Introduction to General Relativity and Gravitational Waves

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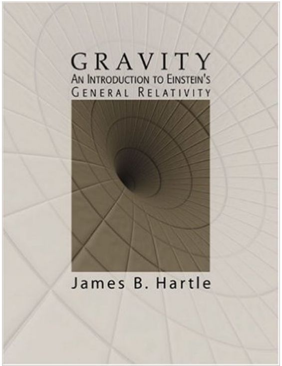
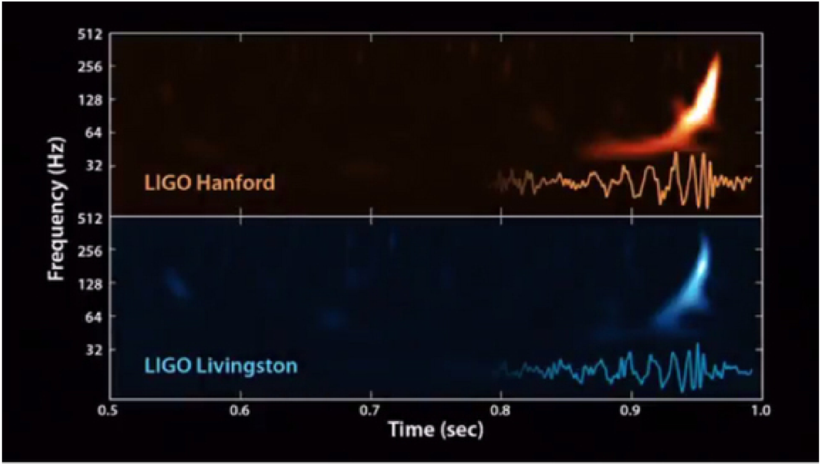
Suggested reading

- James B. Hartle, *Gravity: An Introduction to Einstein's General Relativity*,
- Eanna E. Flanagan and Scott A. Hughes, *The basics of gravitational wave theory*, New J.Phys. 7 (2005) 204
- *Living Reviews in Relativity* – an open-access online journal of invited reviews

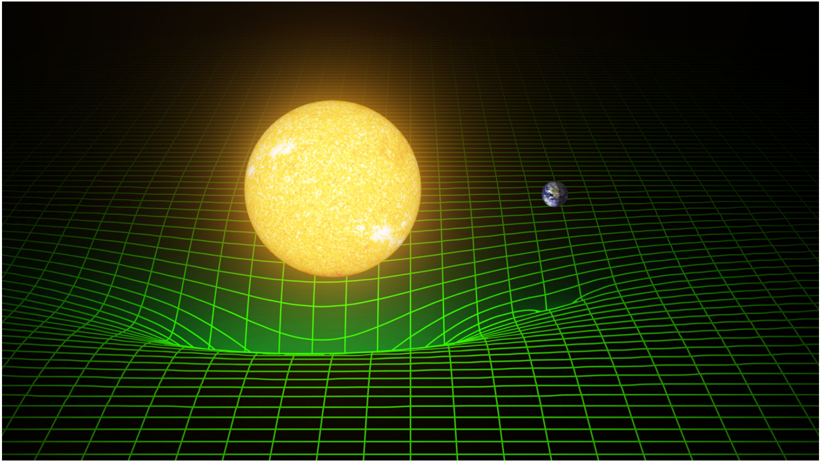


GW150914



Frequency (Hz)
512
256
128
64
32
LIGO Hanford
512
256
128
64
32
LIGO Livingston
0.5
0.6
0.7
0.8
0.9
1.0
Time (sec)

GR: Gravity as Geometry



Geometry = measuring distances

- Pythagoras's formula: the line element of flat space.

$$
\begin{aligned}
ds^2 &= dx^2 + dy^2 + dz^2 \\
&= (dx\ dy\ dz) \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} dx \\ dy \\ dz \end{pmatrix}
\end{aligned} \tag{1}
$$

- Flat spacetime.

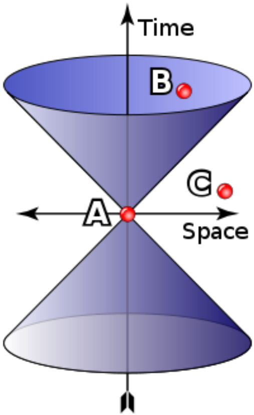
$$
\begin{aligned}
ds^2 &= -(c\,dt)^2 + dx^2 + dy^2 + dz^2 \\
&= (cdt\ dx\ dy\ dz) \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} cdt \\ dx \\ dy \\ dz \end{pmatrix}
\end{aligned} \tag{2}
$$

- Note: *Everyone* uses units in which $c = 1$ (and $G = 1$).

Minkowski spacetime

Types of spacetime interval:

$$ds^2 \begin{array}{lll} > & 0 & \rightarrow \text{ spacelike } (\text{A} - \text{C}) \\ = & 0 & \rightarrow \text{ null } (\text{light cones}) \\ < & 0 & \rightarrow \text{ timelike } (\text{A} - \text{B}) \end{array}$$



Key physical concept: proper time along a worldline.

$$\tau = \int d\tau \equiv \int \sqrt{-ds^2} \qquad (3)$$

The proper time is the time elapsed as measured by an observer moving on that worldline.

Curved spacetime

- Switch from Cartesian to general coordinates $x^\alpha = (x^0, x^1, x^2, x^3)$ with line element

$$ds^2 = \begin{pmatrix} dx^0 & dx^1 & dx^2 & dx^3 \end{pmatrix} \begin{pmatrix} g_{00} & g_{01} & g_{02} & g_{03} \\ g_{10} & g_{11} & g_{12} & g_{13} \\ g_{20} & g_{21} & g_{22} & g_{23} \\ g_{30} & g_{31} & g_{32} & g_{33} \end{pmatrix} \begin{pmatrix} dx^0 \\ dx^1 \\ dx^2 \\ dx^3 \end{pmatrix} \qquad (4)$$

- The matrix $g_{\mu\nu}(x^\alpha)$ is called the metric. Properties:
  - symmetric: : $g_{\mu\nu} = g_{\nu\mu}$
  - a function of position in spacetime: $g_{\mu\nu} = g_{\mu\nu}(x^\alpha)$.
  - All information on the geometry of the spacetime is contained in the metric.
- The inverse matrix is denoted with raised indices: $g^{\mu\nu} \equiv (g_{\mu\nu})^{-1}$.

Coordinate transformations

- There are no preferred coordinates in General Relativity.
- Spacetime intervals ds^2 are invariant under coordinate transformations.
- **Exercise:** Use the invariance of ds^2 to show that under the coordinate transformation $x^\alpha \to x'^\alpha$ the metric transforms as

$$g'_{\alpha\beta} = g_{\mu\nu} \frac{\partial x^\mu}{\partial x'^\alpha} \frac{\partial x^\nu}{\partial x'^\beta} \tag{5}$$

- **Exercise:** Use eqn (5) to show that the line element of flat spacetime in spherical coordinates is

$$ds^2 = -dt^2 + dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2) \tag{6}$$

Examples

- Schwarzschild spacetime: a non-rotating, uncharged black hole of mass M

$$ds^2 = -\left(1 - \frac{2M}{r}\right) dt^2 + \left(1 - \frac{2M}{r}\right)^{-1} dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2) \tag{7}$$

- Friedmann-Lemaitre-Robertson-Walker spacetime: a homogeneous isotropic universe with scale factor $a(t)$

$$ds^2 = -dt^2 + a^2(t)\left[\frac{dr^2}{1 - kr^2} + r^2(d\theta^2 + \sin^2\theta d\phi^2)\right] \tag{8}$$

Physical Consequences of Curved Spacetime

- Hypothesis: freely falling test masses move along worldlines of extremal proper time.

$$\tau = \int \sqrt{-ds^2} = \int \sqrt{-g_{\mu\nu} dx^\mu dx^\nu} \tag{9}$$

- **Exercise:** Show that the Euler-Lagrange equations become this geodesic equation:

$$\frac{d^2 x^\alpha}{d\tau^2} + \Gamma^\alpha_{\beta\gamma} \frac{dx^\beta}{d\tau} \frac{dx^\gamma}{d\tau} = 0 \tag{10}$$

where the Christoffel symbols are

$$\Gamma^\alpha_{\beta\gamma} = \frac{1}{2} g^{\alpha\delta} \left(\partial_\beta g_{\gamma\delta} + \partial_\gamma g_{\delta\beta} - \partial_\delta g_{\beta\gamma} \right) \tag{11}$$

and ∂_α is short-hand for $\partial / \partial x^\alpha$.

Einstein Field Equations

- The metric components $g_{\mu\nu}(x^\alpha)$ are determined by the Einstein field equations,

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R + \Lambda g_{\mu\nu} = 8\pi T_{\mu\nu} \,, \tag{12}$$

where:

  - $T_{\mu\nu}$ is the stress-energy-momentum tensor that describes all of the matter and fields in the spacetime;
  - Λ is the cosmological constant;
  - $R_{\mu\nu}$ and R are the Ricci tensor and Ricci scalar,

$$R = g^{\mu\nu} R_{\mu\nu} \,, \quad R_{\mu\nu} = R^{\alpha}{}_{\mu\alpha\nu} \,; \tag{13}$$

  - $R^{\mu}{}_{\nu\alpha\beta}$ is the Riemann tensor:

$$R^{\mu}{}_{\nu\alpha\beta} = \frac{\partial}{\partial x^\alpha} \Gamma^{\mu}{}_{\nu\beta} - \frac{\partial}{\partial x^\beta} \Gamma^{\mu}{}_{\nu\alpha} + \Gamma^{\mu}{}_{\lambda\alpha} \Gamma^{\lambda}{}_{\nu\beta} - \Gamma^{\mu}{}_{\lambda\beta} \Gamma^{\lambda}{}_{\nu\alpha} \,. \tag{14}$$

- **Exercise:** Show that in vacuum ($T_{\mu\nu} = 0$) the Einstein equations reduce to

$$R_{\mu\nu} = 0\,. \qquad (15)$$

- The Einstein equations are a set of 10 coupled, non-linear, second-order, hyperbolic-elliptic partial differential equations for the metric components $g_{\alpha\beta}$.
- There is no systematic way to solve such systems. Very few analytic solutions exist. These correspond to situations with a high degree of symmetry.
  - E.g.: The exact solution for the two-body problem is not known.
- Analytic solutions exist for the Einstein equation linearised around flat spacetime; e.g., for spacetimes describing weak static gravitational fields or weak gravitational waves.

Linearised gravity

- A weak gravitational field in GR is a spacetime for which there exist global coordinates x^α such that

$$g_{\alpha\beta} = \eta_{\alpha\beta} + h_{\alpha\beta} \, , \quad |h_{\alpha\beta}| \ll 1 \tag{16}$$

  where $\eta_{\alpha\beta} = \mathrm{diag}(-1, 1, 1, 1)$ are the components of the Minkowski metric. Thus, a weak gravitational field differs only slightly from flat spacetime. The quantities $h_{\alpha\beta}$ are perturbations or deviations of the metric away from flat spacetime.

A word about coordinate transformations:

- It is always possible to find coordinates for which the above decomposition is not valid—e.g., flat spacetime in spherical polar coordinates does not satisfy (16), even though the gravitational field is identically zero!
- The set of coordinates x^α in which (16) holds is not unique. It is possible to make an infinitesimal coordinate transformation $x^\alpha \to x'^\alpha$ for which the decomposition with respect to the new set of coordinates still holds.
- We'll often refer to these infinitesimal coordinate transformations as gauge transformations.

- Note that for weak gravitational fields, one typically raises and lowers indices with the background Minkowski metric $\eta^{\alpha\beta}$ and $\eta_{\alpha\beta}$, and not with $g^{\alpha\beta}$ and $g_{\alpha\beta}$. For example,

$$h^{\alpha}{}_{\beta} \equiv \eta^{\alpha\mu} h_{\mu\beta}\,, \quad h^{\alpha\beta} \equiv \eta^{\alpha\mu}\eta^{\beta\nu} h_{\mu\nu}\,. \tag{17}$$

  The only exception is $g^{\alpha\beta}$, which still denotes the inverse of $g_{\alpha\beta}$, not $\eta^{\alpha\mu}\eta^{\beta\nu} g_{\mu\nu}$. To first order,

$$g^{\alpha\beta} = \eta^{\alpha\beta} - h^{\alpha\beta}\,. \tag{18}$$

- **Exercise:** Show that to first order in $h_{\alpha\beta}$ the Riemann tensor has components

$$R_{\alpha\beta\mu\nu} = \frac{1}{2}\left(\partial_\mu\partial_\beta h_{\alpha\nu} - \partial_\mu\partial_\alpha h_{\beta\nu} + \partial_\nu\partial_\alpha h_{\beta\mu} - \partial_\nu\partial_\beta h_{\alpha\mu}\right) \tag{19}$$

- **Exercise:** Show that to first order in $h_{\alpha\beta}$ the Ricci tensor has components:

$$R_{\alpha\beta} = \frac{1}{2}\left(-\Box h_{\alpha\beta} + \partial_\alpha V_\beta + \partial_\beta V_\alpha\right) \tag{20}$$

where

$$\Box := \eta^{\alpha\beta}\partial_\alpha\partial_\beta = -\frac{\partial^2}{\partial t^2} + \nabla^2 \tag{21}$$

is the D'Alembertian (or wave operator) and

$$V_\alpha := \partial_\beta h^\beta{}_\alpha - \frac{1}{2}\partial_\alpha h^\beta{}_\beta \tag{22}$$

- Simplification: It is always possible to find a set of coordinates for which

$$V_\alpha := \partial_\beta h^\beta{}_\alpha - \frac{1}{2}\partial_\alpha h^\beta{}_\beta = 0 \qquad (23)$$

  - This is sometimes called the Loren(t)z condition (in analogy with the gauge condition in electromagnetism).

- Importance: If $V_\alpha = 0$ then the vacuum Einstein equation for a weak gravitational field in this gauge is simply

$$\Box h_{\alpha\beta} = 0 \qquad (24)$$

Thus, the metric perturbations satisfy the flat space wave equation. The solutions can therefore be interpreted as gravitational waves.

Existence Proof for the Lorenz Gauge

- Consider an infinitesimal coordinate transformation

$$x'^{\alpha} := x^{\alpha} + \xi^{\alpha}(x) \tag{25}$$

where ξ^{α} are slowly varying functions: $|\partial_{\alpha}\xi^{\beta}| \ll 1$.

- To first-order, the transformation matrix from x'^{α} to x^{μ} is

$$\frac{\partial x^{\mu}}{\partial x'^{\alpha}} = \delta^{\mu}{}_{\alpha} - \frac{\partial \xi^{\mu}}{\partial x^{\alpha}} \tag{26}$$

- Thus, to first order, the metric components transform as

$$g'_{\alpha\beta} = \frac{\partial x^{\mu}}{\partial x'^{\alpha}} \frac{\partial x^{\nu}}{\partial x'^{\beta}} g_{\mu\nu} = g_{\alpha\beta} - \partial_{\alpha}\xi_{\beta} - \partial_{\beta}\xi_{\alpha} \tag{27}$$

$$h'_{\alpha\beta} = h_{\alpha\beta} - \partial_{\alpha}\xi_{\beta} - \partial_{\beta}\xi_{\alpha} \tag{28}$$

Since $|\partial_{\alpha}\xi_{\beta}| \ll 1$, it follows that $|h'_{\alpha\beta}| \ll 1$, so the new coordinates x'^{α} are also valid coordinates for a weak gravitational field.

- **Exercise:** Show that under this infinitesimal coordinate transformation

$$V'_\alpha = V_\alpha - \Box\xi_\alpha \tag{29}$$

so that

$$V'_\alpha = 0 \quad \Longleftrightarrow \quad \Box\xi_\alpha = V_\alpha \tag{30}$$

- Since $\Box$ is just the wave operator in flat spacetime, one can always find a solution of $\Box\xi_\alpha = V_\alpha$. Thus, if $V_\alpha \neq 0$ in the original coordinates x^α, we can always find new coordinates x'^α for which the Lorenz condition $V'_\alpha = 0$ is satisfied.

- **Exercise:** Show that under an infinitesimal coordinate transformation the components of the Riemann tensor $R_{\mu\alpha\nu\beta}$ given by eqn. (19) are unchanged to first-order.

- This shows that the curvature of a weak-field spacetime, and so any physical predictions such as geodesic deviation, are unchanged to first-order by an infinitesimal coordinate transformation.

Solving the Wave Equation

- The most general solution to $\Box h_{\alpha\beta} = 0$ is a linear combination of sinusoidal plane wave solutions:

$$h_{\alpha\beta} = a_{\alpha\beta} \exp(i\mathbf{k} \cdot \mathbf{x}) \tag{31}$$

where $a_{\alpha\beta}$ and k^α are constants satisfying:

$$\eta_{\alpha\beta} k^\alpha k^\beta = 0\,, \tag{32}$$

$$k_\beta a^\beta{}_\alpha - \frac{1}{2} k_\alpha a^\beta{}_\beta = 0\,. \tag{33}$$

The first condition, from the wave equation, says that a gravitational wave propagates along a null direction (i.e., with the speed of light); the second condition is just the Lorenz condition (23) expressed in terms of $a_{\alpha\beta}$ and k^α.

Transverse traceless gauge

- The Lorenz gauge does not completely fix the coordinates. A further infinitesimal coordinate transformation

$$x^\alpha \to x'^\alpha = x^\alpha + \eta^\alpha(x) \quad (34)$$

with

$$\Box\eta^\alpha = 0 \quad (35)$$

preserves the Lorenz gauge condition.

- We can exploit this additional coordinate freedom to set

$$h'_{ti} = 0\,, \quad \eta^{\alpha\beta}h'_{\alpha\beta} = 0 \quad (36)$$

in these coordinates. Such a choice of coordinates is called the transverse traceless gauge (or TT gauge for short).

- **Exercise:** Consider the infinitesimal coordinate transformation defined by

$$\eta_\alpha = B_\alpha \exp(i\mathbf{k}\cdot\mathbf{x}) \tag{37}$$

where $\mathbf{k}$ is the same null vector as in (31). Show that under this coordinate transformation

$$h_{\alpha\beta} \to h'_{\alpha\beta} = a'_{\alpha\beta} \exp(i\mathbf{k}\cdot\mathbf{x}) \tag{38}$$

with

$$a'_{\alpha\beta} = a_{\alpha\beta} - ik_\alpha B_\beta - ik_\beta B_\alpha \tag{39}$$

- **Exercise:** Explicitly find B_α satisfying the TT gauge conditions

$$a'_{ti} = 0\,, \quad \eta^{\alpha\beta} a'_{\alpha\beta} = 0 \tag{40}$$

[Hint: Contract $a'_{ti} = 0$ with k^i and solve for $B_i k^i$ in terms of B_t; then substitute this expression for $B_i k^i$ into $\eta^{\alpha\beta} a'_{\alpha\beta} = 0$ to solve for B_t; finally, substitute the solution for B_t back into $a'_{ti} = 0$ to find B_i.]

- In the TT gauge, the Lorenz condition eqn. (23) reduces to $\partial_\beta h^\beta{}_\alpha = 0$.
- Thus, in the TT gauge there are 8 conditions on the 10 independent components of $h_{\alpha\beta}$:

$$h_{ti} = 0\,, \quad \eta^{\alpha\beta} h_{\alpha\beta} = 0\,, \quad \partial_\beta h^\beta{}_\alpha = 0 \tag{41}$$

  This leaves only 2 independent components of $h_{\alpha\beta}$.
- In terms of $a_{\alpha\beta}$ and k^α, we have

$$a_{ti} = 0\,, \quad \eta^{\alpha\beta} a_{\alpha\beta} = 0\,, \quad k_\beta a^\beta{}_\alpha = 0 \tag{42}$$

  The remaining two independent components of $a_{\alpha\beta}$ correspond to the two independent polarisation states of a gravitational wave, typically denoted h_+ and $h_\times$.

- For example, take $k^\alpha = (\omega, 0, 0, \omega)$, corresponding to a plane monochromatic gravitational wave with angular frequency ω propagating in the $+z$-direction. Then eqns (42) become

$$a_{ti} = 0\,, \quad a_{zi} = 0\,, \quad a_{tt} = 0\,, \quad a_{xx} + a_{yy} = 0\,. \tag{43}$$

- These show that the perturbations are transverse to the direction of propagation. The metric perturbations $h_{\alpha\beta}$ in the TT gauge are thus

$$h_{\alpha\beta} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & h_+ & h_\times & 0 \\ 0 & h_\times & -h_+ & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{44}$$

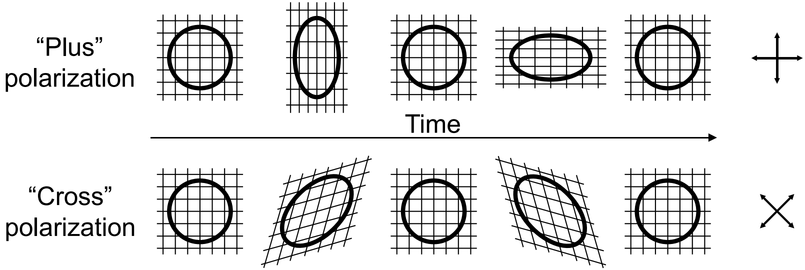
- The corresponding line element is

$$ds^2 = -dt^2 + (1 + h_+)\, dx^2 + (1 - h_+)\, dy^2 + 2h_\times\, dx\, dy + dz^2 \tag{45}$$

- The most general solution of the linearised field equation is a superposition of solutions of the form (44) having different propagation directions, frequencies, and amplitudes for h_+, $h_\times$.

Interpretation: The Effect of GWs

Gravitational waves are deformations of space itself, stretching it first in one direction, then in the perpendicular direction.



Exercise: Consider two particles at rest at $(x, y, z) = (0, 0, 0)$ and $(L, 0, 0)$. A plus polarized gravitational wave of frequency f and amplitude $h_0 \ll 1$ passes by, propagating in the z direction:

$$h_{ab}(t, x, y, z) = h_0 \sin(2\pi f[t - \frac{z}{c}]) \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} . \qquad (46)$$

Show that the distance d measured along the x-axis between the two particles as the wave passes is given by

$$d = \left[1 + \frac{1}{2} h_0 \sin(2\pi f t)\right] L . \qquad (47)$$

Expected sources of gravitational waves

Inspiraling binary systems: Systems that spiral toward one another and eventually coalesce due to the energy lost in GWs.

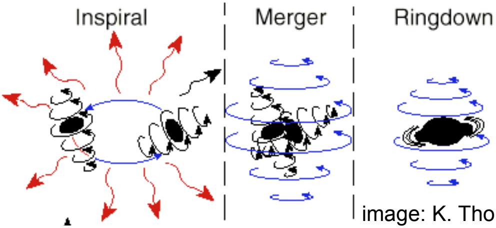
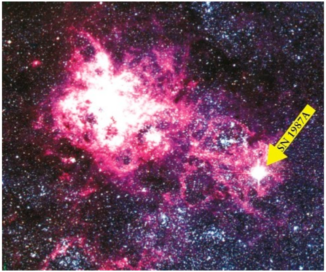


image: K. Thorne

The component objects need to be compact (e.g., neutron stars or black holes) and the inspiral needs to be in its final stages (last few minutes) in order for the GWs to be detectable by Earth-based interferometers.

Unmodeled burst sources: GWs produced by supernovae, gamma ray bursters, or other sources for which we do not know the gravitational waveform. The waveform may be too difficult to calculate due to complicated (or unknown) initial conditions, or numerical relativity has not yet been able to solve the Einstein field equations for the strong-field case of interest.



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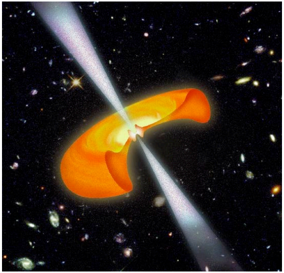
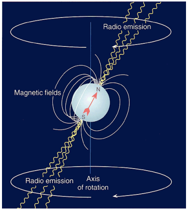
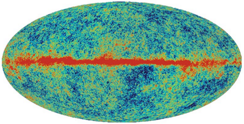


GRB / accreting BH



Periodic sources: Continuous sources of GWs such as pulsars with a non-trivial quadrupole moment (e.g., there is a “mountain” on the surface of a neutron star that is not aligned with the axis of rotation).

Stochastic (random) GWs: Remnant gravitational waves from the big bang, or the superposition of GWs produced by many unresolved astrophysical sources (e.g. distant supernovae or inspiral events).



WMAP 2003 data

Stress-Energy-Momentum Tensor

- Schematically:

$$T^{\alpha\beta} = \left(\begin{array}{c|c} \text{energy} & \text{energy} \\ \text{density} & \text{flux} \\ \hline \text{momentum} & \text{stress} \\ \text{density} & \text{tensor} \end{array} \right) \tag{48}$$

- The stress energy tensor is symmetric: $T^{\alpha\beta} = T^{\beta\alpha}$.
- Momentum density is equivalent to energy flux.
- Conservation law: $\nabla_\alpha T^{\alpha\beta} = 0$.

“Trace-Reversed” Amplitude

When solving the linearised equations in vacuum, it was useful to introduce the Lorenz condition

$$V_\alpha := \partial_\beta h^\beta{}_\alpha - \frac{1}{2}\partial_\alpha h^\beta{}_\beta = 0\,.$$

The equations simplify if we introduce the “trace-reversed” amplitude

$$\bar{h}_{\alpha\beta} \equiv h_{\alpha\beta} - \frac{1}{2}\eta_{\alpha\beta} h^\gamma{}_\gamma\,. \tag{49}$$

Then, the Lorenz condition simplifies to

$$\partial_\beta \bar{h}^\beta{}_\alpha = 0\,. \tag{50}$$

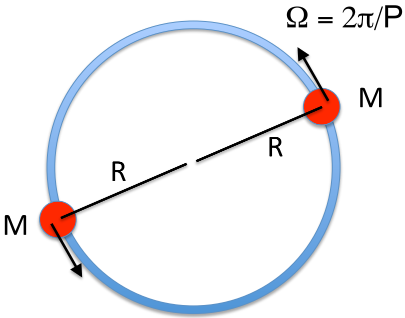
Generation of Gravitational Waves

- With a source term, the linearised Einstein equation is
$$\Box \bar{h}^{\alpha\beta} = -16\pi T^{\alpha\beta} \tag{51}$$

- Using the Green's function for the d'Alembertian gives
$$\bar{h}^{\alpha\beta}(t,\vec{x}) = 4\int \frac{T^{\alpha\beta}(t-|\vec{x}-\vec{x}'|,\vec{x}')}{|\vec{x}-\vec{x}'|} d^3x' \tag{52}$$
$$\sim \frac{4}{r}\int T^{\alpha\beta}(t-r,\vec{x}') d^3x' \tag{53}$$
where $r = |\vec{x}|$.

- **Exercise:** Using the conservation law for the stress tensor, $\nabla_\beta T^{\alpha\beta} = 0$, show that the spatial components are
$$h^{ij}(t,\vec{x}) \sim \frac{2}{r}\frac{d^2}{dt^2}\int \rho(t-r,\vec{x}')\, x'^i x'^j\, d^3x' \,, \tag{54}$$
where $\rho = T^{00}$ is the mass-energy density of the source.

Example: Binary Systems



Order-of-magnitude estimate of GW amplitude:

$$I \sim 2MR^2 \quad (55)$$
$$\ddot{I} \sim 2MR^2\Omega^2 \quad (56)$$

Kepler's third law for a circular binary:

$$M_1 + M_2 = \Omega^2 (R_1 + R_2)^3 \quad (57)$$

$$h \sim \frac{M^2}{rR} \sim \frac{M^{5/3}}{r} \left(\frac{4\pi}{P}\right)^{2/3} \quad (58)$$

It can be shown that the dominant frequency of the GWs is twice the orbital frequency, $f_{\mathrm{GW}} = 2f_{\mathrm{orbit}} = 2/P$.

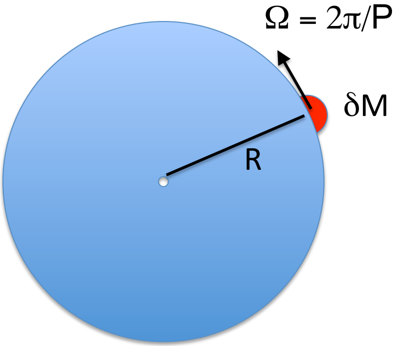
- **Exercise:** For a neutron-star binary ($M \simeq 1.4M_{\odot}$) at 5 kpc with $P = 1$ hr show that $h \sim 10^{-22}$.
- **Exercise:** For the same system with $P = 0.02$ s (giving $f_{\rm GW} = 2f_{\rm orbit} = 100$ Hz, in the sensitive band of LIGO) show that $h \sim 10^{-22}$ at a distance of 15 Mpc – approximately the distance of the Virgo cluster of galaxies.
- **Exercise:** Show the orbital separation $R \sim 100$ km when $P = 0.02$ s. Thus, we can only hope to detect inspirals of compact binary systems (e.g., NS-NS, NS-BH, or BH-BH) with Earth-based interferometers like LIGO.

Example: Distorted Pulsar

Consider a spinning neutron star of radius R with a non-spherical deformation ("mountain") of mass δM on the equator. If the angular velocity is Ω, then

$$I \sim \delta M R^2 \quad (59)$$

$$\ddot{I} \sim \delta M R^2 \Omega^2 \quad (60)$$



The GW amplitude is approximately

$$h \sim \frac{2\delta M R^2 \Omega^2}{r} \quad (61)$$

Exercise: For a star at 1 kpc with $\delta M = 10^{-6} M_\odot$, a spin frequency of 50 Hz, and a stellar radius of 10 km, show that the GW amplitude at Earth is $h \sim 10^{-26}$.

Energy in GWs

- The energy flux (power/area) or energy density in GWs can be estimated using the formula

$$F \sim \frac{c^3}{32\pi^2 G} |\dot{h}|^2 \sim \frac{c^3}{8G} h^2 f^2 \qquad (62)$$

where $f = \omega/2\pi$ is the frequency of the GW (assumed monochromatic) and h is the RMS amplitude.

  - Recall that energy flux or energy density in electromagnetism is $\propto |\vec{E}|^2 + |\vec{B}|^2$. In GR, the metric components play the role of gravitational potential, so their derivatives play the role of the field; hence $F \sim |\dot{h}|^2$.

- **Exercise:** Show that

$$\frac{c^5}{G} = 3.63 \times 10^{52} \text{ Watts} \qquad (63)$$

This equals 1 in geometric units ($c = 1 = G$).

- **Exercise:** GW150914 had a peak amplitude of $h \simeq 10^{-21}$ at $f \simeq 200\,\mathrm{Hz}$. Show that the corresponding energy flux is

$$F \sim \times 10^{-3}\ \frac{\mathrm{W}}{\mathrm{m}^2} \qquad (64)$$

This is approximately the energy flux in electromagnetic waves that we receive from the full moon – despite GW150914 being at an estimated distance of $\sim 400\,\mathrm{Mpc}$!

A final word: GWs vs. EM waves

Electromagnetic waves	Gravitational Waves
Accelerating charge	Accelerating aspherical mass
Wavelength small compared to sources ➔ images	Wavelength large compared to sources ➔ no spatial resolution
Absorbed, scattered, dispersed by matter	Very small interaction; matter is transparent
10 MHz and up	10 kHz and down

Very different information, mostly mutually exclusive.

Difficult to predict GW sources based on EM observations.