The Townes soliton … and beyond

The rich physics of non-miscible Bose mixtures

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Course on "Quantum Mixtures with Ultra-cold Atoms"

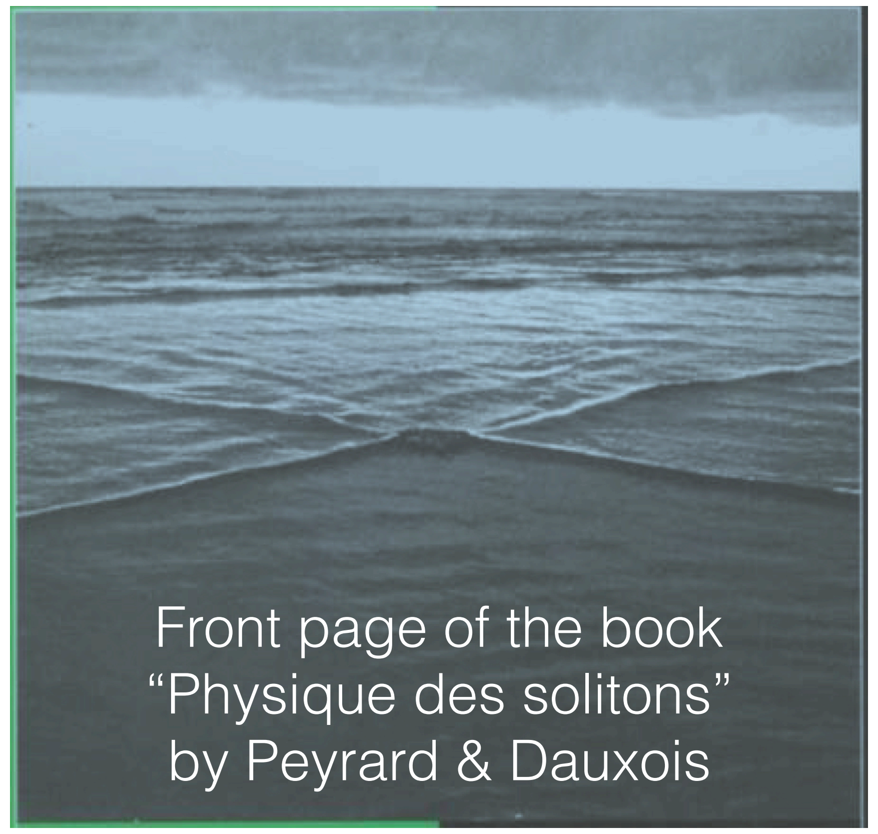
The general goal of these lectures

Start from the concept of "soliton" in a 2D system

Describe its implementation in a binary mixture of quantum gases

Study the transition from the solitonic to a droplet regime, and compare it with "quantum droplets"

http://www.ma.hw.ac.uk/solitons/soliton1b.html



Front page of the book "Physique des solitons" by Peyrard & Dauxois

No significant "beyond mean-field physics" in this lecture

Outline of Lecture 1

1. Solitons in 2D ?

    The constraints imposed by scale invariance

2. The Townes soliton

    Arbitrary size, but a single possible atom number

3. The binary mixture approach to the Townes soliton

    Evolution of a minority component inside an infinite bath

4. A first look at experimental results

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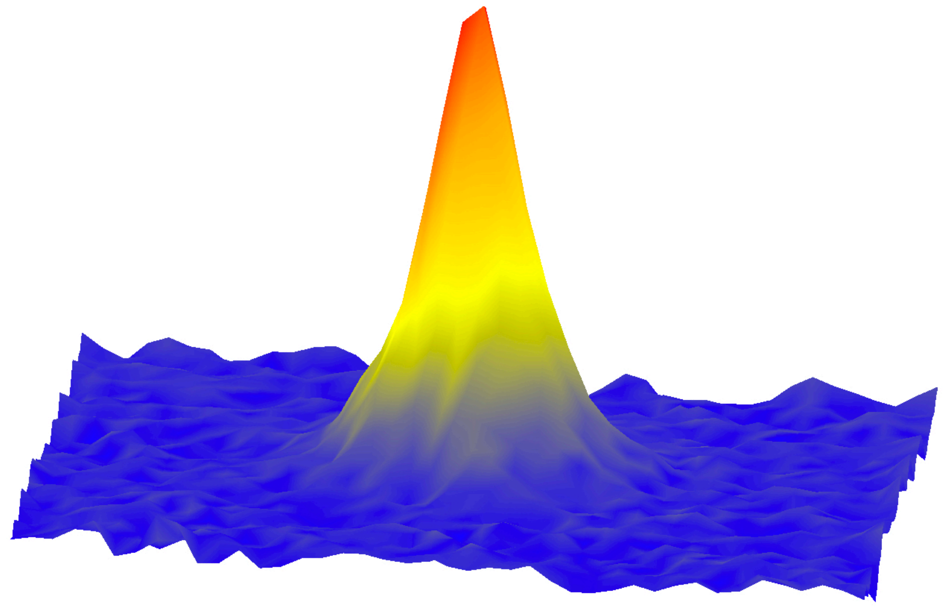
The constraints imposed by scale invariance



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Solitons for the Gross-Pitaevskii equation

Wikipedia: a soliton is a self-reinforcing wave packet that maintains its shape while it propagates

Cancellation of nonlinear and dispersive effects in the medium

Stationary wave function solution of the variational problem $\delta\left[E(\psi)\right] = 0$ for an attractive non-linearity $g < 0$

$$E[\psi] = \frac{1}{2}\int\left(\left|\nabla\psi\right|^2 + g\left|\psi\right|^4\right)\,\mathrm{d}^D r \qquad \int|\psi|^2 = N \qquad \hbar = m = 1$$

Relevant in optics, atomic physics, condensed matter…

Dimensional analysis for a wave packet of size ℓ: $|\psi|^2 \sim \frac{N}{\ell^D}$ $\qquad \frac{E(\ell)}{N} \sim \frac{1}{\ell^2} - \frac{N|g|}{\ell^D}$

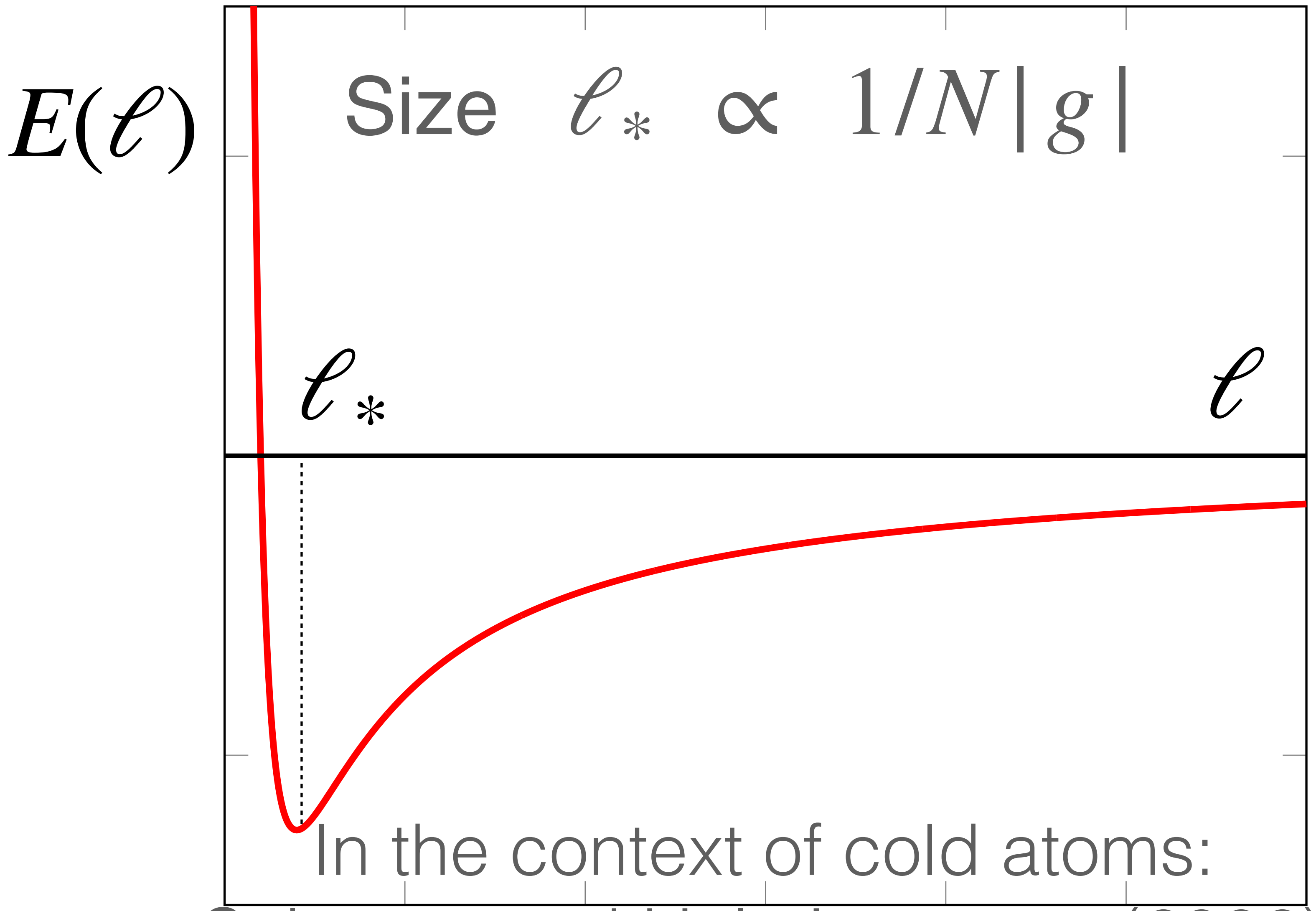
Crucial role of dimensionality

Solitons in 1D and 3D

$g < 0$

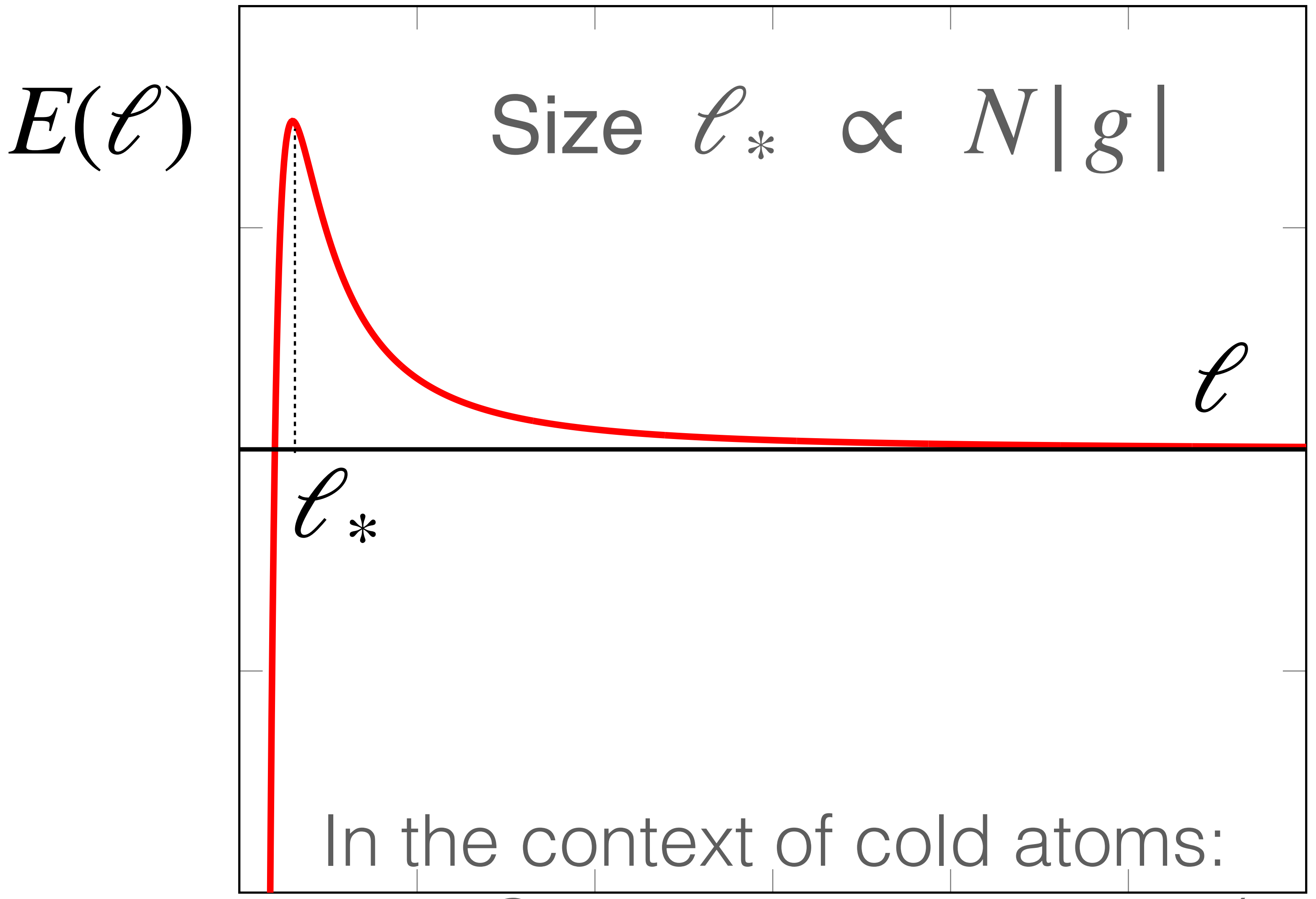
Dimensional analysis for a wave packet of size ℓ: $$\frac{E(\ell)}{N} \sim \frac{1}{\ell^2} - \frac{N|g|}{\ell^D}$$

In 1D: Stable solution for any N and any g



In the context of cold atoms:
Salomon and Hulet's groups (2002)

In 3D: Dynamically unstable extremum



In the context of cold atoms:
BoseNova: Cornell-Wieman group (2001)

Solitons in 2D

$g < 0$ $\int |\psi|^2 = N$

Dimensional analysis for a wave packet of size ℓ in two dimensions:

$$E[\psi] = \frac{1}{2}\int \left(\left|\nabla\psi\right|^2 + g\left|\psi\right|^4 \right) \mathrm{d}^2 r \longrightarrow \frac{E(\ell)}{N} \sim \frac{1}{\ell^2} - \frac{N|g|}{\ell^2}$$

2D is a critical dimension:

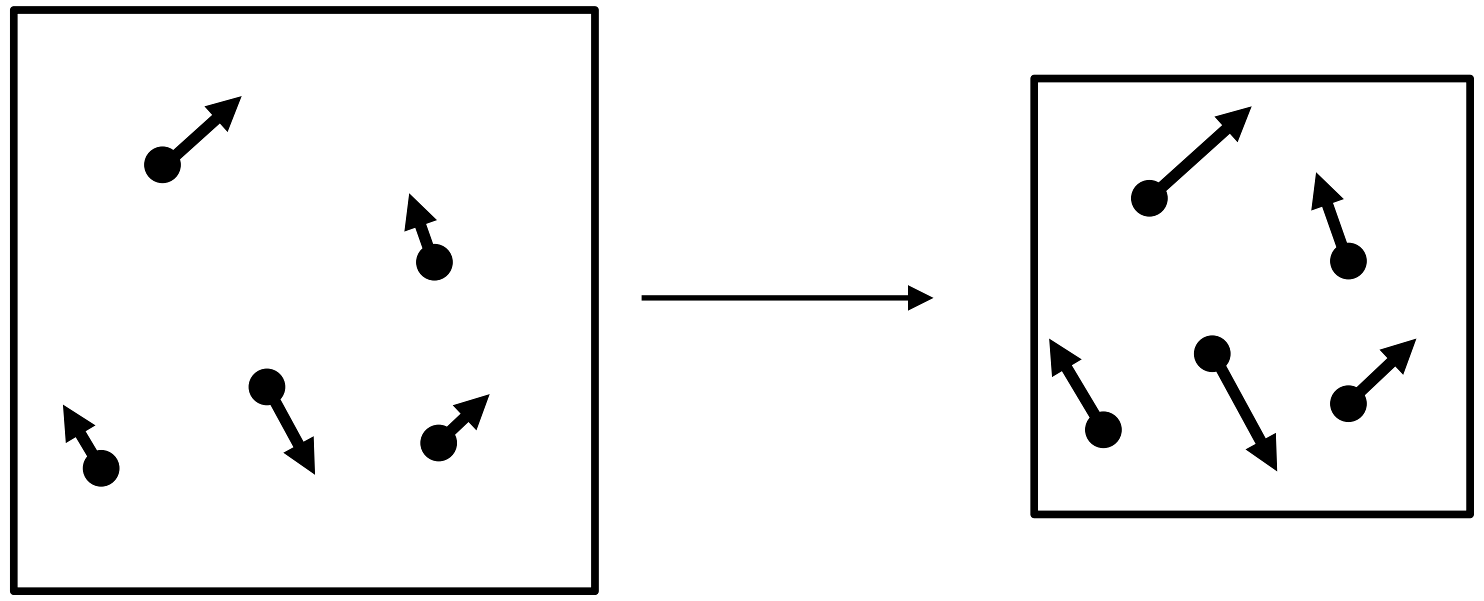
- Stationary solutions can be expected only for discrete values of $N|g|$
- For such a value of $N|g|$, no length scale emerges from the minimization of $E[\psi]$

A manifestation of scale invariance

Scale invariant fluids

Consider a fluid whose equations of motion, *i.e.* its action $\int E \, \mathrm{d}t$, are invariant in the following rescaling:

Positions: $\boldsymbol{r} \;\to\; \boldsymbol{r}/\lambda$ Time: $t \;\to\; t/\lambda^2$



Velocity: $\boldsymbol{v} \;\to\; \lambda \boldsymbol{v}$

Considerable simplification of the study of equilibrium properties and dynamics

Clearly $E_{\mathrm{kin}} \to \lambda^2 E_{\mathrm{kin}}$, implying that $\int E_{\mathrm{kin}} \, \mathrm{d}t$ is invariant

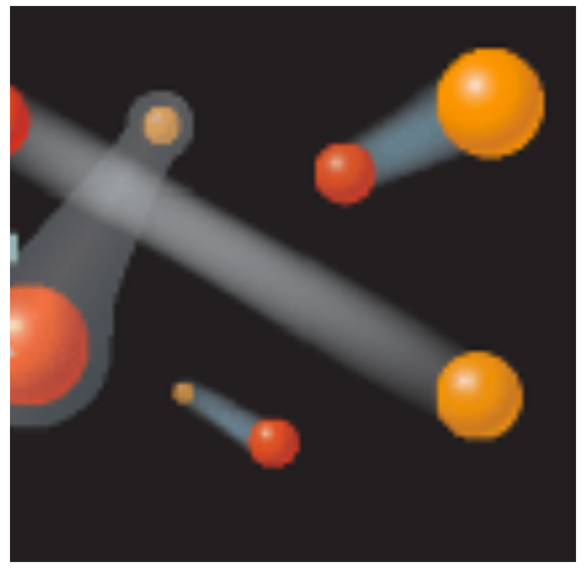
What about interactions? Can we achieve $E_{\mathrm{int}} \to \lambda^2 E_{\mathrm{int}}$ when $\boldsymbol{r} \to \boldsymbol{r}/\lambda$?

Cold atomic gases with scale invariant interactions

$\boldsymbol{r} \to \boldsymbol{r}/\lambda \qquad E_{\text{int}} \to \lambda^2 E_{\text{int}}$

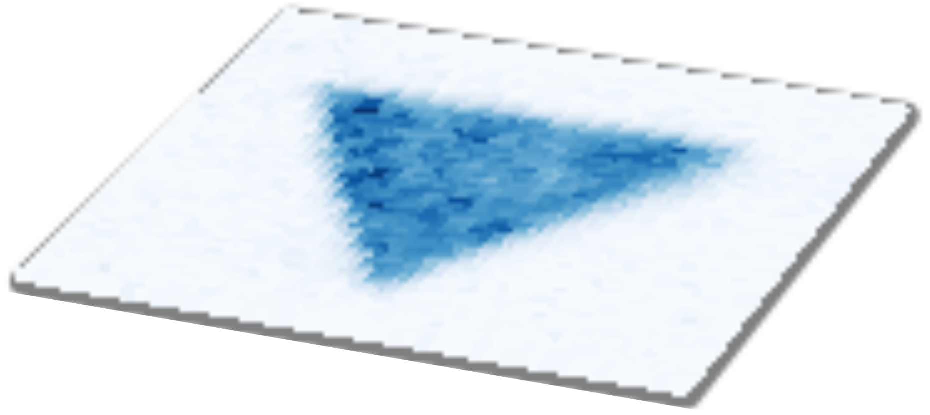
- An interaction potential varying as $V(r) = \dfrac{g}{r^2}$: emerges in some specific situations (Efimov)

- 3D Fermi gas in the unitary regime (infinite scattering length, hence no length scale associated to interactions)



- Contact interaction in a 2D Bose gas:

$$\boldsymbol{r} \to \boldsymbol{r}/\lambda \qquad g\,\delta(\boldsymbol{r}) \to g\,\delta(\boldsymbol{r}/\lambda) = \lambda^2\, g\,\delta(\boldsymbol{r})$$

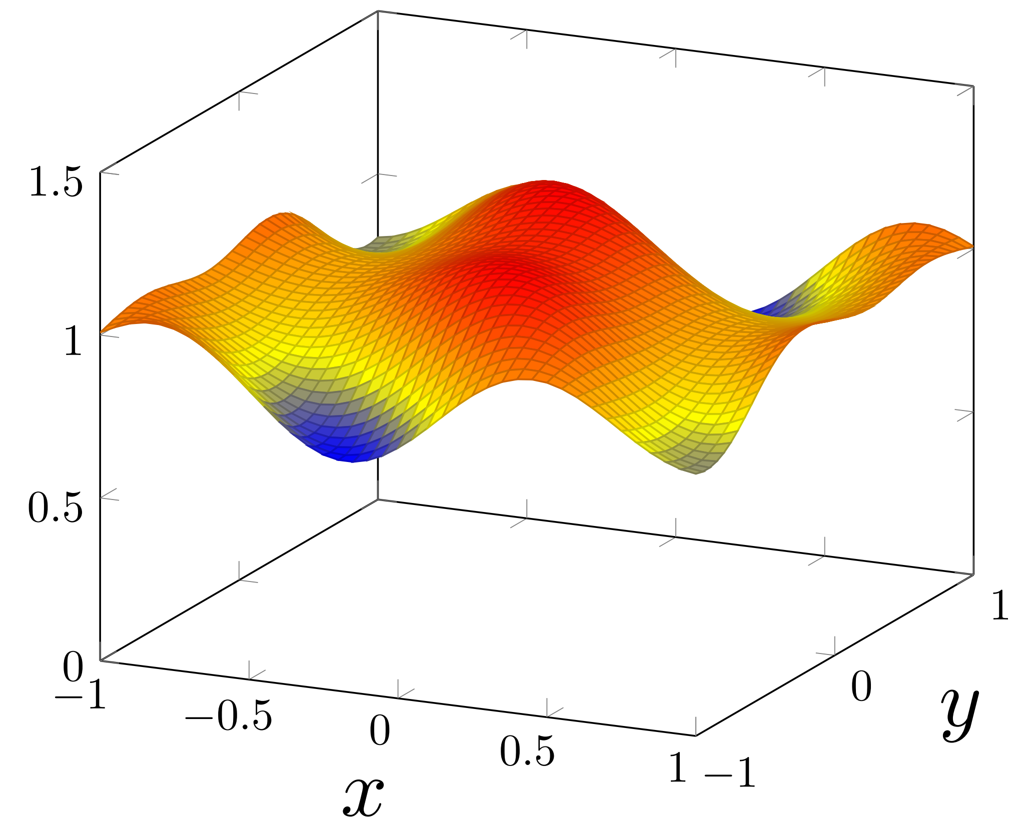


Valid only for relatively weak interactions, so that a classical field description (Gross-Pitaevskii equation) is valid (otherwise, quantum anomaly from the regularisation of $\delta(\boldsymbol{r})$ *)*

Classical field approach to the 2D Bose gas

Describe the gas by a classical field $\psi(\boldsymbol{r}, t)$ obeying the Gross-Pitaevskii equation

Energy of the gas: $E(\psi) = E_{\mathrm{kin}}(\psi) + E_{\mathrm{int}}(\psi)$



$$E_{\mathrm{kin}}(\psi) = \frac{\hbar^2}{2m} \int |\boldsymbol{\nabla}\psi|^2 \qquad E_{\mathrm{int}}(\psi) = \frac{\hbar^2}{2m} \tilde{g} \int |\psi|^4$$

$\tilde{g}$: interaction strength

No singularity at the classical field level

In 3D, $\tilde{g} = 4\pi a^{(3D)}$ where $a^{(3D)}$ is the scattering length

In 2D, the interaction strength $\tilde{g}$ is dimensionless: no length scale associated with interactions

$$\tilde{g} = \sqrt{8\pi} \frac{a^{(3D)}}{\ell_z}$$

Frozen direction z : $\ell_z = \sqrt{\hbar / m\omega_z}$

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Arbitrary size, but a single possible atom number

  

Ray Chiao, Esla Garmire & Charles Townes, 1964

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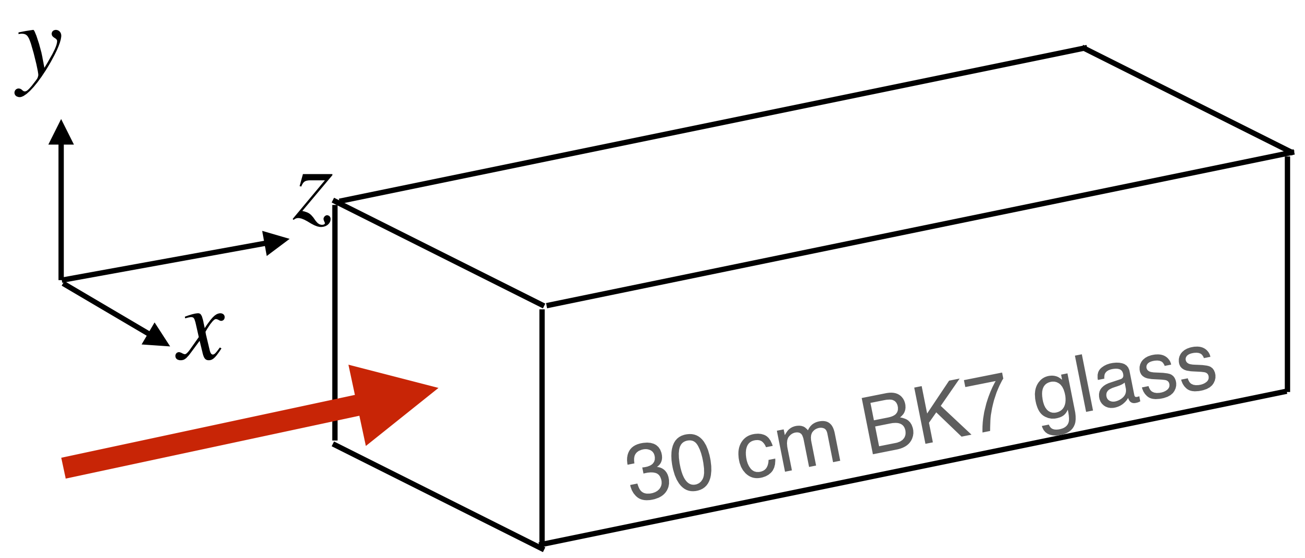
Townes soliton in practice

Initially proposed in the context of non-linear optics

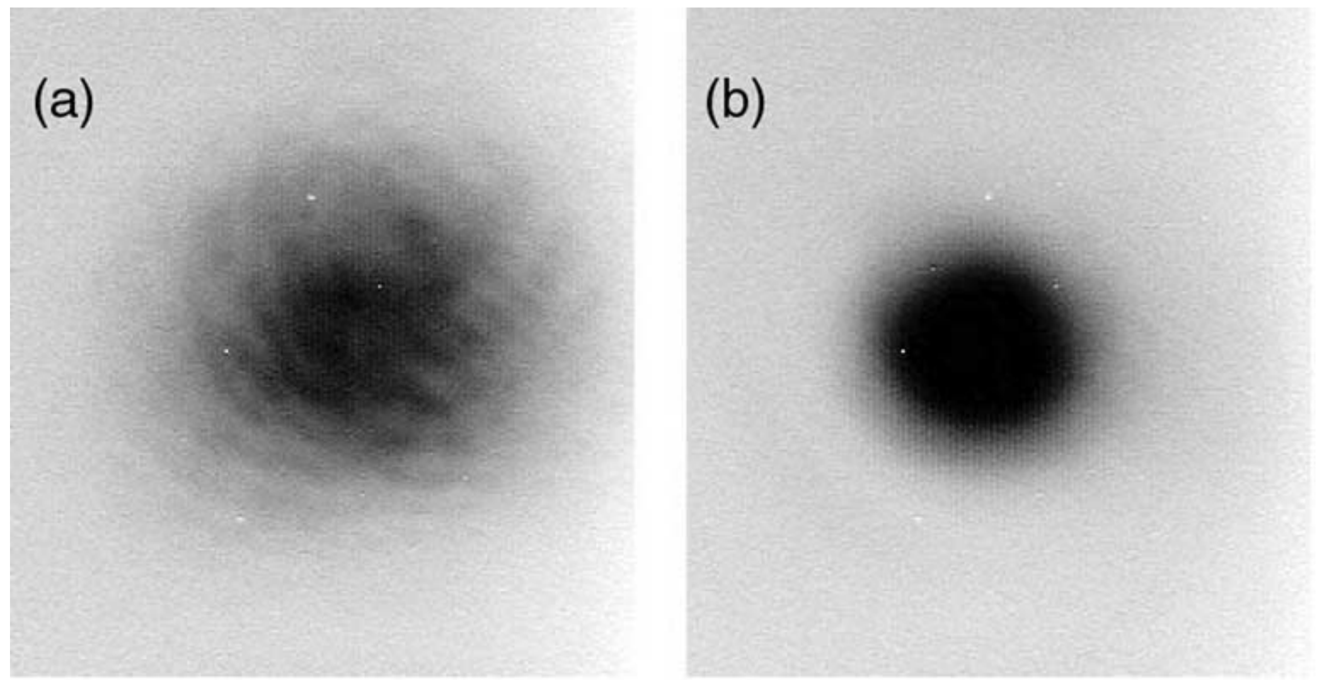
Chiao, Garmire & Townes, “Self-Trapping of Optical Beams,” PRL **13**, 479 (1964)

Moll, Gaeta & Fibich, Self-Similar Optical Wave Collapse: Observation of the Townes Profile, PRL **90**, 203902 (2003)



The axis propagation (z) plays the role of time

Competition between self-focusing and diffraction



Low power: randomly distorted beam

$\sim$ critical power: self-cleaned beam

Many subsequent experiments in bulk photonic systems or waveguides (filamentation, light bullets,…), as well as in polariton systems

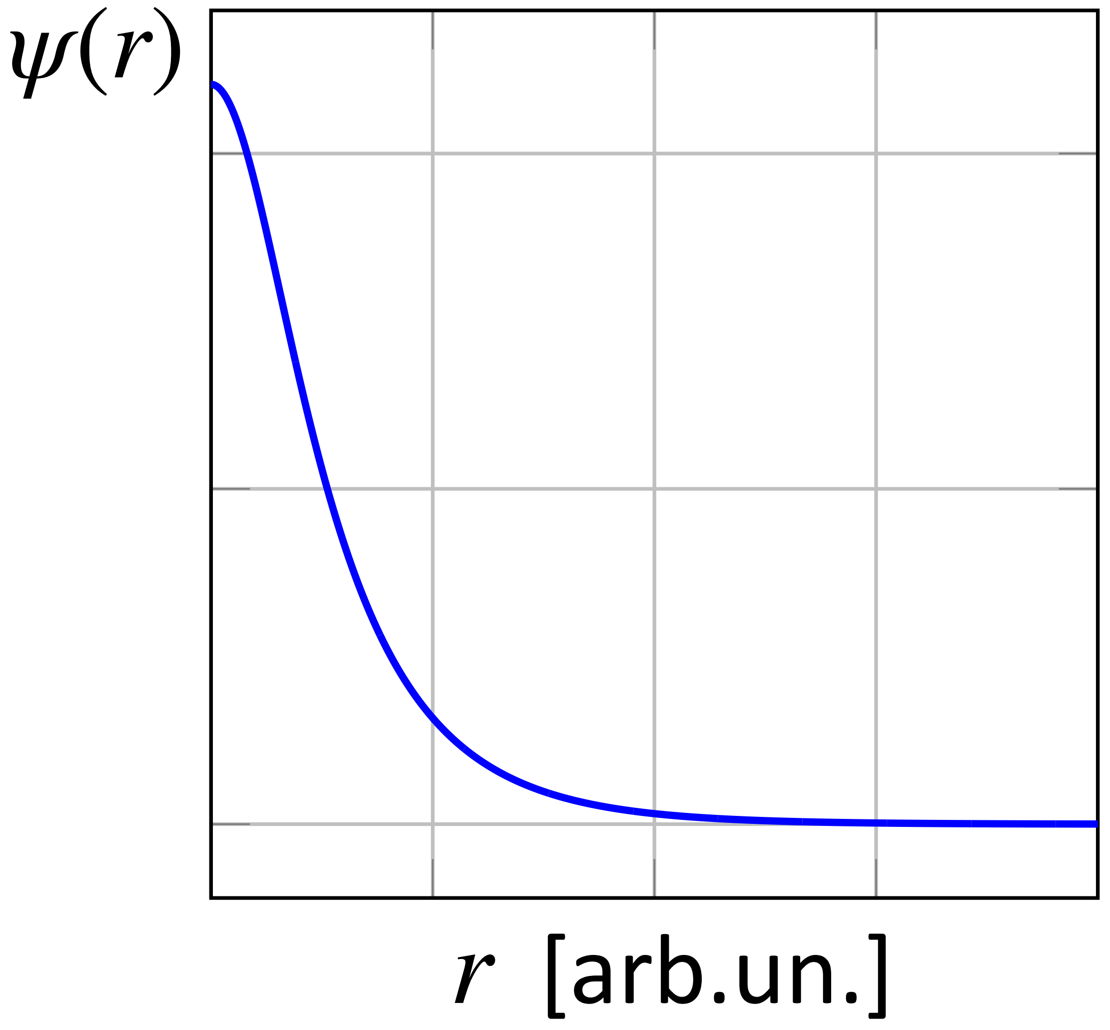
Kartashov et al, Nature Reviews Phys. 1, 185 (2019)

The solution for the Townes soliton

$$E[\psi] = \frac{1}{2}\int \left(\left| \nabla\psi \right|^2 + g \left|\psi\right|^4 \right) \mathrm{d}^2 r$$

Radially symmetric, node-less solution of:

$$-\frac{1}{2}\nabla^2\psi + g\,\psi^3 = \mu\,\psi \qquad \int |\psi|^2 = N$$



Such a solution exists only if $\left(Ng\right)_{\mathrm{Townes}} = -\,5.85...$

It has $E = 0$ and $\mu < 0$

Once a particular solution is known, scale invariance provides a continuous family of solutions

$$\phi(\boldsymbol{r}) = \lambda\,\psi(\lambda\boldsymbol{r}) \qquad \mu_\phi = \lambda^2\mu \qquad \lambda \text{ real}$$

No particular length scale for the Townes soliton when it exists

However: Instable with respect to a change in shape or in Ng

A few known results on Townes soliton

$$E[\psi] = \frac{1}{2}\int\left(\left|\nabla\psi\right|^2 + g\left|\psi\right|^4\right)\,\mathrm{d}^2r$$

- Variance identity, valid for any shape of the wave packet:

$$\frac{\mathrm{d}^2\langle r^2\rangle}{\mathrm{d}t^2} = \frac{4E}{m}$$

a consequence of the scale/conformal invariance of the problem

SO(2,1) symmetry (Niederer, Pitaevskii & Rosch)

- Negative energy $\Longrightarrow$ Collapse

$\langle r^2\rangle$ *becomes negative at a finite time*

But the reciprocal statement is not true: there exist wave packets with $E > 0$ *that collapse*

- Small $N|g|$ $\Longrightarrow$ Regularity

Threshold at the Townes critical number: $N|g| = 5.80\ldots$

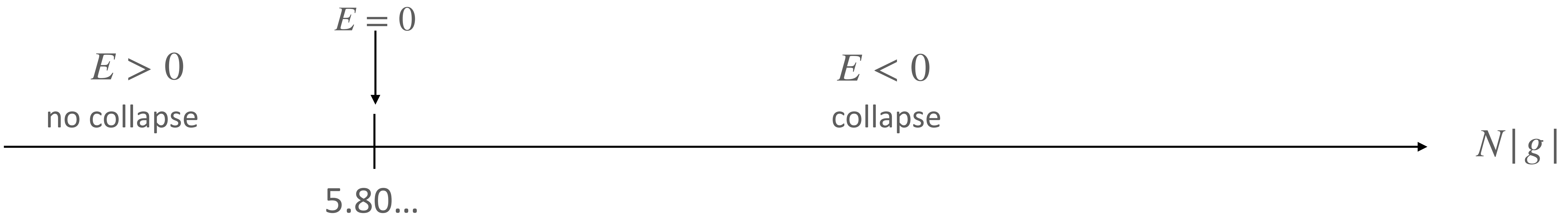
But the reciprocal statement is not true: there exist wave packets with large atom numbers that do not collapse

C. Sulem & P.-L. Sulem, *The nonlinear Schrödinger equation, self-focusing and wave collapse*, Springer 1999

Example: Townes profile vs. Gaussian

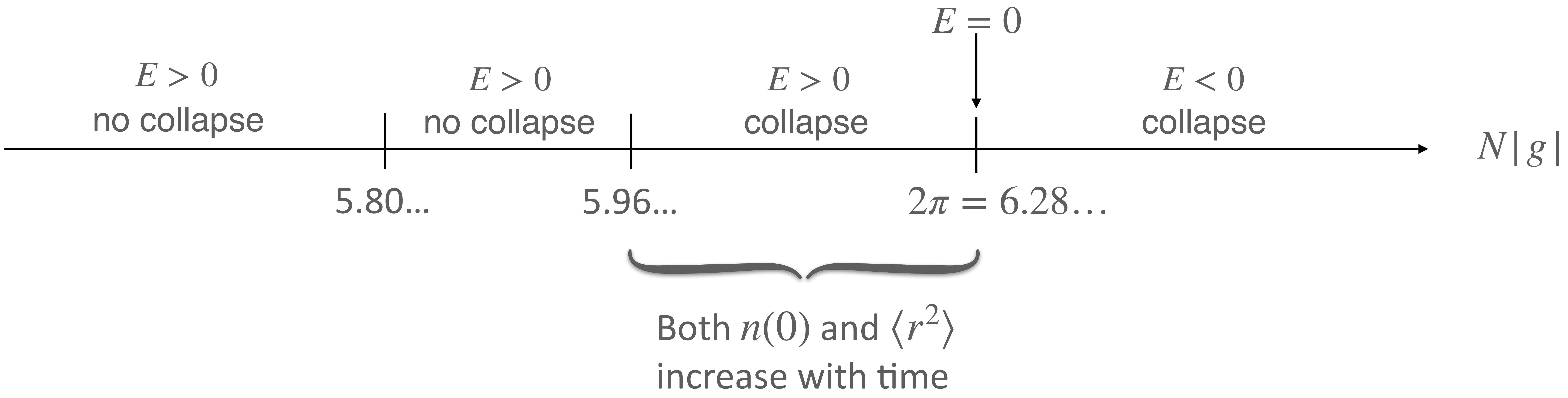
$$E[\psi] = \frac{1}{2}\int\left(\left|\nabla\psi\right|^2 + g\left|\psi\right|^4\right) d^2r$$

For an initial Townes profile



For an initial Gaussian profile

Fibich & Gaeta, 2000

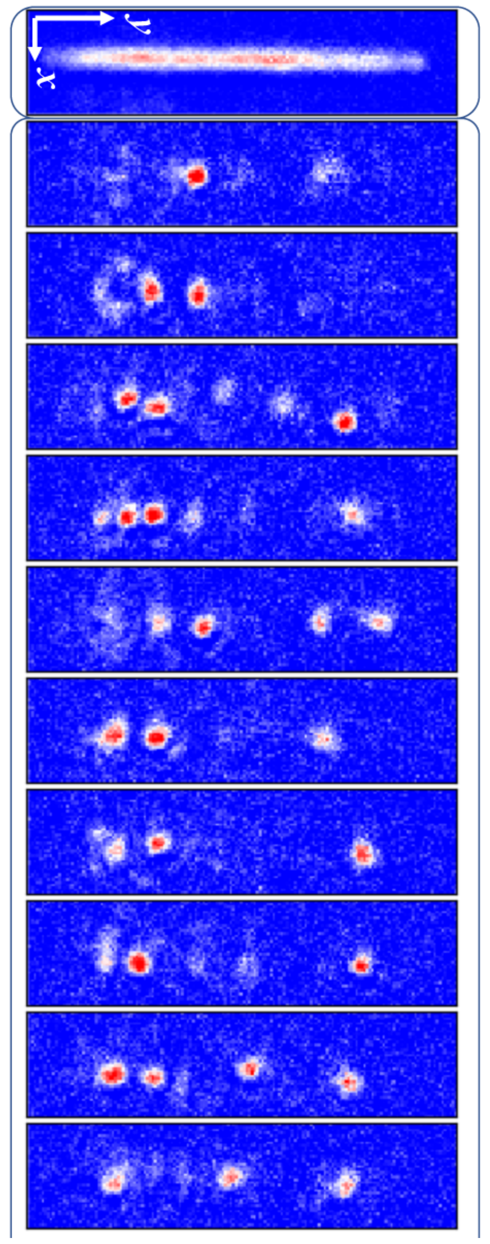


Observation of Townes soliton with cold atomic gases

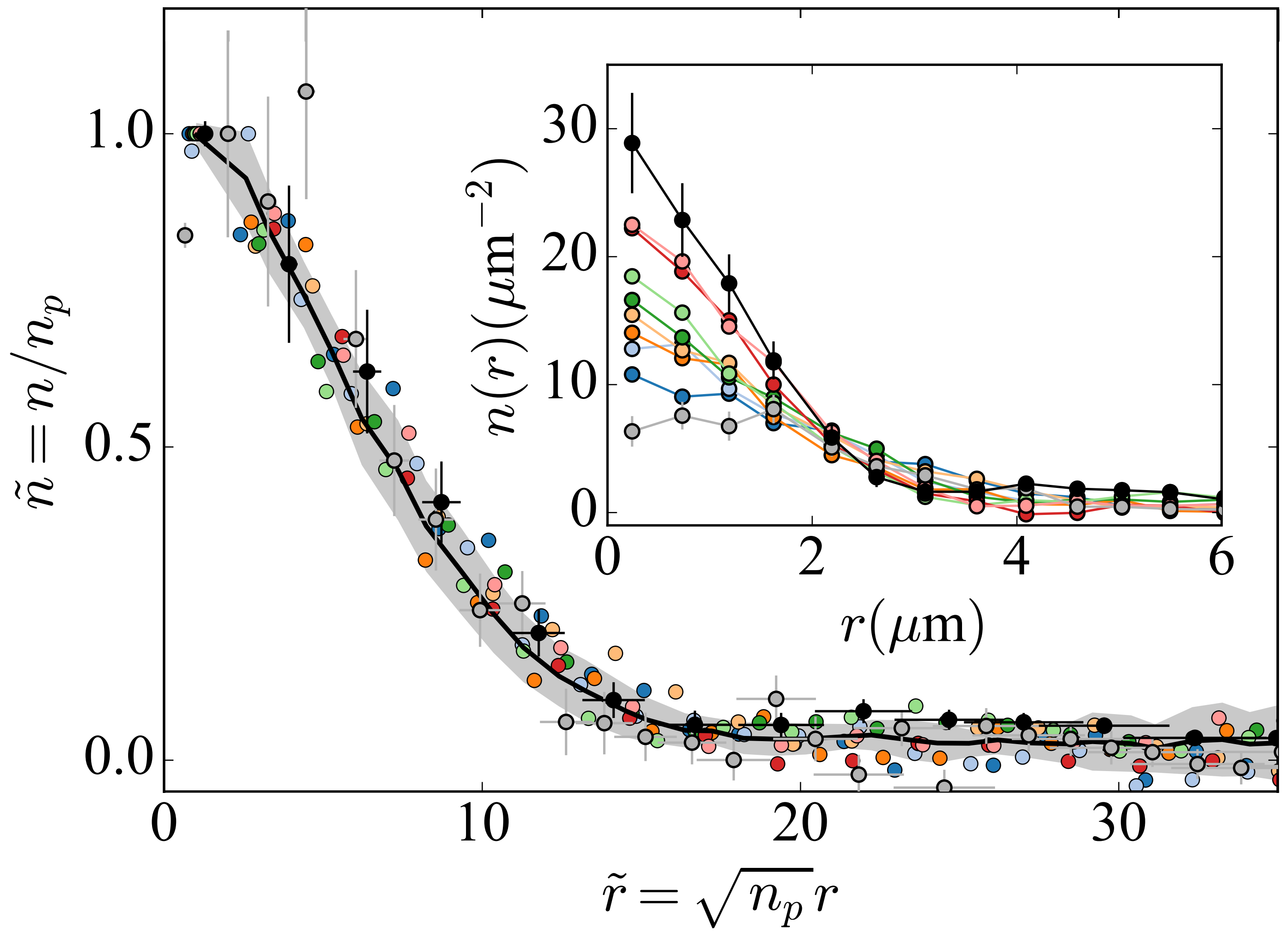
One needs to achieve an effective attractive interaction $\tilde{g} < 0$

Paris group: Phys. Rev. Lett. 127, 023603 (2021), use of a two-component gas with ^{87}Rb
Purdue group: Phys. Rev. Lett. 127, 023604 (2021), use of a Feshbach resonance with ^{133}Cs

Quench $\tilde{g} : +0.13 \rightarrow -0.0215$
and switch from 1D to 2D



Rescale all "droplets" together



Atom number/droplet: $\langle N\tilde{g} \rangle = -6.0\,(8)$

to be compared with $(N\tilde{g})_{\text{Townes}} = -5.85...$

from Chen & Hung, PRL **127**, 023604 (2021)

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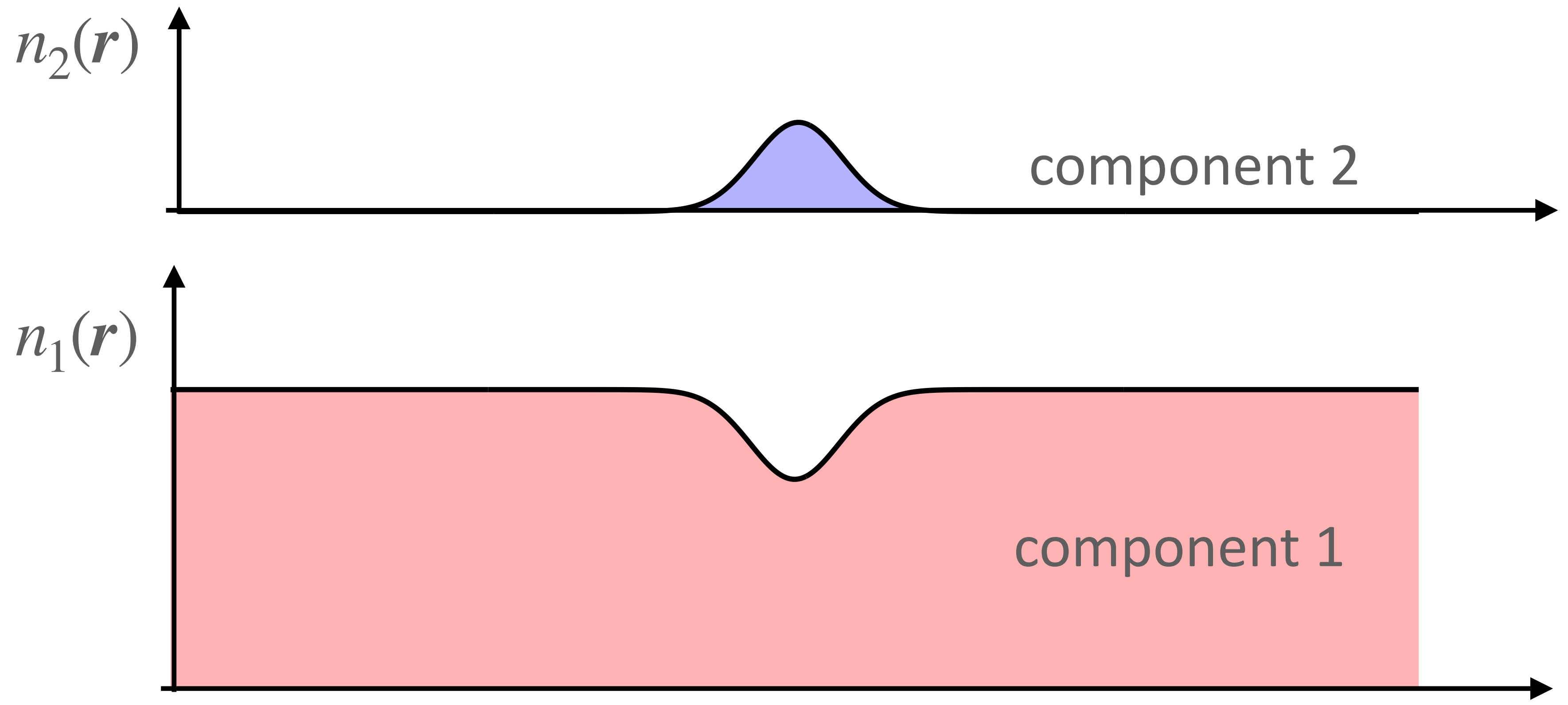
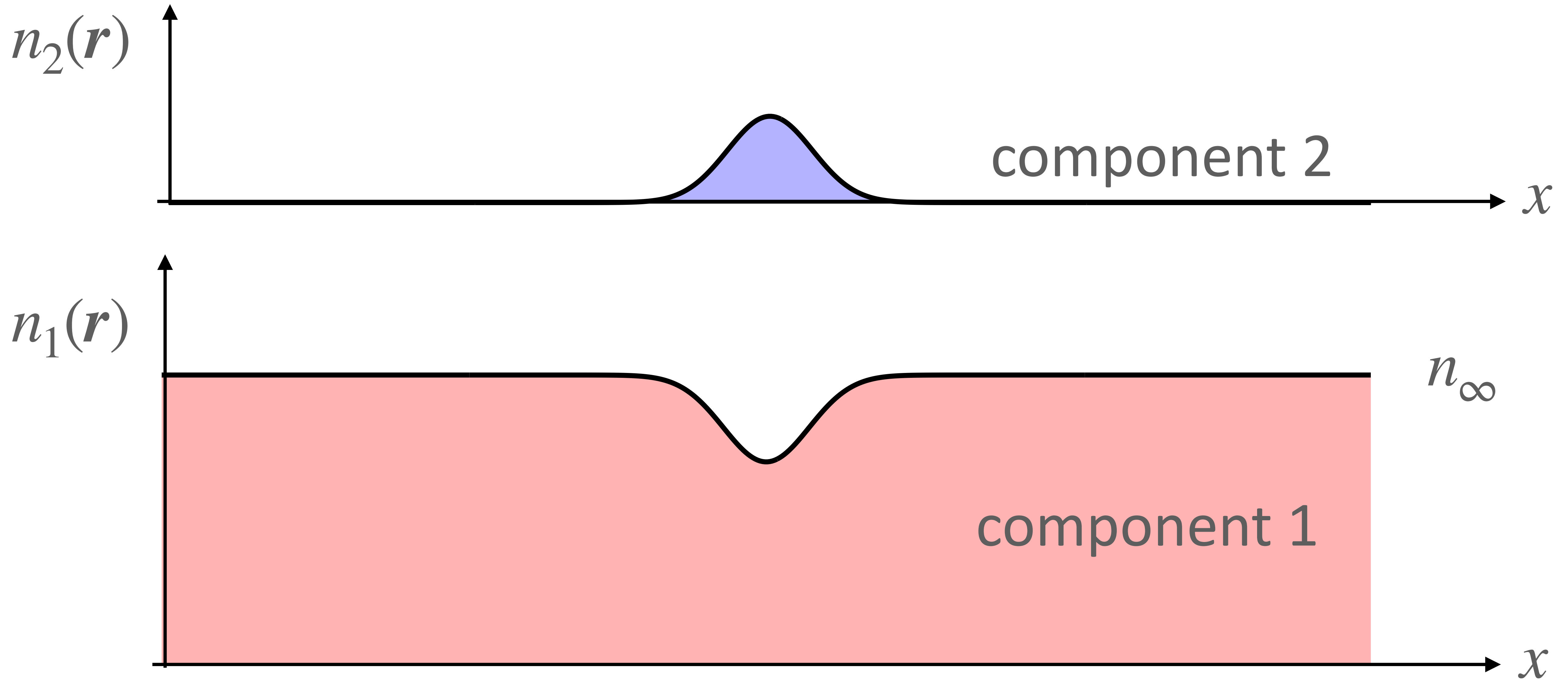
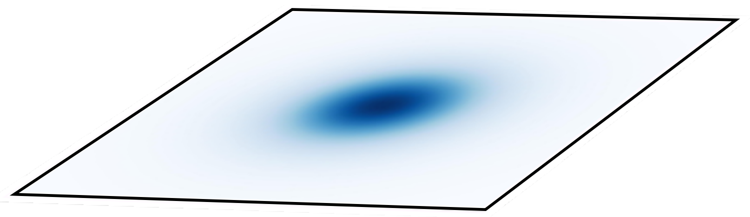
Evolution of a minority component inside an infinite bath



4. A first look at experimental results

A two-component fluid



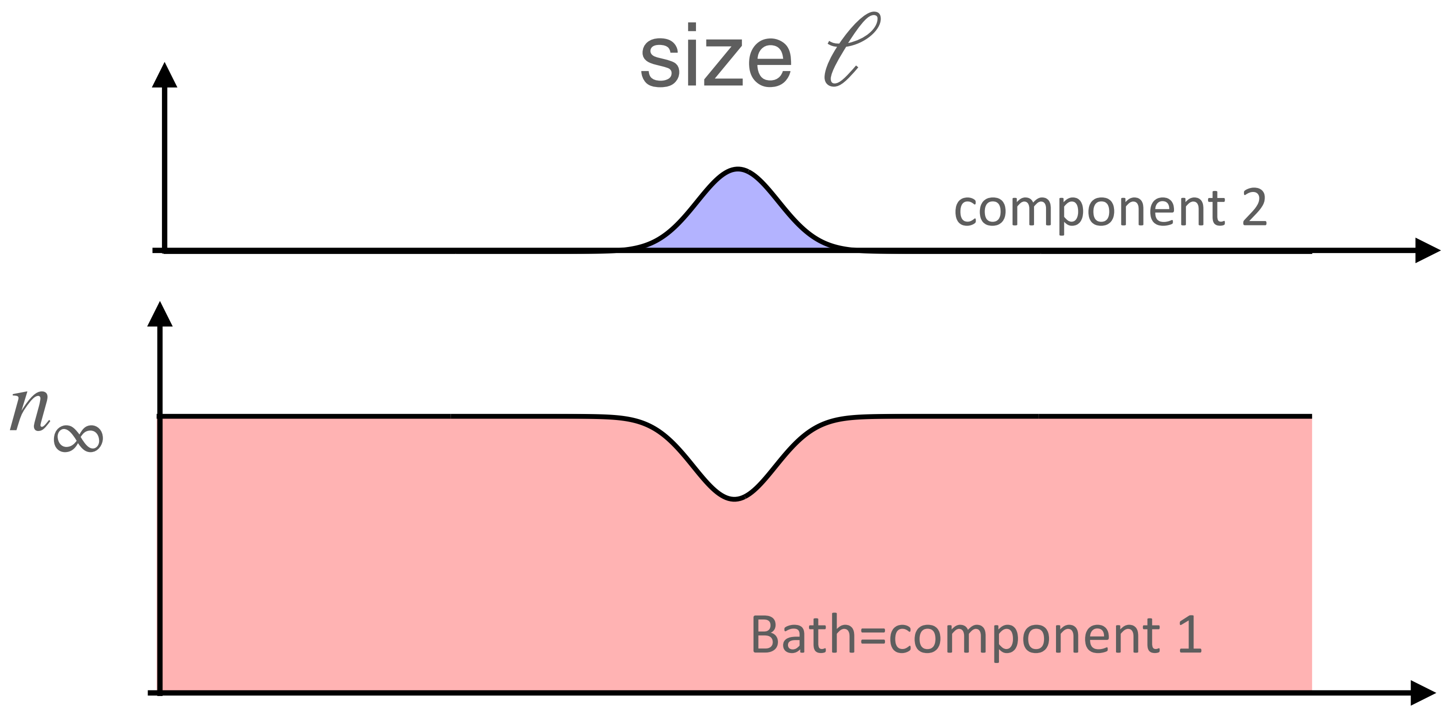


Each fluid is described by a 2D Gross-Pitaevski equation and is stable: $g_{ii} > 0$ for $i = 1{,}2$

- Component 1 extends to infinity with the asymptotic density n_∞
- Component 2 contains N_2 atoms

The two fluids are (slightly) non-miscible: $g_{12} > \sqrt{g_{11} g_{22}}$

The weakly-depleted bath



Bath healing length $\xi_1 = \dfrac{1}{\sqrt{2g_{11}n_\infty}}$

$$\mu_2 \psi_2 = \left(-\frac{\hbar^2}{2m}\nabla^2 + g_{12}n_1 + g_{22}n_2 \right)\psi_2$$

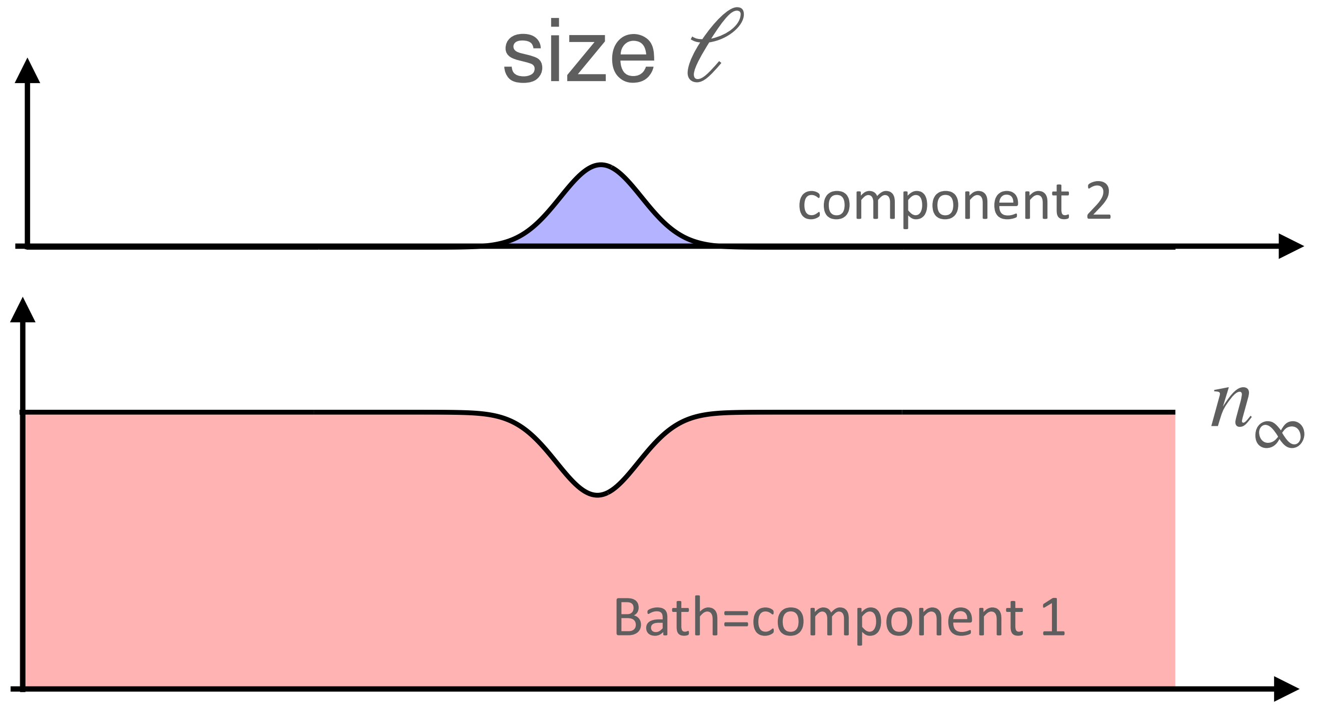
$$\mu_1 \psi_1 = \left(-\frac{\hbar^2}{2m}\nabla^2 + g_{11}n_1 + g_{12}n_2 \right)\psi_1 \qquad \mu_1 = g_{11}n_\infty$$

Assume that $n_2 \ll n_1 \approx n_\infty$ everywhere (weak depletion of comp. 1) and that $\ell \gg \xi$ (large extension of comp. 2)

Thomas-Fermi approximation for the bath (component 1):

$$\mu_1 = g_{11}n_1 + g_{12}n_2 \longrightarrow n_1 = n_\infty - \frac{g_{12}}{g_{11}}n_2$$

The minority component



$$\mu_2 \psi_2 = \left(-\frac{\hbar^2}{2m}\nabla^2 + g_{12}n_1 + g_{22}n_2 \right) \psi_2$$

$$n_1 = n_\infty - \frac{g_{12}}{g_{11}} n_2$$

Simple equation for the component 2:

$$\mu\, \psi_2 = \left(-\frac{\hbar^2}{2m}\nabla^2 + g_{\text{eff}}\, n_2 \right) \psi_2$$

$$\mu = \mu_2 - g_{12} n_\infty$$

$$g_{\text{eff}} = g_{22} - \frac{g_{12}^2}{g_{11}}$$

Bare interaction (repulsive)

Interaction mediated by the bath:

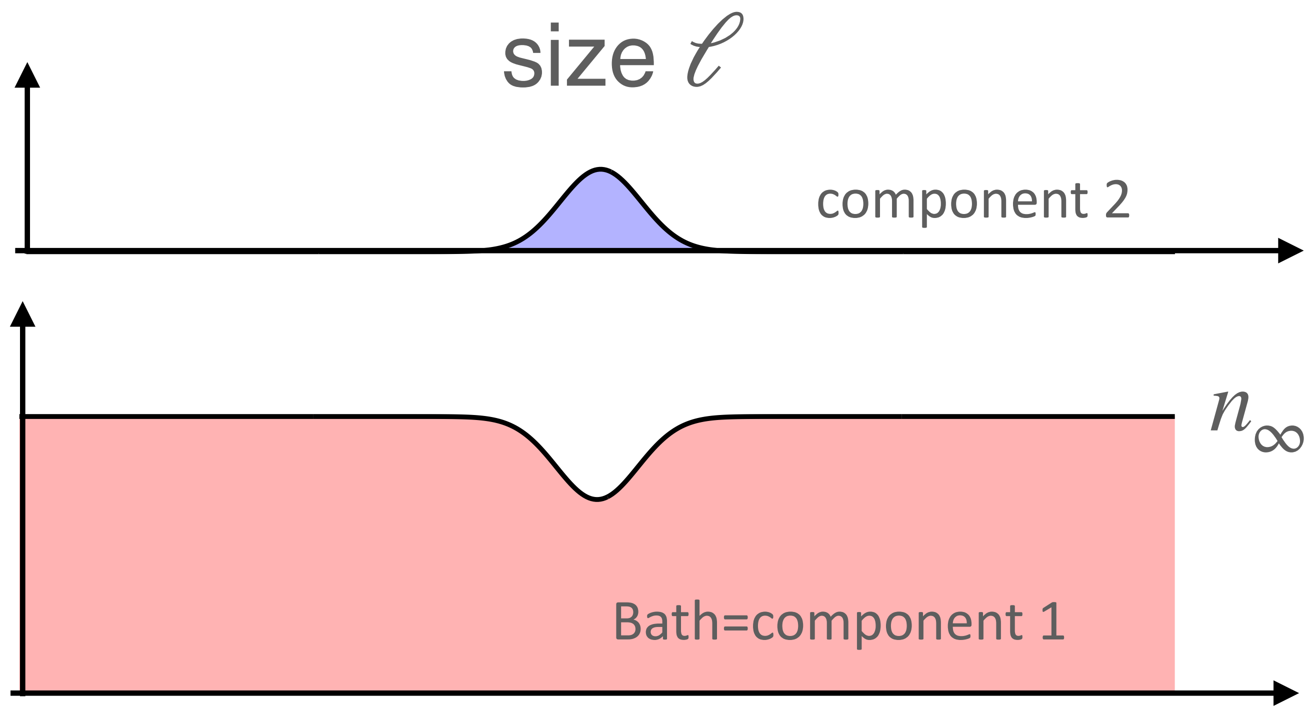
- always attractive
- independent of the bath density

Non-miscibility criterion:

$$g_{12}^2 > g_{11} g_{22} \quad \Leftrightarrow \quad g_{\text{eff}} < 0$$

Validity of this approach

In the equation for the bath $\mu_1 \psi_1 = \left(-\frac{\hbar^2}{2m}\nabla^2 + g_{11}n_1 + g_{12}n_2 \right)\psi_1$



we have neglected $-\frac{\hbar^2}{2m}\nabla^2$ but we have kept $g_{12}n_2$. Is it valid?

The wave function $\psi_1 = \sqrt{n_\infty}$ of the bath is slightly distorted by a quantity $\delta\psi_1(\boldsymbol{r})$ and we require ($\hbar = m = 1$):

$$\frac{\delta\psi_1}{\ell^2} \ll g_{12}n_2\psi_1 \ll g_{11}n_1\psi_1$$

neglected **kept** **kept**

Valid if: $\frac{\delta\psi_1}{\psi_1} \ll g_{12}\, n_2\ell^2 = \frac{g_{12}}{|g_{\text{eff}}|} \; |g_{\text{eff}}|N_2$

$\frac{g_{12}}{|g_{\text{eff}}|} \gtrsim 1$

$|g_{\text{eff}}|N_2 = 5.80$ for the Townes soliton

OK if $\frac{\delta\psi_1}{\psi_1} \ll 1$

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4. A first look at experimental results

PhD students : Brice Bakkali-Hassani, Chloé Maury, Guillaume Chauveau, Franco Rabec+ Raphaël Saint-Jalm, Edouard Le Cerf

Postdocs: Yiquan Zou, Patricia Castilho

Permanent: Jérôme Beugnon, Sylvain Nascimbene, Jean Dalibard

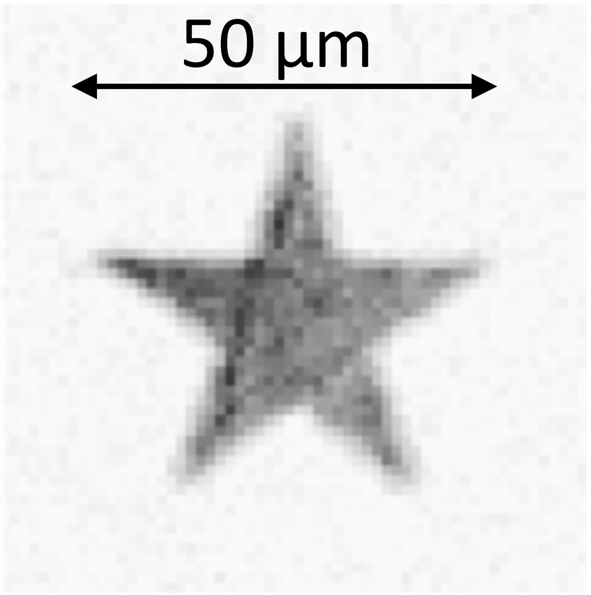
Our experimental setup (rubidium)

Frozen motion along the vertical direction z

$$\omega_z/2\pi = 4\,\mathrm{kHz}$$

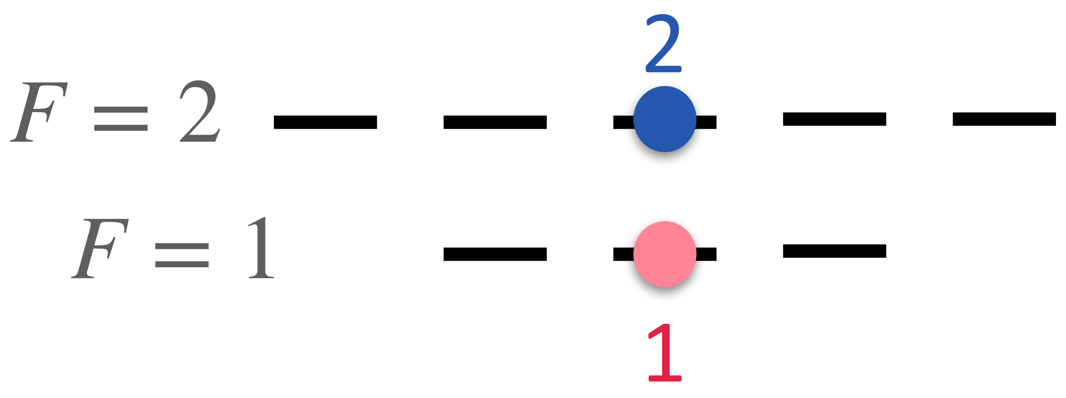
Initial confinement in the xy plane:

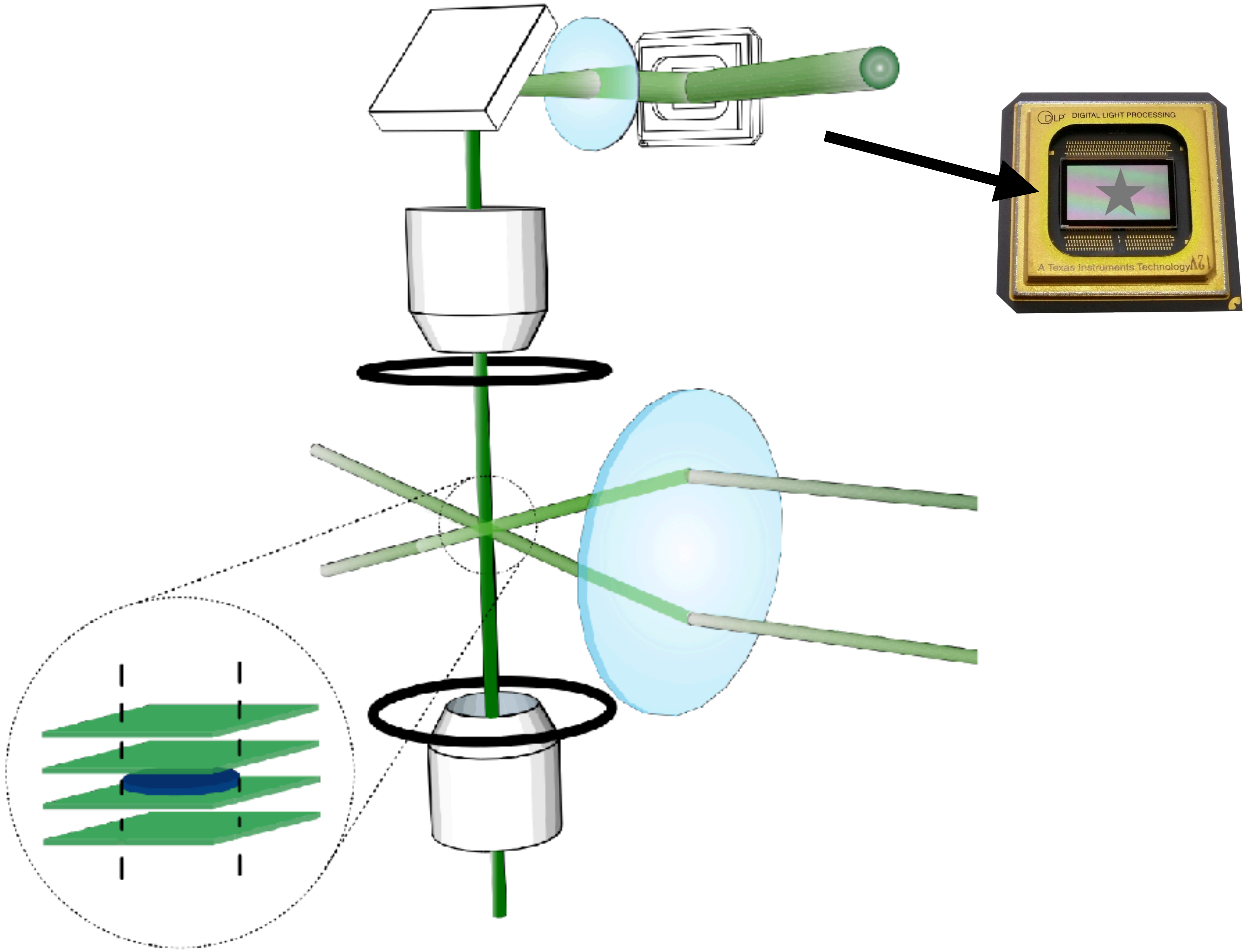
Box-like potential with arbitrary shape



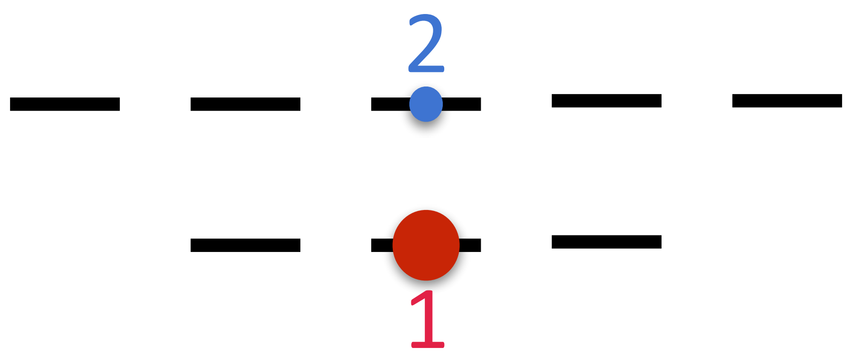
Uniform gas with up to 10^5 atoms

Density up to 100 atoms/μm^2



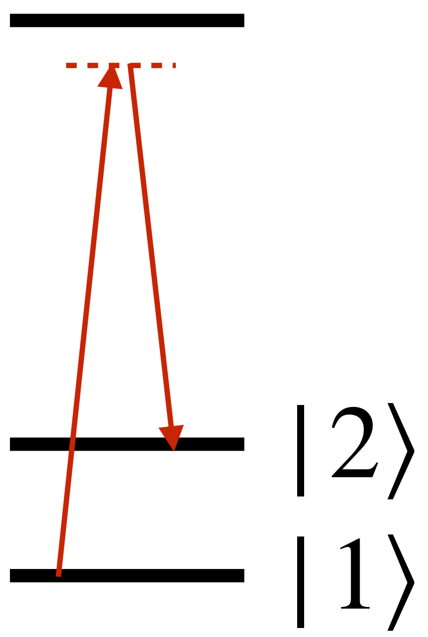


Our approach to Townes soliton creation

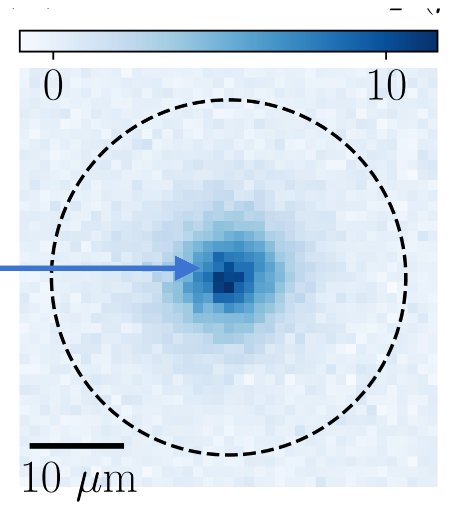


- Prepare a uniform ^{87}Rb gas in the internal state $|1\rangle$

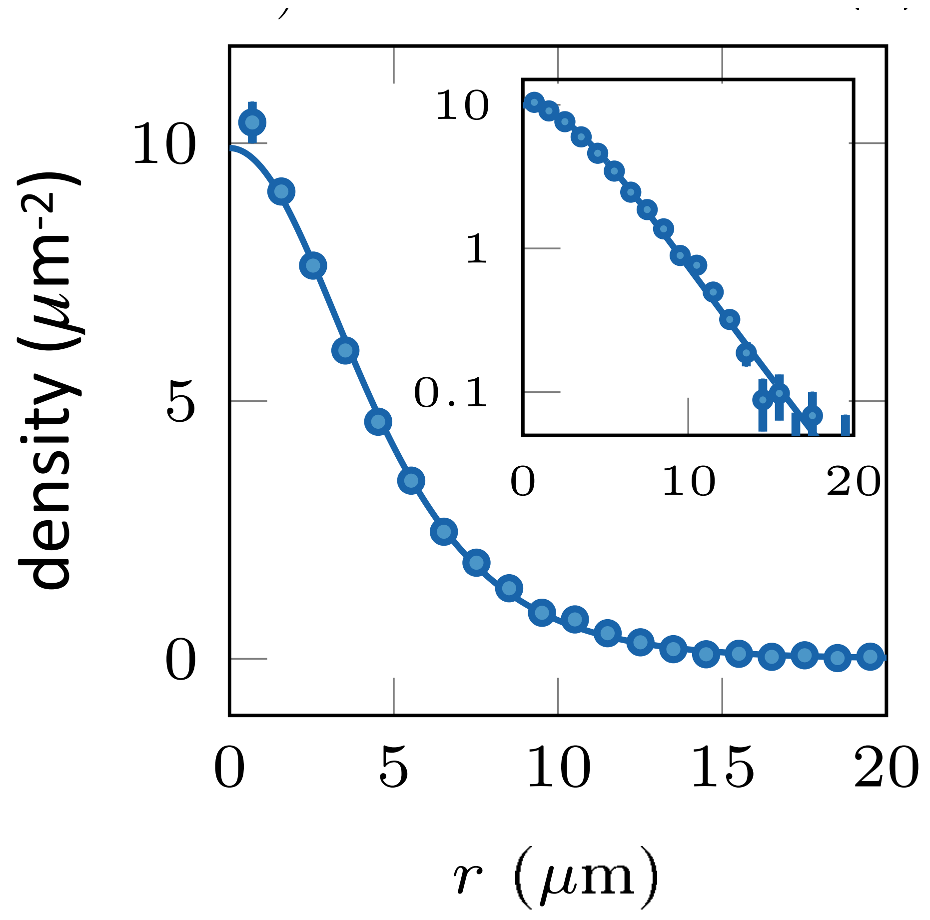
- Transfer in a spatially resolved way a small fraction of atoms in state $|2\rangle$



Atoms in $|2\rangle$



Atoms in $|1\rangle$ are still here, but not imaged



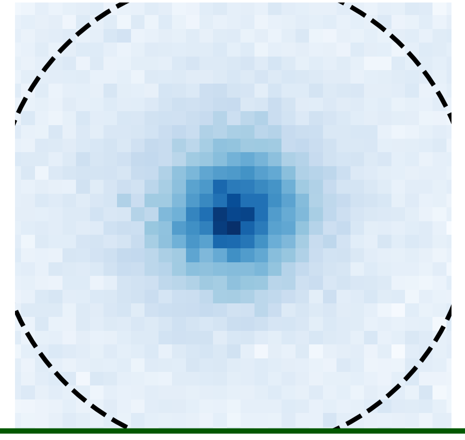
Townes profile with very good precision

- Look at the evolution of this “bubble” of atoms $|2\rangle$ immersed in a bath of $|1\rangle$

3D scattering lengths: $a_{11}=100.9\ a_0$, $a_{12}=100.4\ a_0$, $a_{22}=94.9\ a_0$ → 2D coupling strengths: $g_{11}=0.160$, $g_{12}=0.159$, $g_{22}=0.151$ →

$$g_{\text{eff}} = g_{22} - \frac{g_{12}^2}{g_{11}} \approx -0.0076$$

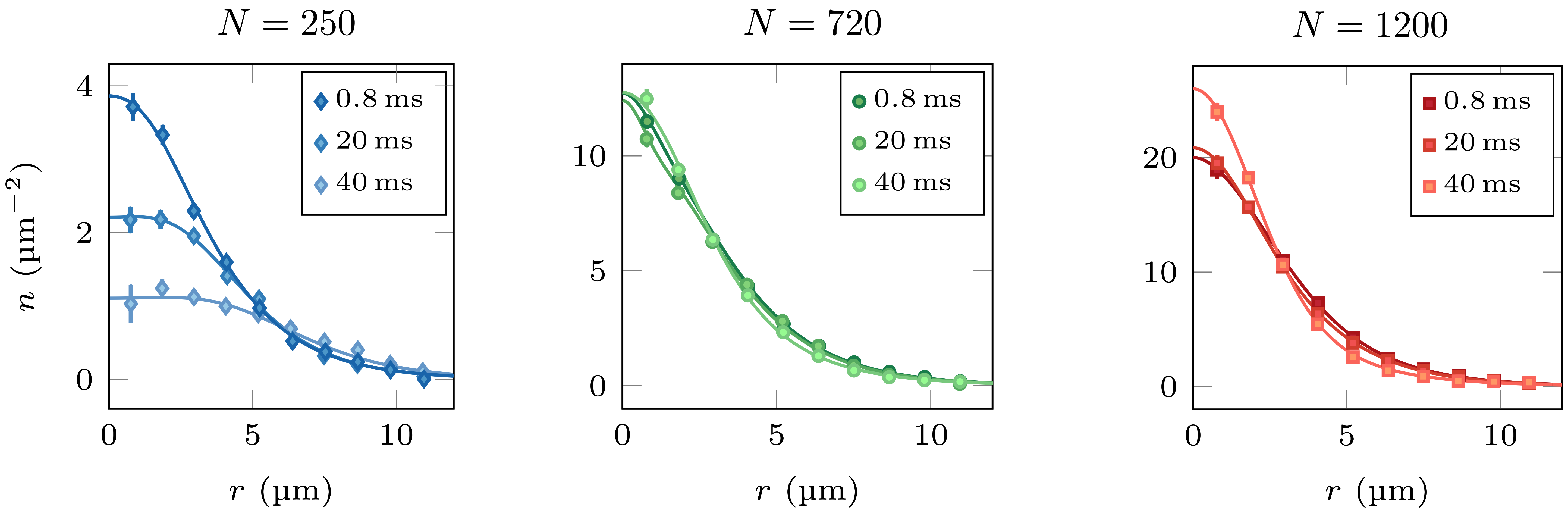
Observation of a Townes soliton

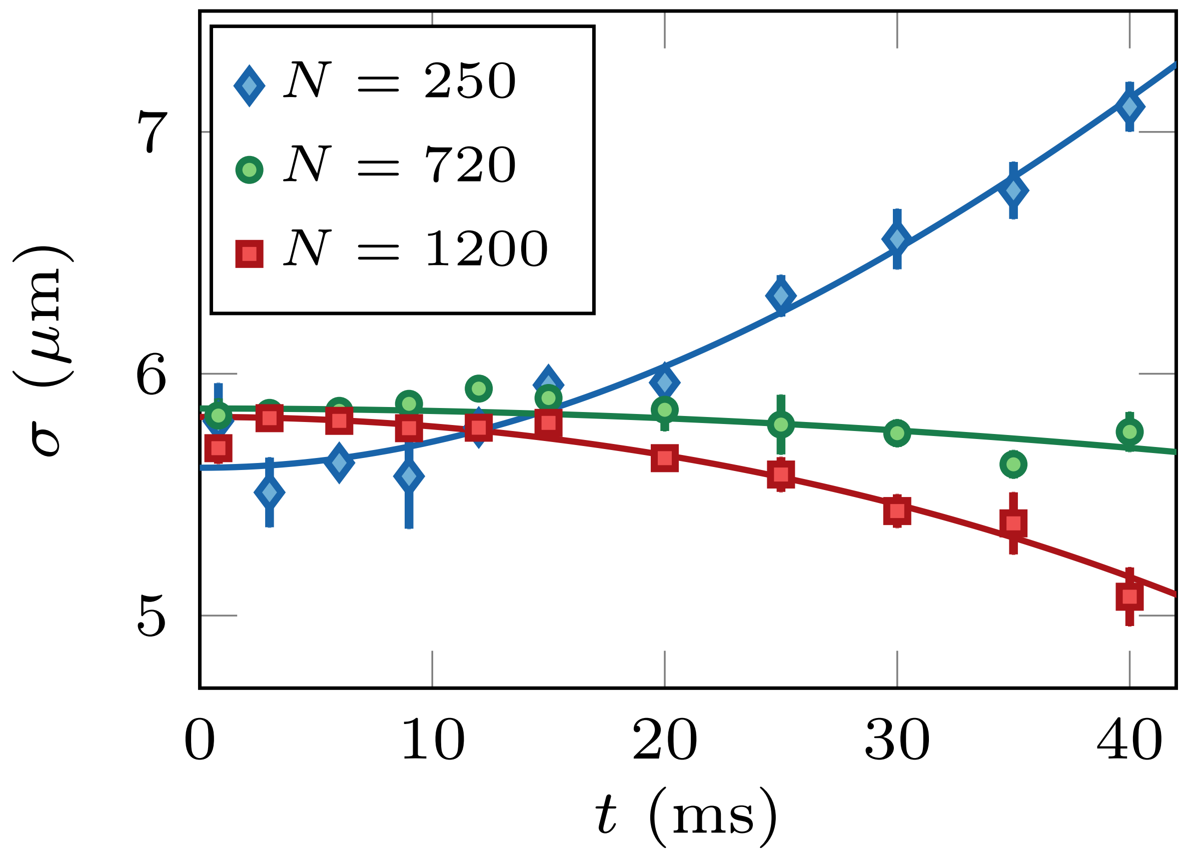


$$g_{\text{eff}} = g_{22} - \frac{g_{12}^2}{g_{11}} \approx -0.0076$$

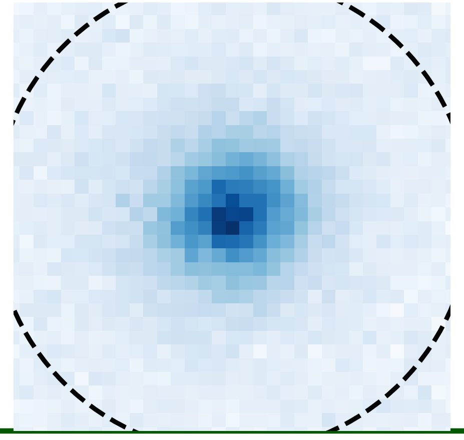
For our parameters, the threshold $N_{\text{Townes}}\,|\,g\,| = 5.85$ corresponds to $N_{\text{Townes}} \approx 770$

Here we print the Townes pattern with a given size $\sigma_0 = 5.7\ \mu\text{m}$, but with different atom numbers



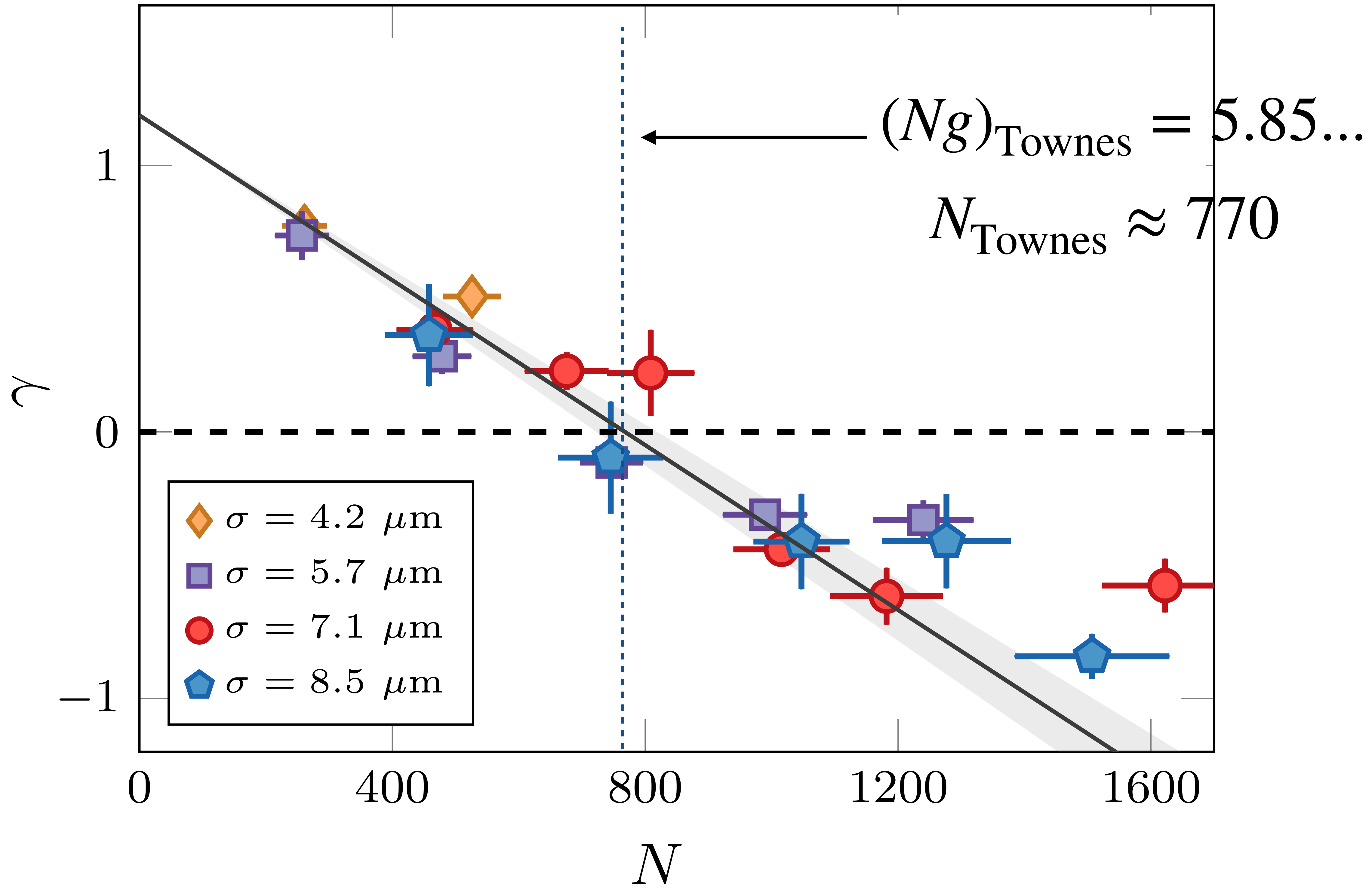


Scale invariance of Townes soliton



Expansion factor

$$\propto \frac{\mathrm{d}}{\mathrm{d}t}\langle r^2 \rangle$$



The stable shape is always obtained for ≈ the same atom number, irrespective of the size

PRL 127, 023603 (2021) see also PRL 127, 023603 (2021) by Chen & Hung

Goals of the next lecture

Revisit theoretically the coupling between the two species by a microscopic analysis

Yukawa potential between atoms of the minority component, mediated by the bath

Finite-range corrections: enriching the Gross-Pitaevskii equation

Study the transition from the soliton to the droplet regime by a mean-field analysis

Similarities and differences with the now well-known Beyond Mean-Field (BMF) "Quantum droplets"

D. S. Petrov, PRL **115**, 155302 (2015)

Experiments with BMF binary mixtures :
C. R. Cabrera et al., Science **359**, 301 (2018), G. Semeghini et al, PRL **120**, 235301 (2018),
C. D'Errico et al., PR Research **1**, 033155 (2019), Guo et al, PR Research **3**, 033247 (2021)

+ dipolar gases (Stuttgart, Innsbruck, Florence)