying to achieve continuity between $n(+T)$ and $n(-T)$
To avoid this the analysis is made applying a window that weighs less the extremities of the time interval, and performing the analysis over time intervals that typically overlap by 50
Summarizing, for a sampling frequency f_s the spectrum is computed for $1/T < f < f_s/2]$ with a number of overlapping samples $(n/2)$.
The precision at each frequency will increase with $\sqrt{n}$

Signal and noise

Assume that noise is added to the signal so that the detector output is $s(t) = n(t) + h(t)$
In a way similar to the minimum χ^2 method one asks that

$$\int_0^T \int_0^t \frac{(s(t-\tau) - \alpha h(t-\tau))(s(t) - \alpha h(t))}{C(t, t-\tau)} \mathrm{d}\tau \mathrm{d}t$$

be minimum. The product is weighted with the equivalent of the covariance, taken between noise at different time differences

$$C(t, t-\tau) =< n(t)n(t-\tau) >= R_n(\tau)$$

Minimizing with respect to α requires that

$$\int_0^T \int_0^t \frac{(s(t-\tau) - \alpha h(t-\tau))h(t)}{C(t, t-\tau)} \mathrm{d}\tau \mathrm{d}t = 0$$

The solution for α is

$$\alpha = \frac{\int_0^T \int_0^t \frac{(s(t-\tau)h(t)}{C(t,t-\tau)} \mathrm{d}\tau \mathrm{d}t}{\int_0^T \int_0^t \frac{h(t-\tau))h(t)}{C(t,t-\tau)} \mathrm{d}\tau \mathrm{d}t} = 0$$

Signal and noise II

This can be translated in the frequency domain
$h(t)$ is a function of several parameters (coalescence time, masses, ...)
The optimal filter for a deterministic signal requires to compute

$$\begin{aligned} s_W &= 2\int_0^{+\infty} \frac{\tilde{s}(f)\tilde{h}^*(f)}{S_n(f)}\,\mathrm{d}f \\ &= \int h(t)\int s(\tau)w(t-\tau)\,\mathrm{d}\tau\,\mathrm{d}t \end{aligned}$$

$w(t)$ weighs $s(t)$ more at frequencies where the detector is less noisy
s_W is Gaussian with mean zero and standard deviation 1 if the template $h(t)$ is properly normalized.
The signal-to-noise ration (SNR) is given by

$$\mathrm{SNR}^2 = 4\int_0^{+\infty} \frac{|\tilde{h}(f)|^2}{S_n(f)}\,\mathrm{d}f$$

SNR^2 is an integral over a frequency band:
one should cover as much as possible of the signal band with a low noise detector
One can perform a best fit varying the signal parameters obtaining best estimates.
Note that resonant detectors can be broadband, it is a matter of how low is the noise, which comes from the electronics
A. Papoulis, Probability, Random Variables and Stochastic Processes, McGrawHill.
S. Kay, Statistical Signal Processing, Estimation Theory, Detection Theory, Prentice-Hall.

Broadband detectors

Detection principle: measure the round trip time between to free falling masses
For ground detectors one can have free masses in the horizontal plane, for signal with frequency higher than the pendulum frequency
Distance is measured using the phase of the light electric field
Available clocks are not stable enough, one has to compare travel time in two orthogonal directions using the particular structure of $h_{\mu\nu}$: an extension along x corresponds to a contraction along y.
The following are the main noise sources that are encountered in interferometric gravitational detectors
Mass position noise

- Seismic noise
- Thermal noise
- Local gravity fluctuations
- Radiation pressure

Broadband detectors II

Measurement noise

- Photon counting noise (shot noise)

Quantum limit

- Reducing shot noise requires to increase the circulating power. This increases the size of the fluctuations in photon number or radiation pressure
- This increase the fluctuations of the position of the mass
- This is the Heisenberg principle
- Elegant workarounds have been tested on km scale detectors, measuring only one of the two non commutating physical quantities, phase ϕ and photon number n_γ
- The uncertainty principle says

$$\sigma_{n_\gamma}\sigma_\phi > \frac{1}{4}$$

  but only ϕ is needed, so one can decrease σ_ϕ with a larger σ_{n_γ}
- This is achieved by modifying the vacuum state of the quantized electromagnetic field

Position noise

Position noise

Seismic noise: elastic waves of the earth crust and interior, propagating with different speed and attenuation length as the nature of the medium varies.
Origin

- Earthquakes, even very light ones.
- Ground vibrations from anthropic activity, manufacturing processes, transportation infrastructures.
- Pressure variations on soil due to sea motion, ocean swell, wind, planes and helicopters
- On time scale of hours: earth crust deformation due to the tide.

The typical spectrum at the Virgo location is

$$\tilde{x}_s(f) = 10^{-6} \left(\frac{1\,\mathrm{Hz}}{f} \right)^2 \mathrm{m}/\sqrt{\mathrm{Hz}}$$

for $f > 1\,\mathrm{Hz}$
Below: presence of a peak at 140 mHz: from wind and sea.

Seismic attenuation

Coupling with the ground has to be severely reduced leaving test masses inertial.
This is achieved by using a chain of masses one connected to the other, that are more and more isolated through the inertia present upstream: a cascade of mechanical filters.
Consider a pendulume with suspension point that moves according to $x_s(t)$. The equation of motion for the pendulum mass is well known:

$$\ddot{x} + \Gamma\dot{x} + \frac{g}{l}x = \frac{g}{l}x_s(t)$$

In the frequency domain the solution for an excitation $x_s(t) = x_0 \exp[j\omega t]$ has an amplitude

$$x(\omega) = x_0 \frac{\omega_0^2}{\sqrt{(\omega^2 - \omega_0^2)^2 + \Gamma^2\omega^2}}, \quad \omega_0^2 = \frac{g}{l}$$

At high frequency, $\omega \gg \omega_0$, this becomes

$$x(\omega) = x_0 \frac{\omega_0^2}{\omega^2}$$

Attenuation is achieved through inertia.
Dissipation must be low, otherwise coupling to the ground is higher. Furthermore dissipation is intrinsically noisy.

Seismic attenuation II

By cascading n pendula one has a system with n normal modes.
Above the highest mode frequency the reaction of pendulum i on pendulum $i-1$ becomes negligible and attenuations can be multiplied.
The total attenuation A can be estimated taking the order of magnitude of the frequency of the highest mode.
For a 1 m pendulum

$$\omega_0^2 = \sqrt{g}l = 10\,\mathrm{rad}^2\,\mathrm{sec}^{-2}$$

while for n pendulums

$$A \sim \left(\frac{\omega_0^2}{\omega^2}\right)^n = \left(\frac{10}{40f^2}\right)^n$$

Seismic attenuation III

Requiring a noise floor at $f = 10$ Hz of $h = 10^{-23}/\sqrt{\text{Hz}}$ with an arm of $L = 3$ km implies

$$A < \frac{h(f)L}{2x_s(f)} = \frac{1.5 \times 10^{-20}}{10^{-8}} = 1.5 \times 10^{-12}$$

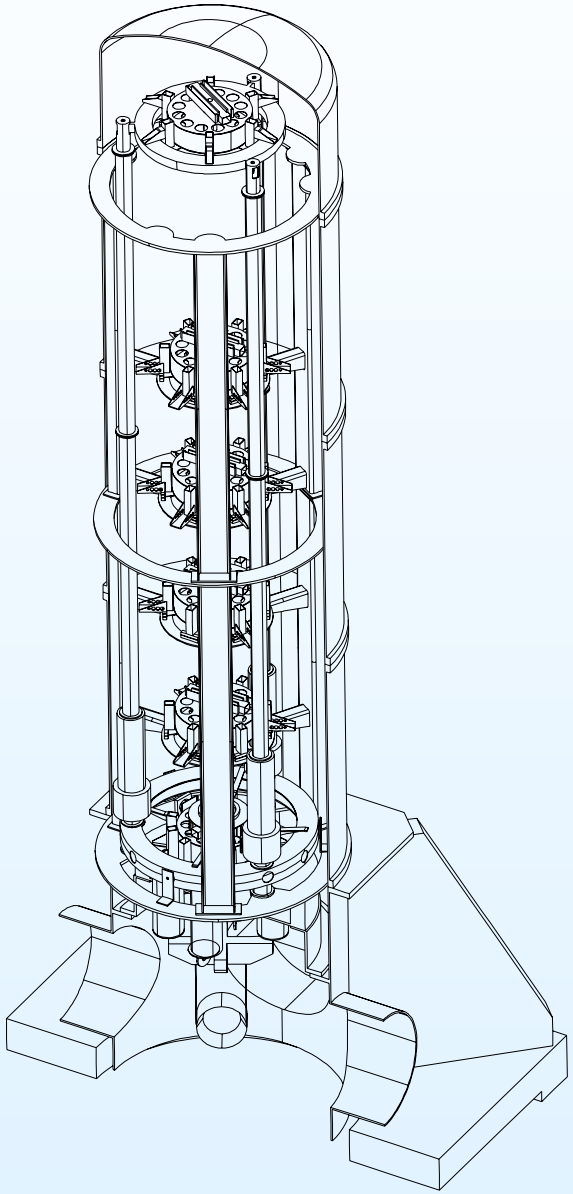
On the other hand

$$A = \left(\frac{\omega_0^2}{(2\pi f)^2}\right)^n = \left(\frac{10}{4000}\right)^n < 1.5 \times 10^{-12}$$

$$n > \frac{-\log 1.5 + 12}{\log 400 + 2\log f} = 4.5$$

Five filters, that include the test mass are necessary In the frequency band of the normal modes, if dissipation is low, motion is amplified, and this may take the interferometer out of its working point.
Through an active control on the first stages one can avoid the excitation of these modes.
This requires a complex feedback system with low noise sensors and actuators.
Acting upstream allows to filter the additional noise introduced at high frequency.
Another requirement on the attenuation system is to to be able to control the mirror position in 6 degrees of freedom.
This must be achieved without reintroducing noise. In particular the actuator should not act by contact and have a low relative position noise with respect to the mirror due to unavoidable force gradients.

The Virgo Superattenuator

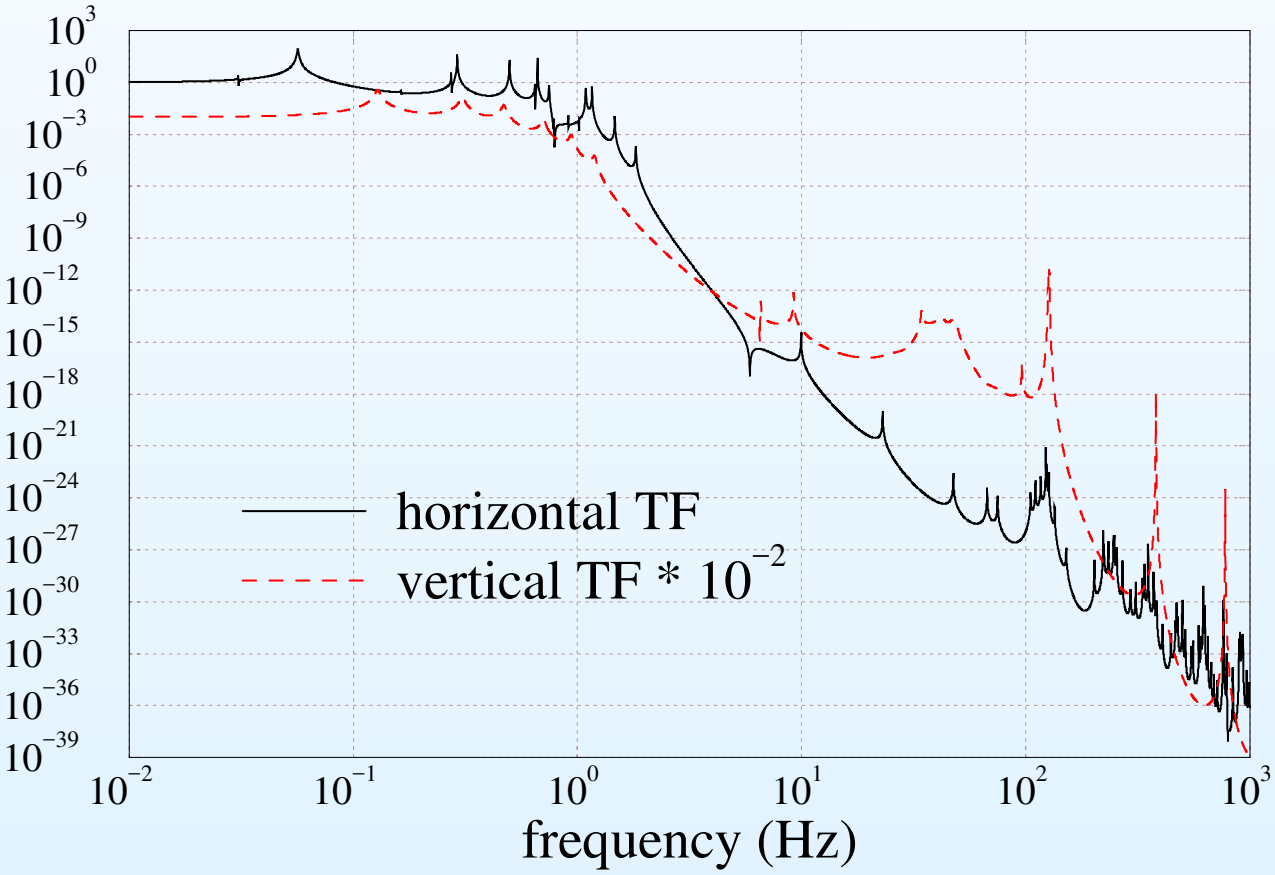


SA Transfer function

The ratio in amplitude (and phase) in the frequency domain between the mirror motion $x_m(\omega)$ and the ground motion $x_s(\omega)$ is called the transfer function:

$$A(f) = \frac{x_m(f}{x_s(f}$$

Below $|A(f)|$ for the Virgo Superattenuator is shown.



Thermal noise

The test mass is a body in thermal equilibrium with the environment.
Position and velocity degrees of freedom have on average an energy of

$$k_B T = 3.9 \times 10^{-21}\ \mathrm{J}$$

For a 40 kg mirror suspended by a 1 m pendulum, the elastic constant of the equivalent harmonic oscillator is

$$mg/l = 400\,\mathrm{N/m}$$

and the corresponding oscillation amplitude x_T is

$$x_T = \sqrt{\frac{k_B T}{k}} = \sqrt{\frac{k_B T l}{mg}} = 3.1 \times 10^{-12}\ \mathrm{m}$$

This is the rms value, integrated over the whole frequency band.
Luckily in terms of power spectrum the energy is not distributed evenly over the band

Fluctuation-Dissipation theorem

The Fluctuation-Dissipation theorem links the power spectrum of noise from thermal origin to the dynamics of the system

$$S_{\dot{x}_T}(\omega) = 2k_B T \Re Y(\omega), \quad Y = \frac{1}{Z(\omega)} = \frac{\dot{x}}{F}$$

$Z(\omega)$ is the impedance of the system, its inverse $Y(\omega)$ is the admittance.
Considering a pendulum in vacuum, with a model for the dissipation inside the wires different from the usual viscous damping, one has

$$\ddot{x} + \omega_0^2[1 + j\phi(\omega)]x = F/m$$

with $\phi(\omega) \sim 10^{-6}$.
The motion is given by

$$x(\omega) = \frac{F}{m} \frac{\omega_0^2 - \omega^2 - j\phi(\omega)\omega_0^2}{(\omega_0^2 - \omega^2)^2 + \phi^2(\omega)\omega_0^4}$$

Applying the Fluctuation-Dissipation theorem

$$\Re Y = \frac{\omega}{m} \frac{\phi(\omega)\omega_0^2}{(\omega_0^2 - \omega^2)^2 + \phi^2(\omega)\omega_0^4}$$

The linear power density for the pendulum thermal noise is given by

$$\tilde{x}_T(\omega) = \frac{\sqrt{S_{\dot{x}_T}(\omega)}}{\omega} = \sqrt{\frac{4k_B T \phi(\omega)\omega_0^2}{m\omega[(\omega_0^2 - \omega^2)^2 + \phi^2(\omega)\omega_0^4]}}$$

for $\omega \geq 0$.

Thermal noise mitigation

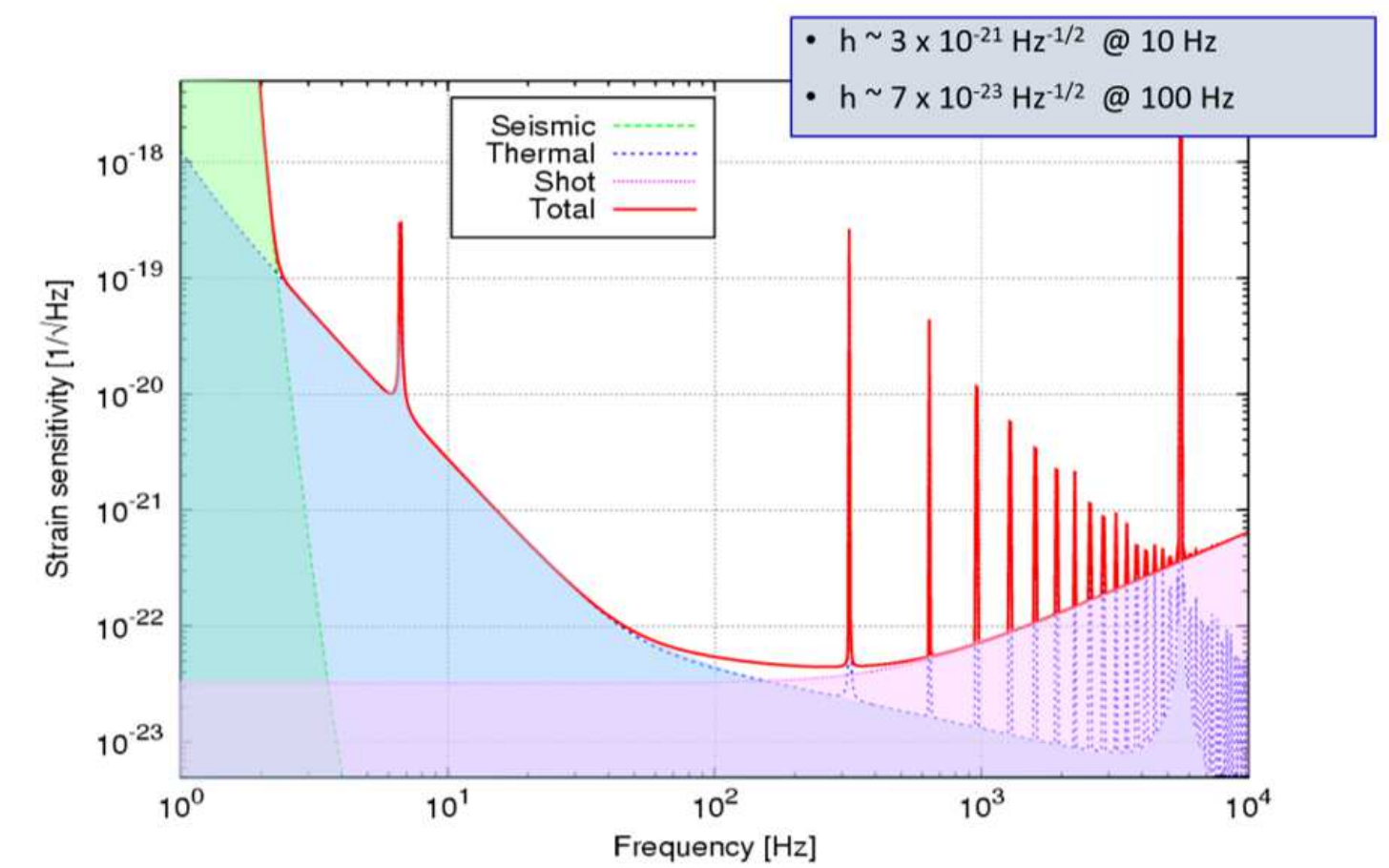
To mitigate the effects of thermal noise one can act on the temperature T
This leads to a number of complications in the experimental apparatus and is not the preferred solution, although some experiments are running some tens of K above the absolute zero.
The second possibility is to act on the dissipation mechanism so that the channel with the environment is narrow.
In this way the motion will be described by a very narrow peak at the resonant frequency, with low tails.
While gravitational motion of the pendulum doesn't dissipate (forget about gravitational radiation), flexion of the suspending wires does.
Normal modes of the mirrors have similar effects.
One has to select materials that are elastic and won't flow or creep before they break, like very pure fused silica.
Currently a serious limitation comes from the coatings deposited on mirrors to obtain the desired transmission coefficient. The dielectric materials used have a much higher dissipation than the mirrors substrates and are currently determining over most of the frequency band the noise of the interferometers.

Gravity fluctuations

Variations with time of the local gravitational field cannot be shielded and can mimic perfectly the effect of gravitational waves
These fluctuations can be generated by the motion of large masses, or by density variations.
This noise is often called Newtonian noise.
Seismic waves can rise the surface of the Earth increasing the mass surrounding a mirror and introducing a force.
This effect can be reduced by having ground completely surrounding the mirror, that is operating underground
Density variations can be found in the atmosphere, due to wind, humidity and rain
One finds also density variation in the propagation of longitudinal seismic waves
Research is going on to understand how, by monitoring the ground or air motion, the effect of local gravity fluctuations can be predicted and therefore subtracted. It is expected that arrays of hundreds of seismic sensors are necessary.

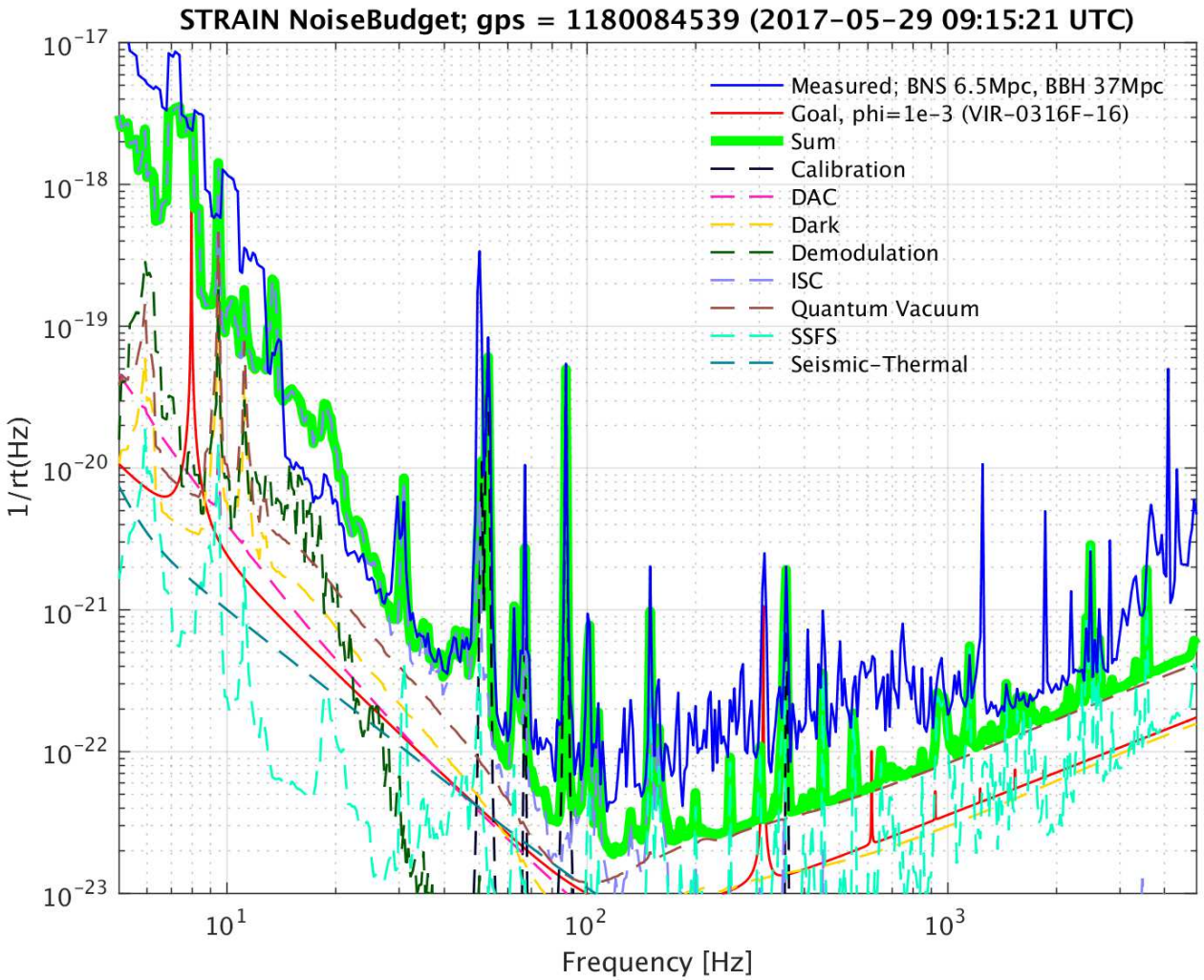
Noise curve

Main noise sources



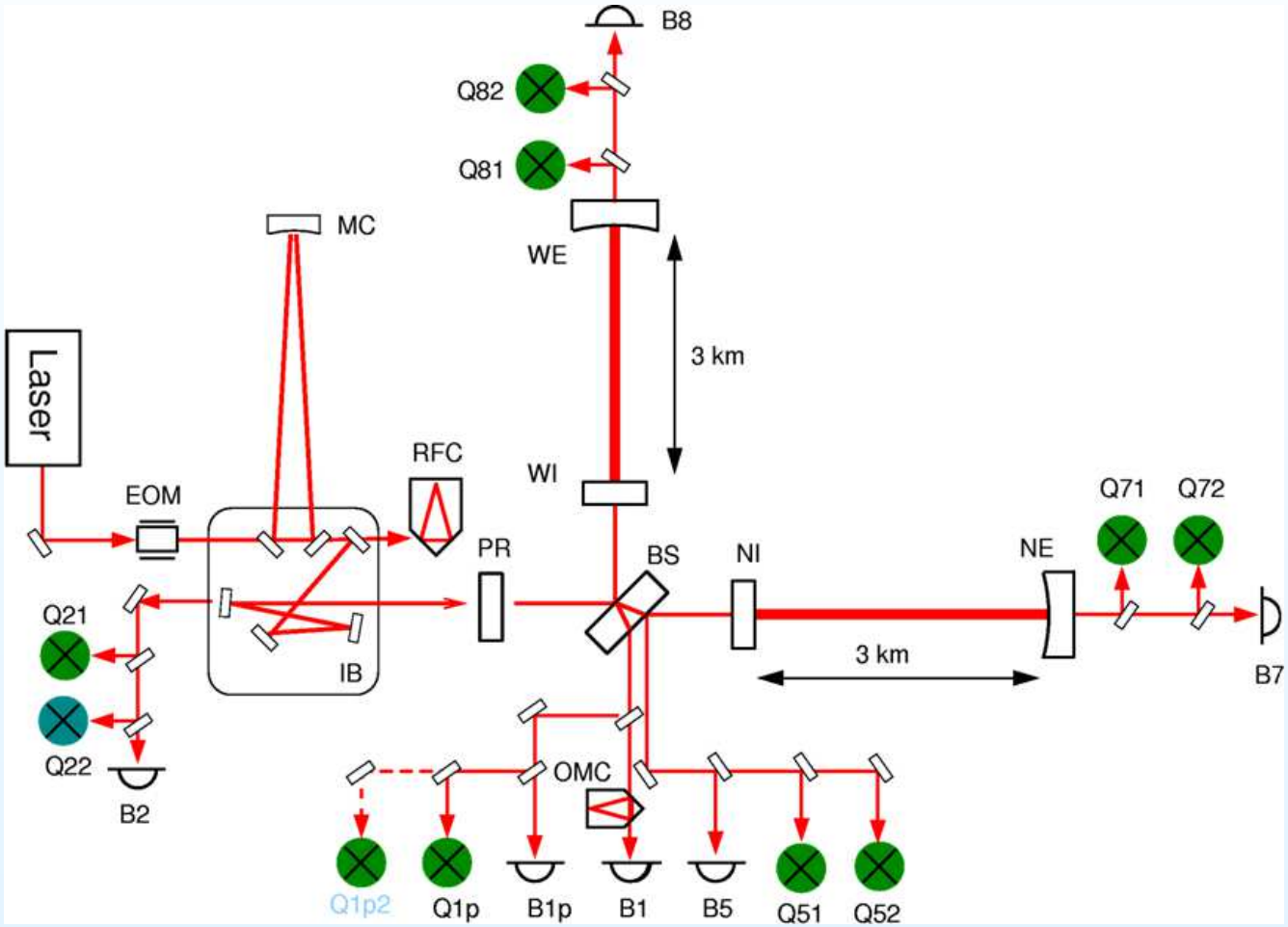
Real noise curve

Noise budget in commissining phase



Measurement noise

Virgo optical scheme



Measurement noise

Consider a Michelson interferometer with incident light power P_{in}, and detected power P_{out}.
Consider a phase difference between arms

$$\Delta\varphi = \pi + \alpha + \phi_{OG}$$

$\pi + \alpha$ is to be chosen
$\alpha = 0$ means destructive interference in transmission, it is usually said that the interferometer is "on the dark fringe".
ϕ_{OG} is the phase change from the gravitational wave.
One has:

$$P_{out} = P_{in} \sin^2\left(\frac{\alpha + \phi_{OG}}{2}\right)$$

For $\phi_{OG} \ll 1$:

$$P_{out} = P_{in}\left(\sin^2\frac{\alpha}{2} + \frac{1}{2}\phi_{OG}\sin\alpha\right)$$

Measurement noise II

The interferometer **sensitivity** is:

$$\frac{\mathrm{d}P_{out}}{\mathrm{d}\phi_{OG}} = P_{in} \sin\left(\frac{\alpha+\phi_{OG}}{2}\right) \cos\left(\frac{\alpha+\phi_{OG}}{2}\right) = \sin\left(\alpha+\phi_{OG}\right),$$

which is maximum for $\alpha = \pi/2$.
The instrument is at half fringe: $P_{out} = P_{\bar{out}} = P_{in}/2$.

Note: this is the usual instrument sensitivity, that is the variation of the output for a given input variation. No reference is made to noise. In the GW community, the word sensitivity is used differently: it is the noise curve including the detector response.

Measurement noise III

One needs to know how P_{out} fluctuates due to photon counting to compute the **precision**.
If one counts on average N pulses per second that are Poisson distributed (Poisson process), the noise spectrum over positive and negative frequencies is

$$S_\gamma(\omega) = 2\pi N^2 \delta(\omega) + N.$$

Leaving aside the DC term $\delta(\omega)$, one has that the variance per Hz is N.
Counting pulses for one second gives a 1 Hz resolution and the Poisson statistics has variance N.
The fluctuation in P_{out} is

$$\sigma_{P_{out}} = \sqrt{\eta N} = \sqrt{\frac{\eta P_{in}}{h\nu}\left(\sin^2\frac{\alpha}{2} + \frac{1}{2}\phi_{OG}\sin\alpha\right)}$$

η is the quantum efficiency of the photon detector.
In absence of signal

$$\sigma_{P_{out}} = \sqrt{\frac{\eta P_{in}}{h\nu}}\left|\sin\frac{\alpha}{2}\right|$$

Measurement noise IV

The signal-to noise ratio in a 1 Hz band, that is for 1 s measurement time is:

$$SNR = \sqrt{\frac{\eta P_{in}}{2h\nu}} \left|\cos\frac{\alpha}{2}\right| \phi_{OG}$$

which is maximum per $\alpha = 0$. It is more convenient to work on the dark fringe.
GW community defines sensitivity as the signal level that has the same spectral amplitude of the detector noise or $SNR = 1$.
The phase sensitivity is

$$\phi_{OG}^{min} = \sqrt{2\frac{h\nu}{\eta P_{in}}}$$

over 1 second.

Measurement noise V

As a case study consider infrared light with $\lambda = 1064\,\mathrm{nm}$, an incident power of 10 W and an ideal detector: $\eta = 1$).
The resulting phase noise is

$$\phi_{OG}^{min} \sim \sqrt{\frac{2h\nu}{\eta P_{in}}} = 2.0 \times 10^{-10}\,\mathrm{rad\,Hz^{-1/2}}$$

For a wave arriving perpendicularly to the interferometer plane with $+$ polarization,

$$\phi_{OG} = \frac{2\pi}{\lambda} 2L \frac{h}{2} \times 2.$$

One has

$$h^{min} = \frac{\lambda}{4\pi L}\sqrt{\frac{2h\nu}{\eta P_{in}}}$$

This is as expected a spectral density in $\mathrm{Hz^{-1/2}}$.

Here are two cases

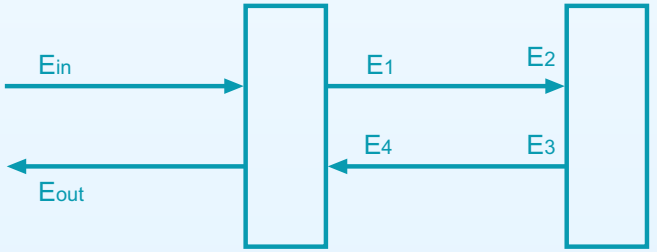
	L	$h_{min}\,\mathrm{Hz^{-1/2}}$
Michelson da tavolo	1 m	1.6×10^{-17}
Virgo	3 km	5.3×10^{-21}

off by several orders of magnitude.

Fabry-Perot cavities

Increasing the optical path allows to increase the observed effect of the gravitational wave.
The simplest way is to have multiple reflections, building a delay line.
However if too many reflections are required mirrors have to be large and light diffused at the reflection point may contaminate the other beams changing in an uncontrolled way the output phase.
The largest interferometers use Fabry-Perot resonant cavities, made by a semitransparent mirror at the input and an essentially totally reflective mirror at the end.
This is a device frequently used when one has to have precise wavelength measurements.
Here it will be used for its phase response to length variations.
Consider the incoming and outgoing electric fields E_{in} e E_{out} at the input mirror, and those inside the cavity E_1, E_2, E_3 e E_4.
Here are the relations between the various fields

$$\begin{aligned} E_1 &= t_1 E_{in} + j r_1 E_4 \\ E_2 &= \exp[jkL] E_1 \\ E_3 &= j r_2 E_2 \\ E_4 &= \exp[jkl] E_3 \\ E_{out} &= j r_1 E_{in} + t_1 E_4 \end{aligned}$$



With transmission phase change equal to $pi/2$ (dielectric materials).

Fabry-Perot cavities II

Solving for E_1

$$E_1 = \frac{t_1}{1 + r_1 r_2 \exp\left[2jkl\right]} E_{in}$$

which is maximum for $2kL = \pi$ giving

$$E_1 = \frac{t_1}{1 - r_1 r_2} E_{in}$$

Power at resonance is

$$P_1 = \frac{\epsilon_0 c}{2} |E_1|^2 = \left(\frac{t_1}{1 - r_1 r_2} \right)^2 P_{in}$$

e in generale

$$P_1 = \left(\frac{t_1}{1 - r_1 r_2} \right)^2 \frac{P_{in}}{1 + \frac{4\mathcal{F}^2}{\pi^2} \sin^2\left[x/2\right]}$$

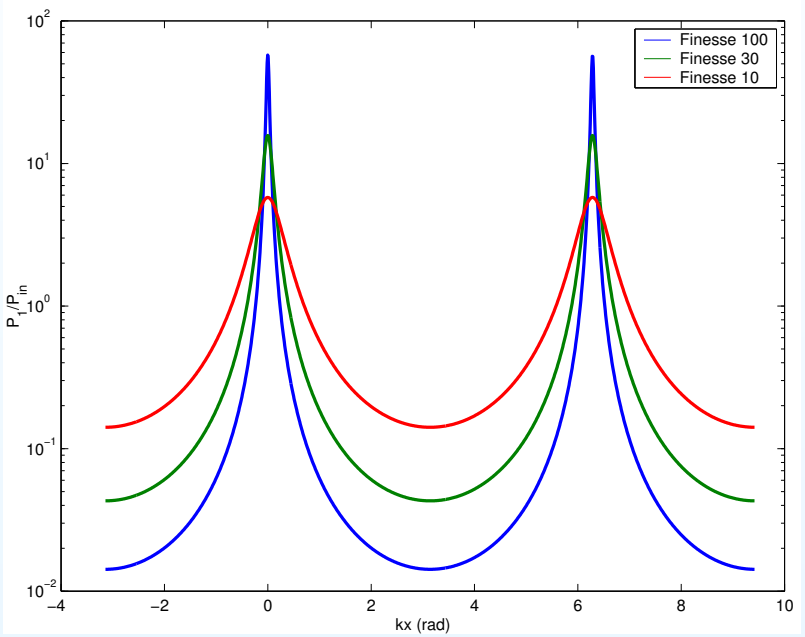
where x is the deviation from resonance expressed in radians and $\mathcal{F}$ defines the **finesse** of the cavity

$$\mathcal{F} = \frac{\pi \sqrt{r_1 r_2}}{1 - r_1 r_2}$$

The various resonances are at

$$\nu_n = \left(n + \frac{1}{2} \right) \frac{c}{2L}$$

Fabry-Perot cavities III



Fabry-Perot cavities IV

The separation between resonances is called **free spectral range** (FSR).
For a 3 km cavity FSR = 50 kHz.
The reflected field E_{out} is:

$$E_{out} = jr_1 E_{in} + jr_2 t_1 \exp[j2kL] E_1 = j\mathcal{R} E_{in}$$

with

$$\mathcal{R} = \frac{r_1 + (1 - p_1) r_2 \exp[j2kL]}{1 + r_1 r_2 \exp[j2kL]}$$

p_1 is the absorption in the cavity, usually very small.
The sensitivity to the phase variation is

$$\frac{\mathrm{d}\Phi}{\delta L} = \frac{8\mathcal{F}}{\lambda}$$

for not too high $\mathcal{F}$ and low absorption.
However this doesn't correspond to the maximum precision, because this corresponds to maximum absorption and therefore little light available for detection.
More complex detection schemes allow to work around this, however one reduces the precision by a factor of 2.

Fabry-Perot cavities V

The phase difference between two cavities leads to the level

$$h^{min} = \frac{\lambda}{8\mathcal{F}L} 2 \sqrt{\frac{h\nu}{\eta P_{in}}} \sqrt{1 + \frac{4\mathcal{F}^2}{\pi^2} \sin^2\left(\frac{\Omega L}{c}\right)}$$

to be compared with the Michelson case

$$h^{min} = \frac{\lambda}{2\pi L} \sqrt{\frac{h\nu}{\eta P_{in}}}$$

At low frequency $\Omega \ll c/L$ the sensitivity gain is

$$\frac{2\mathcal{F}}{\pi}$$

Forr $\mathcal{F} = 450$ the gain is 286, bringing the sensitivity of Advanced Virgo to

$$1.9 \times 10^{-23}\,\mathrm{Hz}^{-1/2}.$$

Power recycling

By increasing the laser power sensitivity can be improved.
However it is possibile to recycle the light coming from the input port of the interferometer, which is intense since we are have a dark fringe at the output port
By positioning another semitransparent mirror before the beam splitter one creates a resonant cavity having the full Michelson with Fabry-Perot arms reflecting light with some phase.
By positioning the recycling mirror one can build another ”Fabry-Peor” cavity in which light power can build up.
The total absorption in the optical system will ultimately limit to the available circulating power
Recycling gains $C = 30 - 50$ are achieve.
La sensibilità risultante con questi dati è data da

$$h^{min} = \frac{\lambda}{4\mathcal{F}L}\sqrt{\frac{h\nu}{\eta C P_{in}}}\sqrt{1 + \frac{4\mathcal{F}^2}{\pi^2}\sin^2\left(\frac{\Omega L}{c}\right)}$$

Standard quantum limit

Fluctuations in the radiation pressure introduce position variations that increase with intensity
On the other hand a high circulating power is required to achieve high sensitivity.
Reflected photons introduce a momentum change of the mirror $M_m \Delta v = 2h\nu$
For an incident power on the mirror P_{in}, the number of photons per second is

$$N_\gamma = \frac{P_{in}}{h\nu}$$

which is equal to its variance (Poisson distribution) in one second
The variance of the acceleration of the mirror is

$$\mathrm{var}(F) = \left(\frac{2h\nu}{M_m}\right)^2 \frac{P_{in}}{h\nu} = \frac{4P_{in}h\nu}{M_m^2}$$

resulting in a position noise spectrum

$$S_x(f) = \frac{4P_{in}h\nu}{16\pi^4 M_m^2 f^4}$$

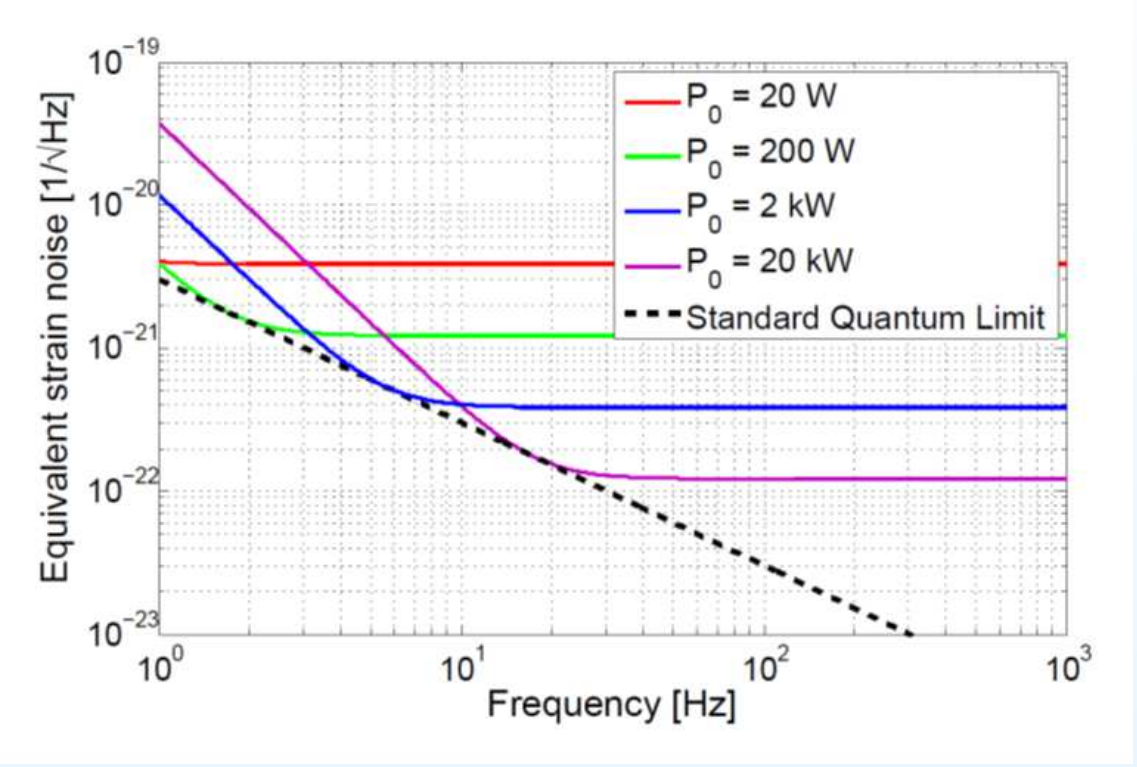
which is frequency dependent.
On the other hand shot noise leads to a position measurement error spectrum

$$S_{x^s} = \frac{c^2}{8\pi^2}\frac{h\nu}{P_{in}}$$

which is frequency independent.

Standard quantum limit II

Varying P_in, there is an envelope that sets a lower noise limit



Quantum noise curves for several values of P_{in} and the Standard Quantum Limit (M. Bloom Thesis)

Null instrument

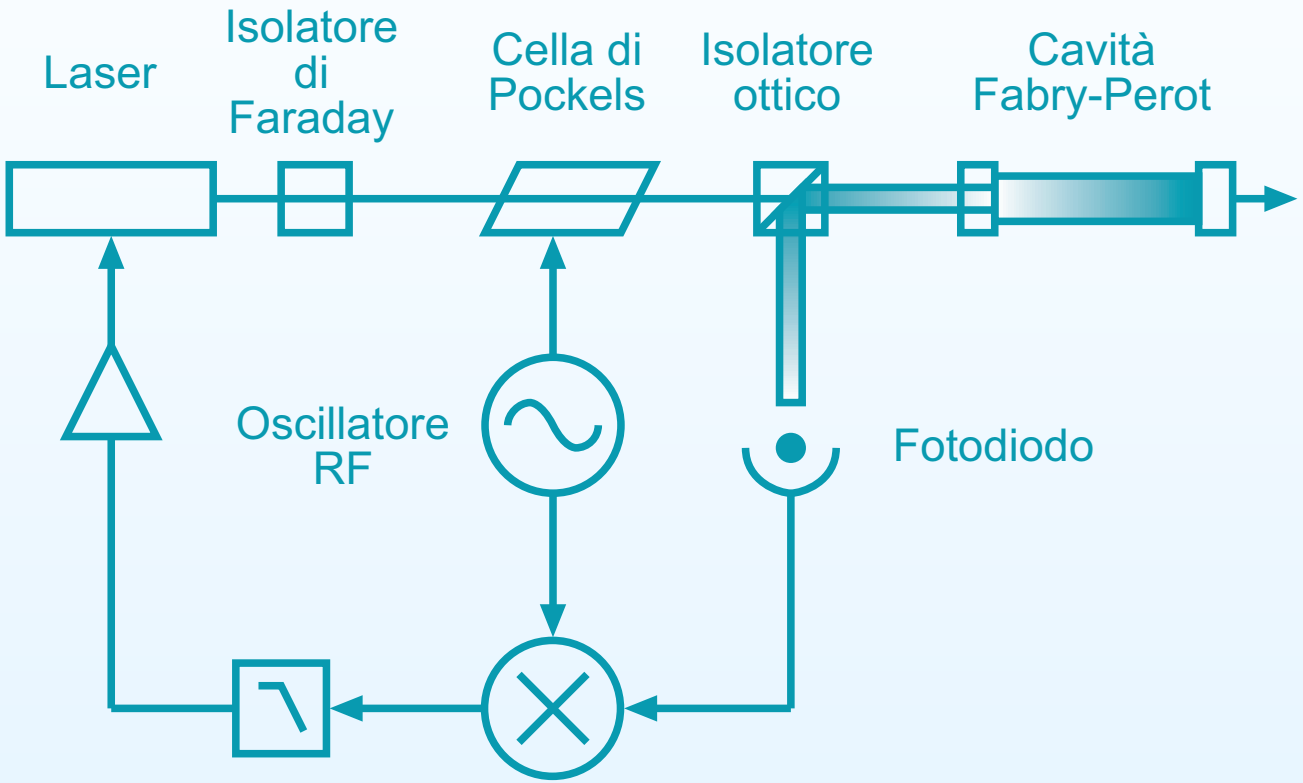
Pound-Drever-Hall technique

The Pound-Drever-Hall technique is used in gw interferometers to interrogate resonant cavities through phase shifts rather that through the intensity, which is quadratic in the field.
La derivative

$$\frac{\mathrm{d}I}{\mathrm{d}\phi}$$

can be obtained through the Pound-Drever-Hall technique
This was used to stabilize a laser in frequency against a reference Fabry-Perot cavity
R. V. Pound, Electronic Frequency Stabilization of Microwave Oscillators, Rev. Sci. Instrum. 17 (1946) 490-505
R. W. P. Drever et al., Laser phase and frequency stabilization using an optical resonator, Appl. Phys. B: Photophys. Laser Chem. 31 (1983) 97-105
Eric D. Black, An introduction to Pound-Drever-Hall laser frequency stabilization, Am. J. Phys. 69 (2001) 79-87

Pound-Drever-Hall technique II



Pound-Drever-Hall technique III

The reflection coefficient is a function of frequency

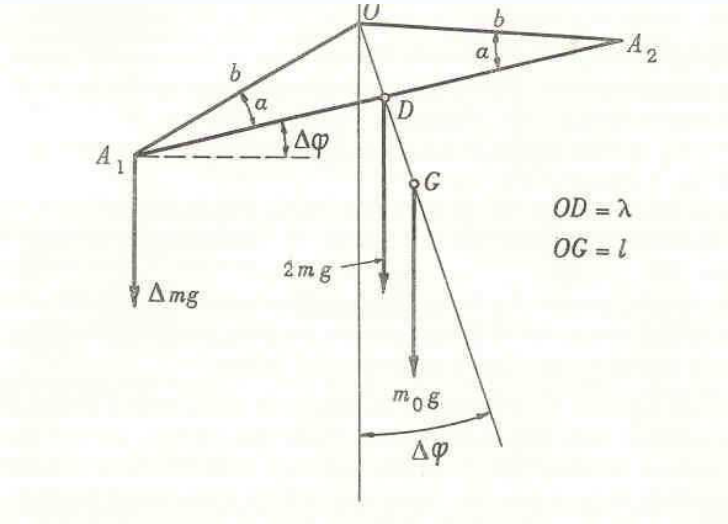
$$F(\omega) = \frac{E_{out}}{E_{in}} = \frac{r\left(\exp[i\omega/\Delta\nu_{fsr}] - 1\right)}{1 - r^2\exp[i\omega/\Delta\nu_{fsr}]}$$

with $\Delta\nu_{fsr} = c/2L$ FSR.
By modulating in phase the laser

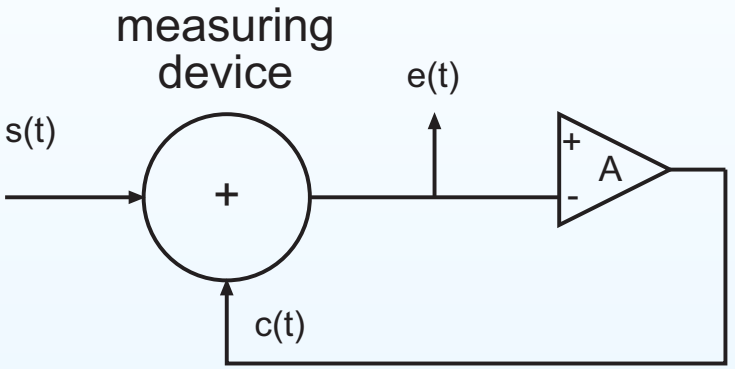
$$\begin{aligned} E_{in} &= E_0\exp[j\omega t + \beta\sin\Omega t] \\ &\simeq E_0\{J_0(\beta)\exp[j\omega t] + J_1(\beta)\exp[j(\omega+\Omega)t] + J_1(\beta)\exp[j(\omega-\Omega)t]\} \end{aligned}$$

These are three beams superimposed of different frequency. The central beam (carrier) resonates in the cavity while the other two are reflected immediately
There is an interference between these beams generating an intensity modulation of the light at Ω (e also 2 Ω, ...)
The error signal is obtained as the sidebands contain the instantaneous laser frequency while the carrier resonates at the cavity frequency.

The null instrument



Balance scale: deviation and zeroing of $\Delta\varphi$.



Equivalent system.

Keeping the instrument in a fixed state reduces systematic errorss
Around an equilibrium position these dependencies will be quadratic.

The null instrument II

For a simple case the state of the system, that is the error signal $e(t)$ is:

$$e(t) = s(t) + c(t)$$

where $s(t)$ is the signal to be measured and $c(t)$ the correction signal.
Here a linear system is assumed.
The closed loop equation is:

$$-Ae(t) + s(t) = e(t)$$

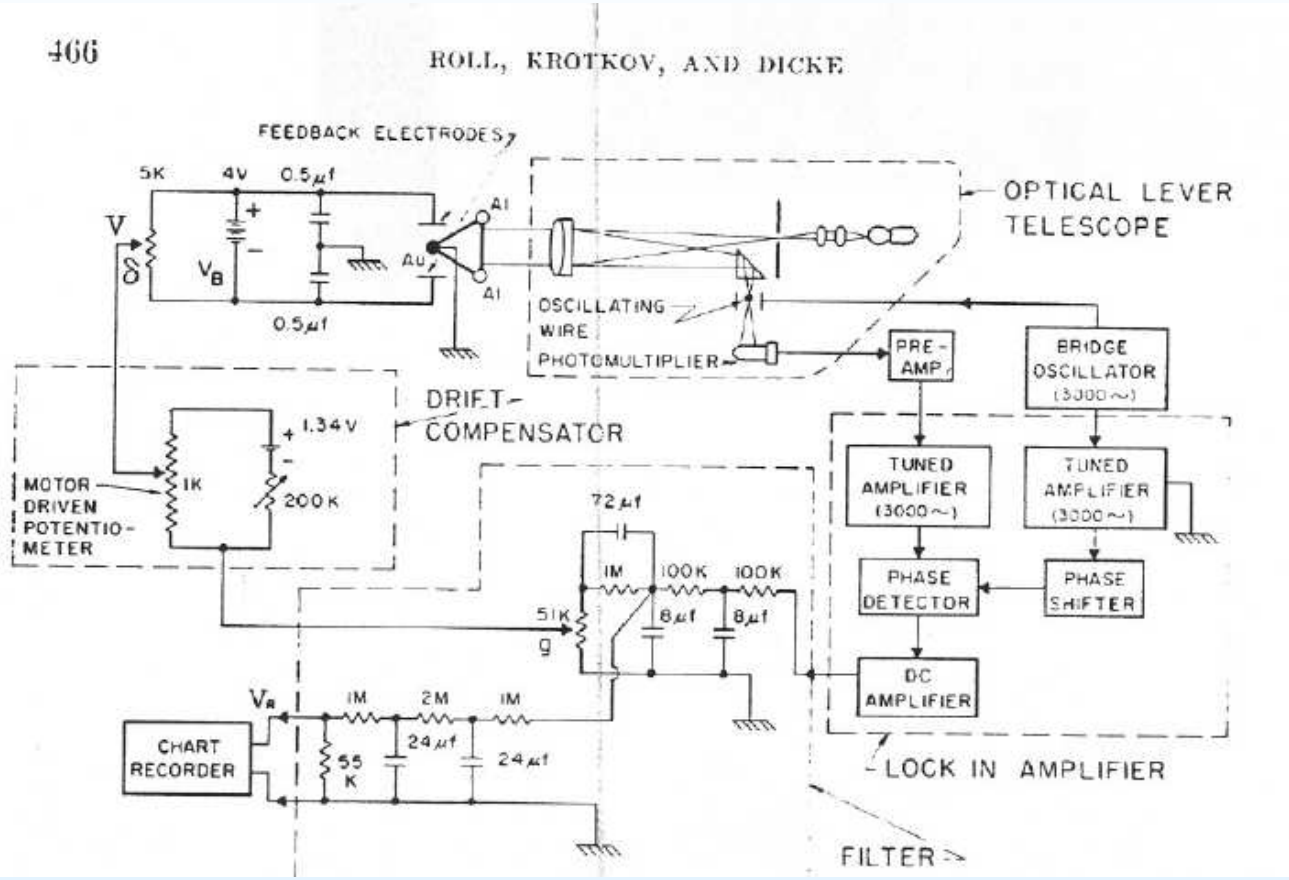
yielding

$$e(t) = \frac{s(t)}{A+1}, \quad c(t) = -\frac{A}{A+1}s(t).$$

For large A $e(t) \simeq 0$ while $-c(t)$ measures $s(t)$.

Null measurement

The experiment of Roll, Krotkov and Dicke.



Feedback in measurement

Summary

Summary

Applied General Relativity is subtle and effects are very small
It is about tracking geodetics, marking the shape of space time
Local effects must be shielded, masses should not move due to seism, thermal energy, gravity, ...
The most precise techniques are needed to perform the measurement: interferometry using an ultrastable unit of length, making comparisons rather than absolute measurements, using powerful optical techniques
Performing a null measurement to be always at the top of precision
All this has been shown to be possible

It is up to you to progress!