# The Townes soliton … and beyond

## *The rich physics of non-miscible Bose mixtures*
## *Lecture II*

Jean Dalibard
Collège de France
Laboratoire Kastler Brossel

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Course on “Quantum Mixtures with Ultra-cold Atoms”

# Outline of this lecture

1. Summary of lecture 1

*…and some responses to questions*

2. A microscopic view on soliton formation

*Yukawa-type interaction, mediated by the bath*

cf. Lectures by Meera Parish

3. Towards the droplet regime

*The role of non-local interactions*

4. The excitation spectrum of the soliton (and its daughter states)

cf. Lectures by Dmitry Petrov

*Similarities and differences with the now well-known "Quantum droplets"*

D. S. Petrov, PRL **115**, 155302 (2015) , C. R. Cabrera et al., Science **359**, 301 (2018),
G. Semeghini et al, PRL **120**, 235301 (2018), C. D'Errico et al., PR Research **1**, 033155 (2019),
Guo et al, PR Research **3**, 033247 (2021) + dipolar gases (Stuttgart, Innsbruck, Florence)

# Summary of lecture 1: The Townes soliton

**Do solitons exist in a 2D fluid described by the Gross-Pitaevskii equation?**

$$E[\psi] = \frac{\hbar^2}{2m}\int \left( \left|\nabla\psi\right|^2 + g\left|\psi\right|^4 \right) \mathrm{d}^2r \qquad \int |\psi|^2 = N \qquad g \text{ (dimensionless)} < 0$$

**Yes, but only for specific values of** $|g|N$

Nodeless, isotropic solution obtained for $|g|N = 5.80$, known as the Townes soliton

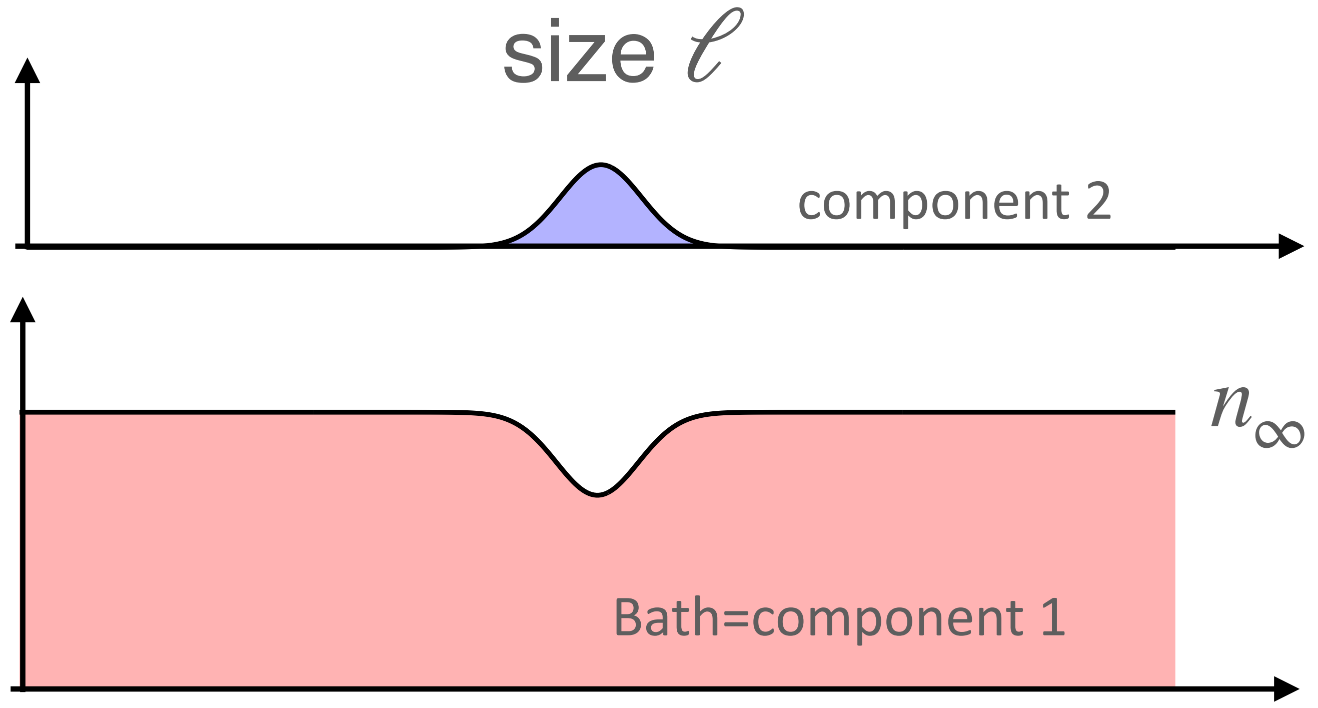
Isotropic solution with one node (two nodes): $|g|N = 38.6\ (97.9)$ [but dynamically unstable]

**For ultra-cold atoms, two recent experiments explored this unusual non-linear object**

Chen & Hung, PRL **127**, 023604 (2021) : single component $^{133}$Cs gas + Feshbach resonance

Bakkali-Hassani, Maury, Zou, Le Cerf, Saint-Jalm, Castilho, Nascimbene, Dalibard & Beugnon
PRL **127**, 023603 (2021) : immiscible mixture of two $^{87}$Rb hyperfine states

# Summary of lecture 1: A mean-field analysis




$$\mu_i \psi_i = \left( -\frac{\hbar^2}{2m}\nabla^2 + g_{ii} n_i + g_{ij} n_j \right) \psi_i \qquad i, j = 1{,}2 \qquad i \neq j$$

Hypotheses: weak depletion of comp. 1 $(n_2 \ll n_1 \approx n_\infty)$ and large extension of comp. 2 $(\ell \gg \xi)$

Simple equation for the component 2: $$\mu \, \psi_2 = \left( -\frac{\hbar^2}{2m}\nabla^2 \; + \; g_{\mathrm{eff}} \, n_2 \right) \psi_2$$

$$g_{\mathrm{eff}} = g_{22} - \frac{g_{12}^2}{g_{11}}$$

Bare interaction (repulsive)

Interaction mediated by the bath:

- always attractive
- independent of the bath density

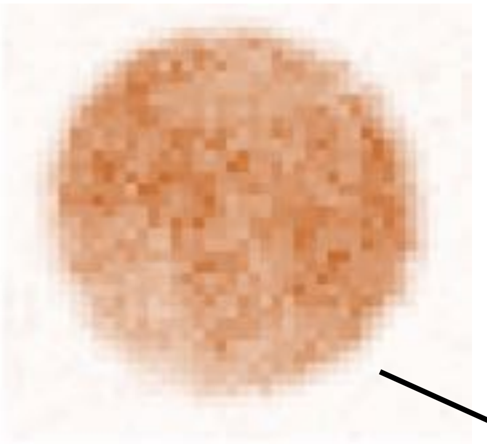
$$g_{\mathrm{eff}} < 0 \;\Leftrightarrow\; g_{12}^2 > g_{11} g_{22}$$

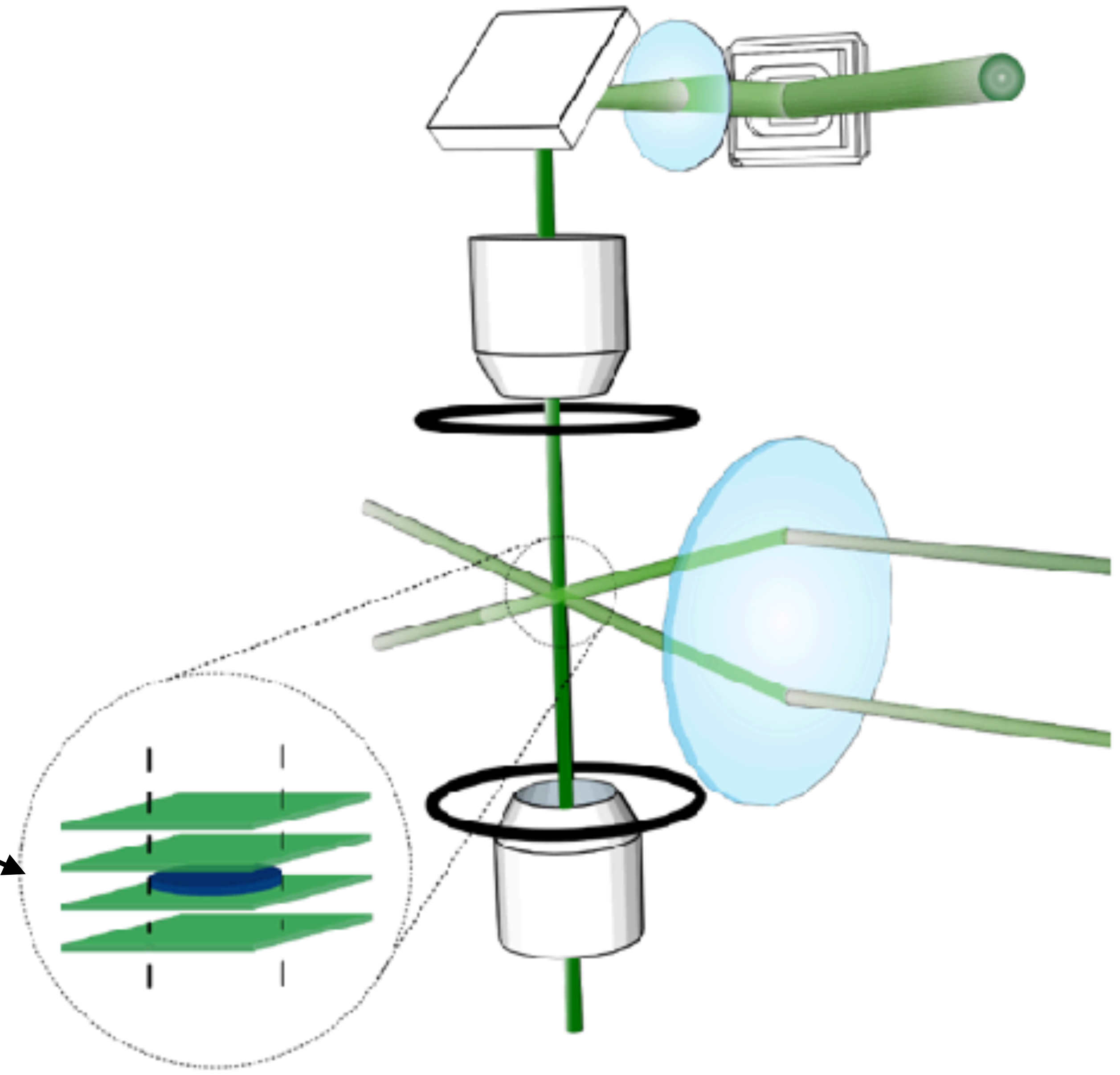
+ stability

non-miscibility

# Summary of lecture 1: The Paris experiment ($^{87}$Rb)

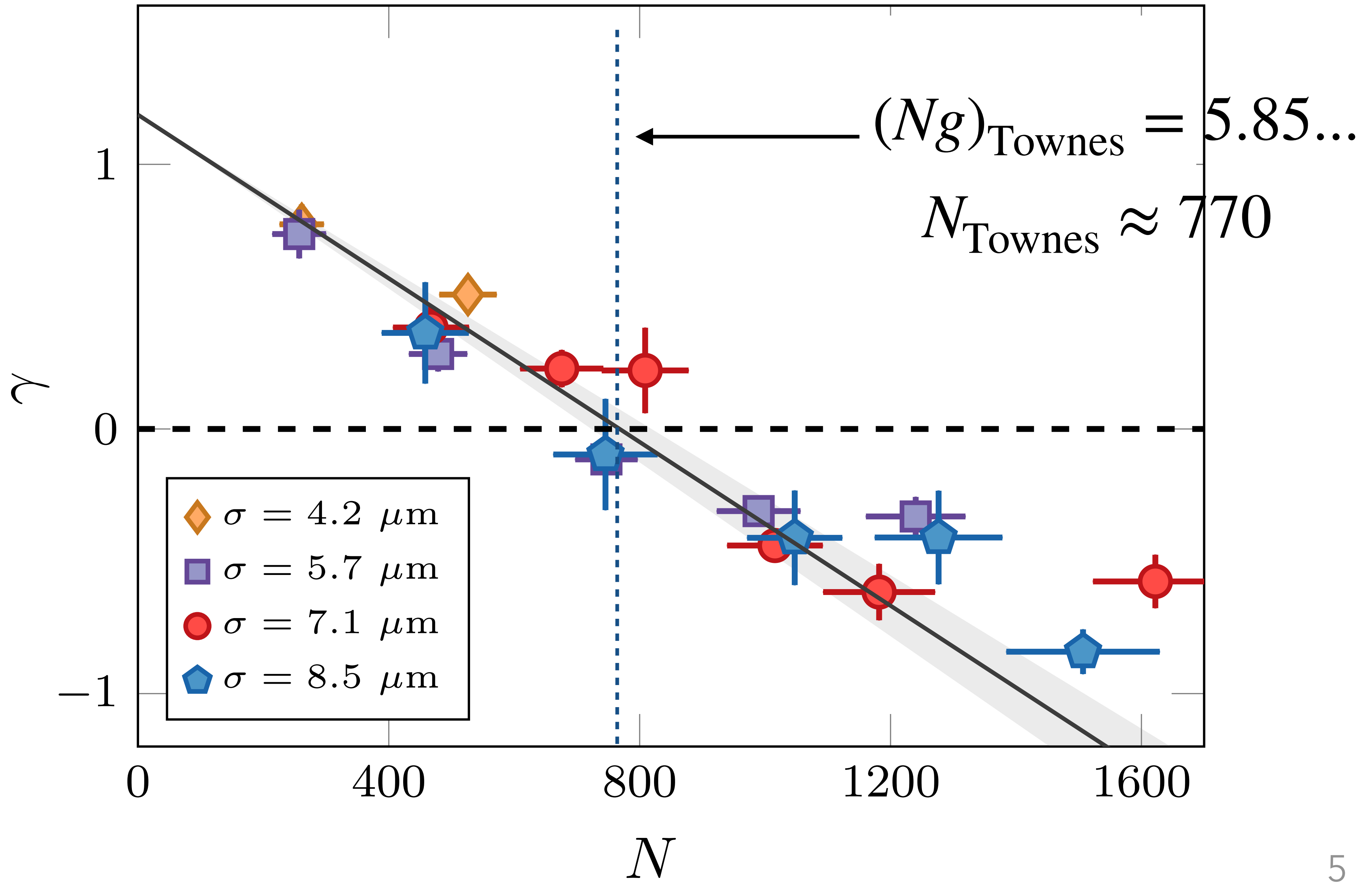
Atoms in $|1\rangle$

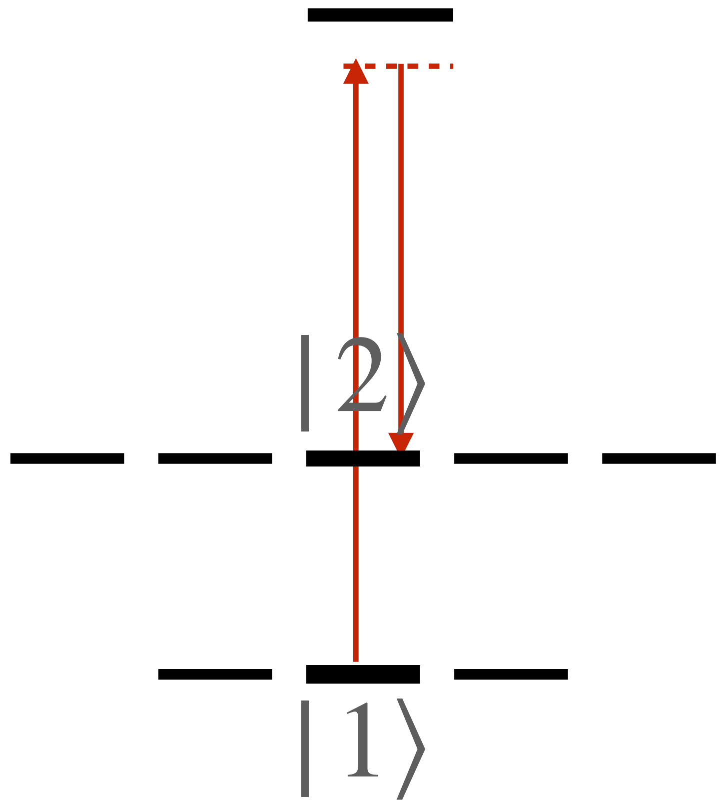




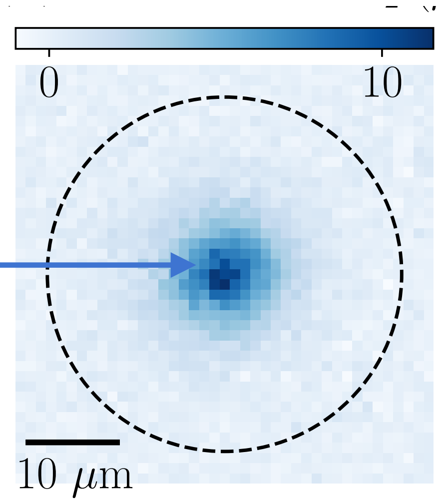
$$g_{\text{eff}} = g_{22} - \frac{g_{12}^2}{g_{11}} \approx -0.0076$$

Expansion factor $\propto \frac{\mathrm{d}}{\mathrm{d}t}\langle r^2 \rangle$




Atoms in $|2\rangle$

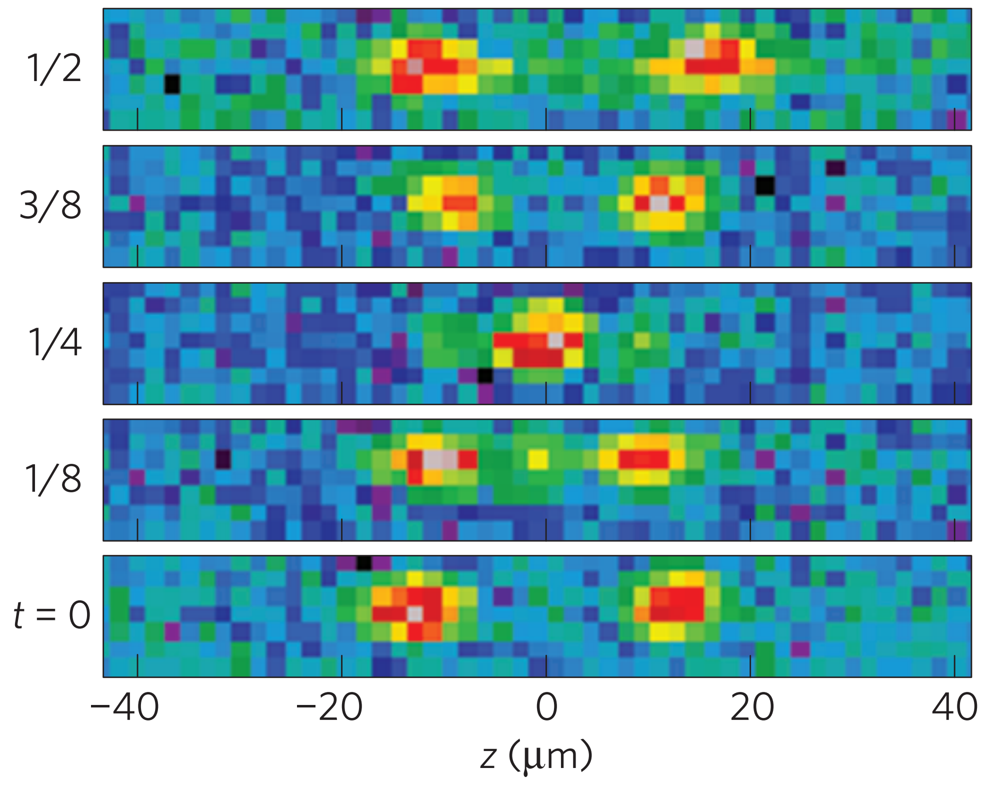



Prepare a wave packet of atoms in $|2\rangle$ with the Townes profile, immersed in the bath of atoms in $|1\rangle$

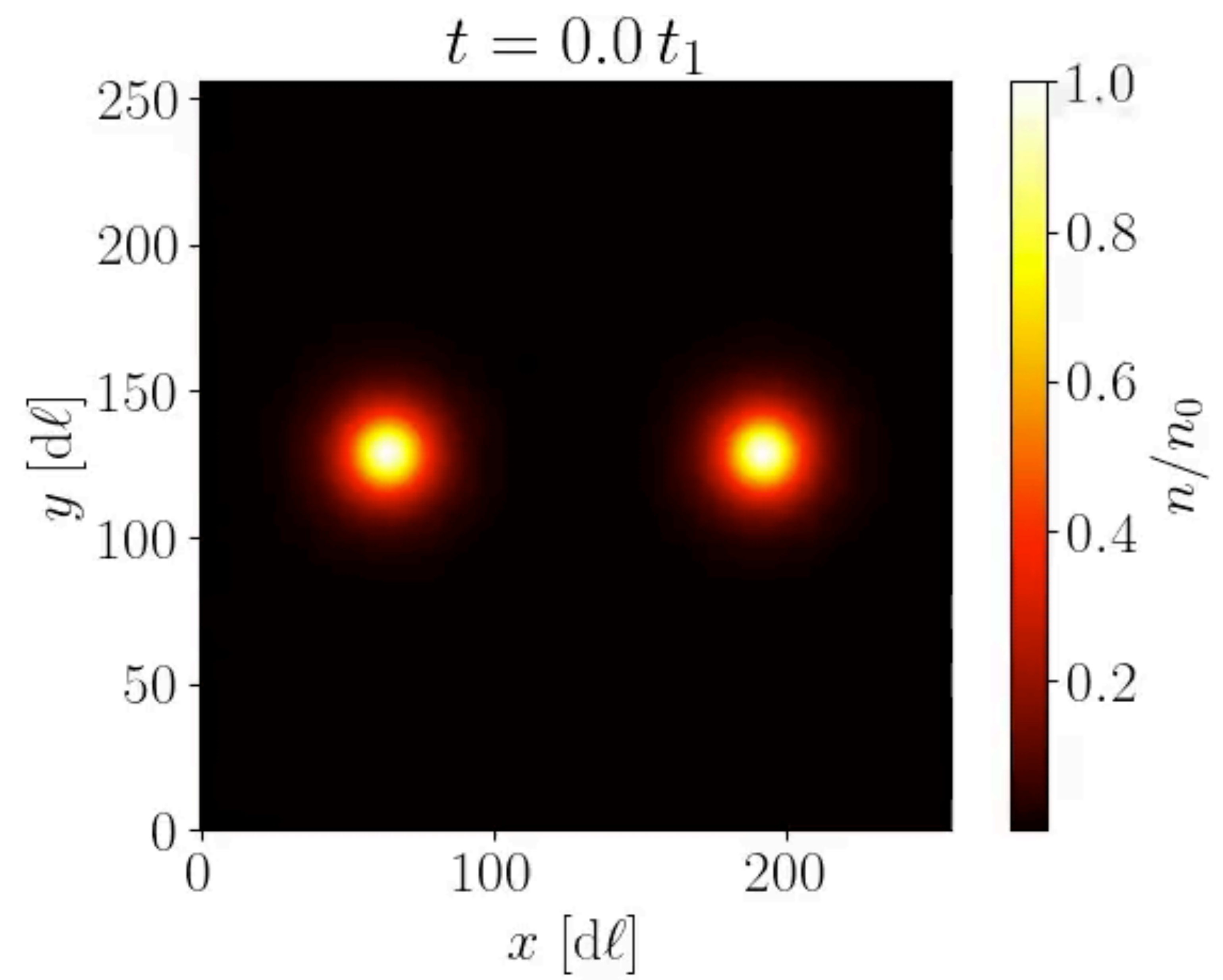
# Collisions between identical solitons

In 1D, the two solitons emerge “unperturbed” from the collision

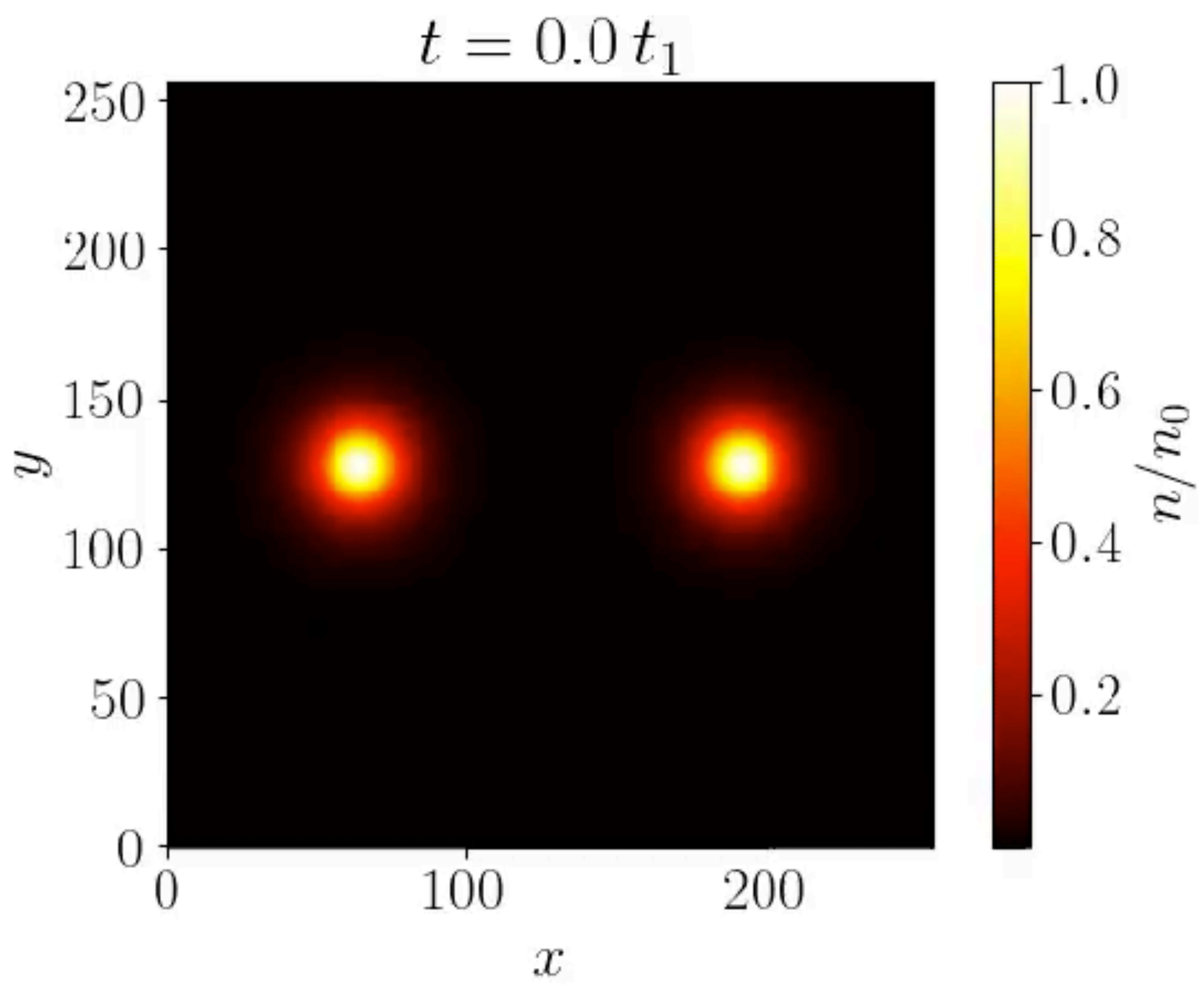
Nguyen, Dyke, Luo, Malomed and Hulet (Nat. Phys. 2014)




In 2D, the initial relative motion of the solitons sets a new length/energy scale




**Head-on collision with a large initial velocity**




**Head-on collision with a small initial velocity**

Simulations by
Brice Bakkali-Hassani

Collisions of self-bound quantum droplets,
Ferioli *et al.*,
PRL **122**, 090401 (2019)
merging/separation

# Breaking the SO(2,1) symmetry with a quantum anomaly

Hammer & Son, PRL **93** 250408 (2004): Going beyond the classical field analysis based on the Gross-Pitaevskii energy

See also Bazak & Petrov, New J. Phys. **20** 023045 (2018)

Introduction of a short-distance (i.e. UV) cutoff at $r \sim R_{\mathrm{vdW}}$ (van der Waals length : nanometer size)

→ There exists a stable solution of size $\sigma_N$ for any value of the atom number $N$

Geometric scaling: $\dfrac{\sigma_N}{\sigma_{N+1}} \sim 3$ $\qquad$ $\sigma_{770} \sim 10^{-9}\, \sigma_{750} \quad !!!$

In practice, for our interaction strength $|g_{\mathrm{eff}}| \ll 1$, the predicted value for $\sigma_N$ is physically reasonable only for $|N - N_{\mathrm{Townes}}| \sim$ a few units

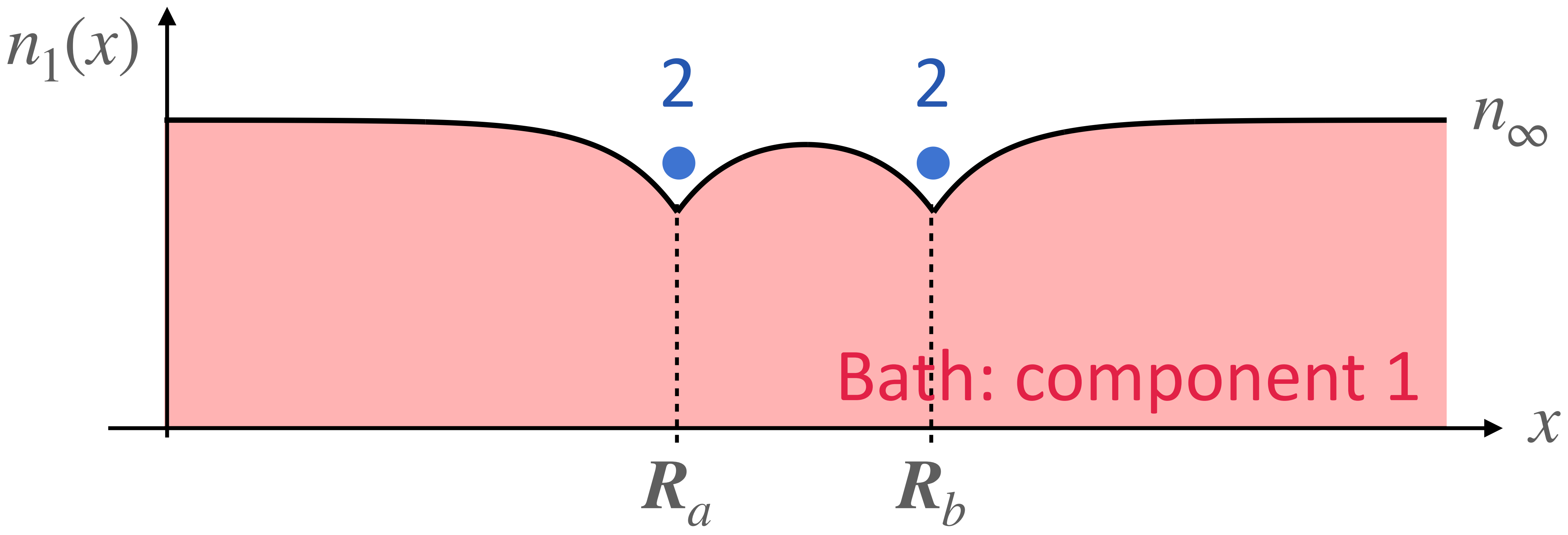
→ In the strongly interacting case $|g| \sim 1$, a realistic droplet size would be achieved with only a few atoms and one could observe the predicted scaling of $\sigma_N$ with $N$

# Outline of this lecture

1. Summary of lecture 1

**2. A microscopic view on soliton formation**

*Yukawa-type interaction between impurity atoms, mediated by the bath*

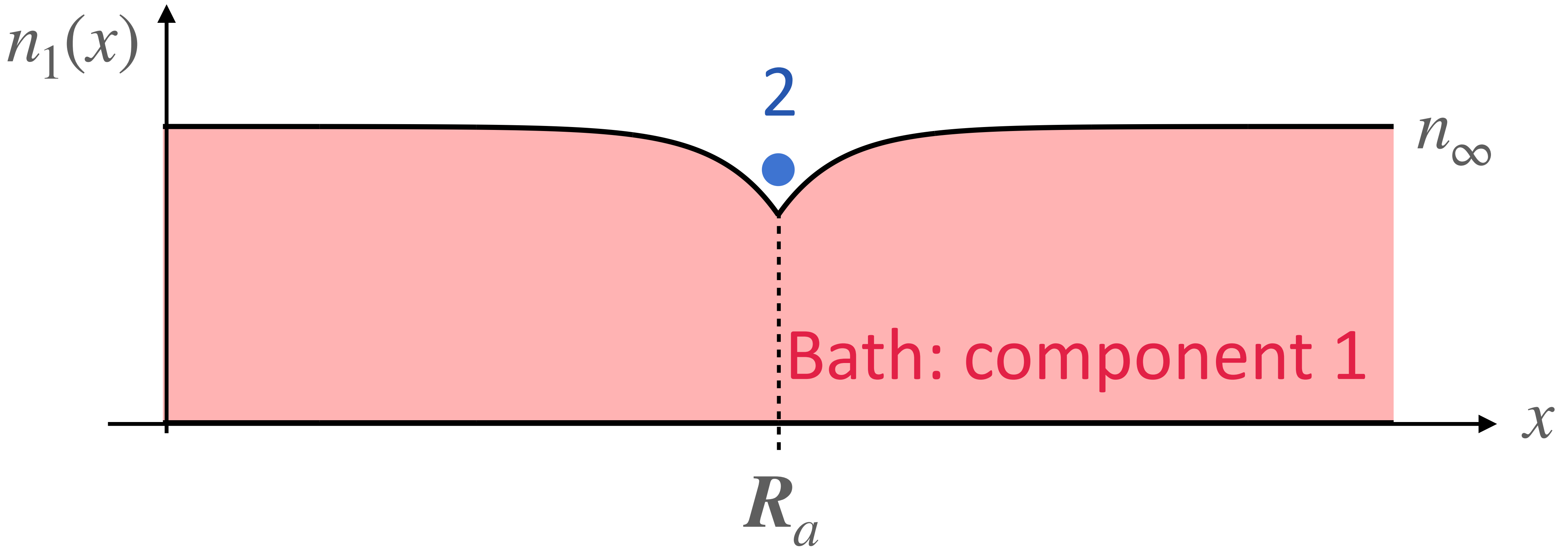



3. Towards the droplet regime

4. The excitation spectrum of the soliton (and its daughter states)

# The Bose polaron problem

For a review, see Grusdt & Demler (2015)




Fröhlich Hamiltonian: $\hat{V}(\boldsymbol{R}_a) = g_{12}\,\hat{\Psi}^\dagger(\boldsymbol{R}_a)\,\hat{\Psi}(\boldsymbol{R}_a)$

$\hat{\Psi}(\boldsymbol{r})$ : field operator of the bath

$g_{12}$ dimensionless ( with $\hbar = m = 1$)

Description of the bath by the Bogoliubov approach: $\hat{\Psi}(\boldsymbol{r}) = \sqrt{n_\infty} + \delta\hat{\Psi}(\boldsymbol{r})$

$$\delta\hat{\Psi}(\boldsymbol{r}) = \sum_{k\neq 0} \frac{e^{i\boldsymbol{k}\cdot\boldsymbol{r}}}{\sqrt{L^2}}\,\hat{a}_{\boldsymbol{k}} \qquad \hat{a}_{\boldsymbol{k}} = u_k \hat{b}_{\boldsymbol{k}} + v_k \hat{b}^\dagger_{-\boldsymbol{k}}$$

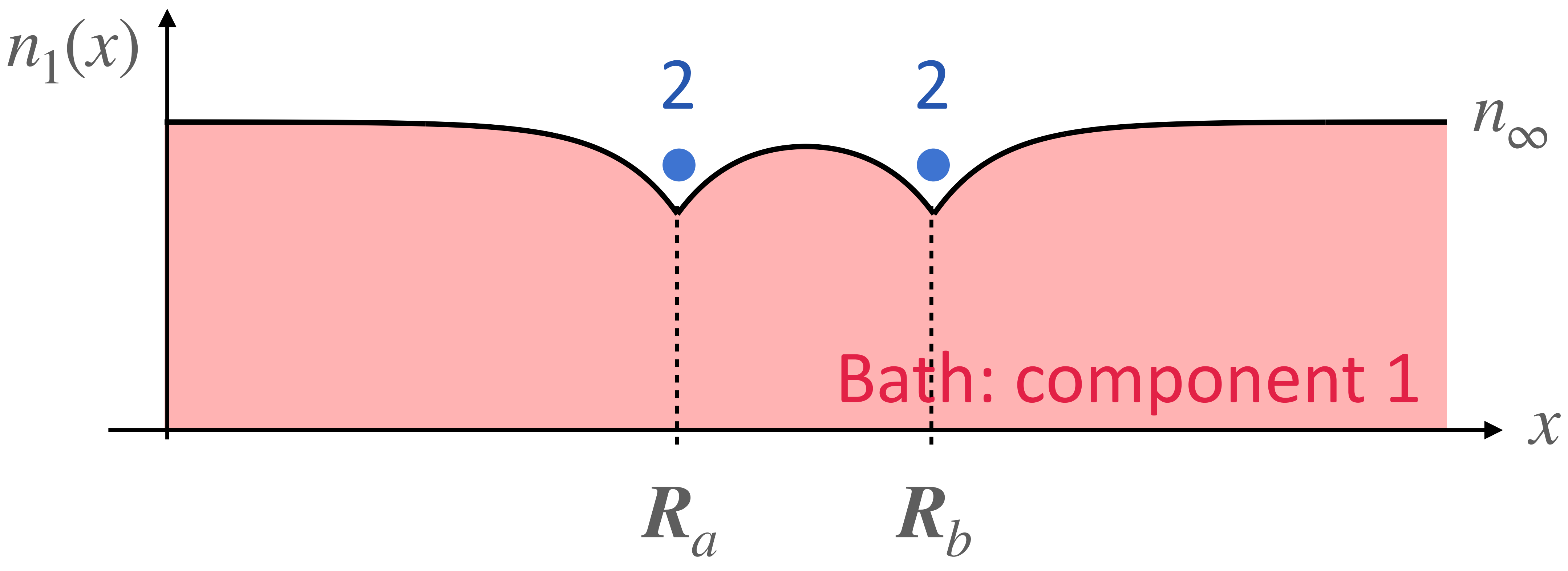
Ground state in the presence of the impurity obtained by perturbation theory: $|\Phi_0'\rangle = |\Phi_0\rangle + \sum_{\boldsymbol{k}} \frac{\langle\Phi_{\boldsymbol{k}}|\hat{V}(\boldsymbol{R}_a)|\Phi_0\rangle}{E_0 - E_k}|\Phi_{\boldsymbol{k}}\rangle$

→ Hole of size $\sim \xi = \dfrac{1}{\sqrt{2g_{11}n_\infty}}$ with a relative depth $\dfrac{\delta n_1}{n_\infty} \sim g_{12} \ll 1$

→ Typically $\delta n_1 \xi^2 \sim \dfrac{g_{12}}{g_{11}} \sim 1$ atom of the bath missing around the location of the impurity

# Yukawa potential

Naidon, 2018




Interaction potential : $\hat{\mathscr{V}} = \hat{V}(\boldsymbol{R}_a) + \hat{V}(\boldsymbol{R}_b)$

*induces at 2nd order of perturbation theory an energy shift $\Delta E$ that depends on the distance $R_{ab}$*

$$\Delta E = -\sum_{\alpha \neq 0} \frac{\langle \Phi_0 | \mathscr{V} | \Phi_\alpha \rangle \langle \Phi_\alpha | \hat{\mathscr{V}} | \Phi_0 \rangle}{E_\alpha - E_0}$$

Interaction between the two impurities mediated by the exchange of virtual phonons in the bath

A simple calculation gives: $U_{\mathrm{med}}(R_{ab}) = -\dfrac{2g_{12}^2 n_\infty}{(2\pi)^D} \displaystyle\int \frac{\mathrm{e}^{i\boldsymbol{k}\cdot(\boldsymbol{R}_a - \boldsymbol{R}_b)}}{\epsilon_k + 2g_{11}n_\infty}\, \mathrm{d}^D k \qquad \epsilon_k = \dfrac{\hbar^2 k^2}{2m}$

Fourier transform of a Lorentzian: Yukawa potential (for $D = 3$) of range $\xi = 1/\sqrt{2g_{11}n_\infty}$

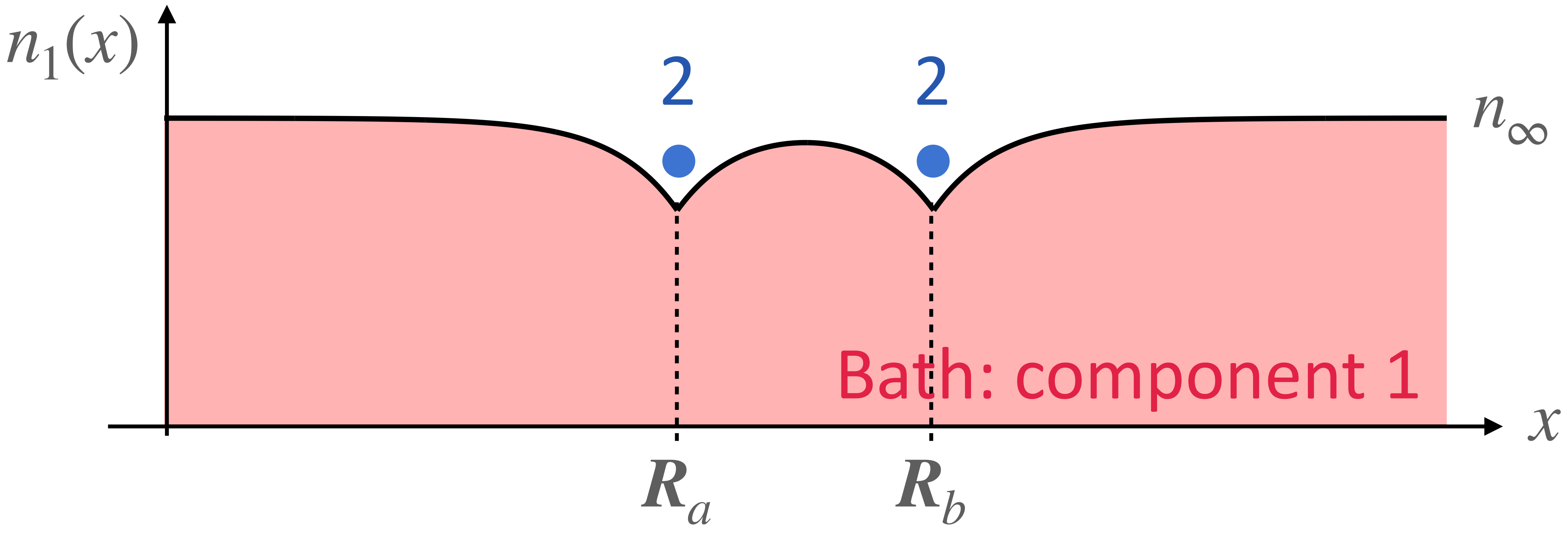
*Note: $\Delta E < 0$ irrespective of the sign of $g_{12}$ , i.e. the mediated interaction is always attractive*

# Born approximation

$$\delta(\boldsymbol{k}) = \frac{1}{(2\pi)^2}\int e^{i\boldsymbol{k}\cdot\boldsymbol{R}}\,d^2R$$

$$U_{\rm med}(R_{ab}) = -\frac{2g_{12}^2 n_\infty}{(2\pi)^2}\int \frac{e^{i\boldsymbol{k}\cdot(\boldsymbol{R}_a-\boldsymbol{R}_b)}}{\epsilon_k + 2g_{11}n_\infty}\,d^2k$$

$$\epsilon_k = \frac{\hbar^2k^2}{2m}$$




Using Born approximation, this leads to the dimensionless interaction strength for the mediated interaction:

$$g_{\rm med} = \int U_{\rm med}(R)\,d^2R = -\frac{2g_{12}^2 n_\infty}{(2\pi)^2}\int\int \frac{e^{i\boldsymbol{k}\cdot\boldsymbol{R}}}{\epsilon_k + 2g_{11}n_\infty}\,d^2k\,d^2R = -2g_{12}^2 n_\infty\int\frac{\delta(\boldsymbol{k})}{\epsilon_k + 2g_{11}n_\infty}\,d^2k = -\frac{g_{12}^2}{g_{11}}$$

We thus recover $g_{\rm eff} = g_{\rm bare} + g_{\rm med} = g_{22} - \dfrac{g_{12}^2}{g_{11}}$

*Warning: The bare interaction cannot be obtained by Born approximation.*
*The procedure above is valid thanks to the separation of length scales :* $R_{\rm vdW} \ll \xi$

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1. Summary of lecture 1

2. A microscopic view on soliton formation

**3. Towards the droplet regime**

*The role of non-local interactions + finite bath-depletion effects*

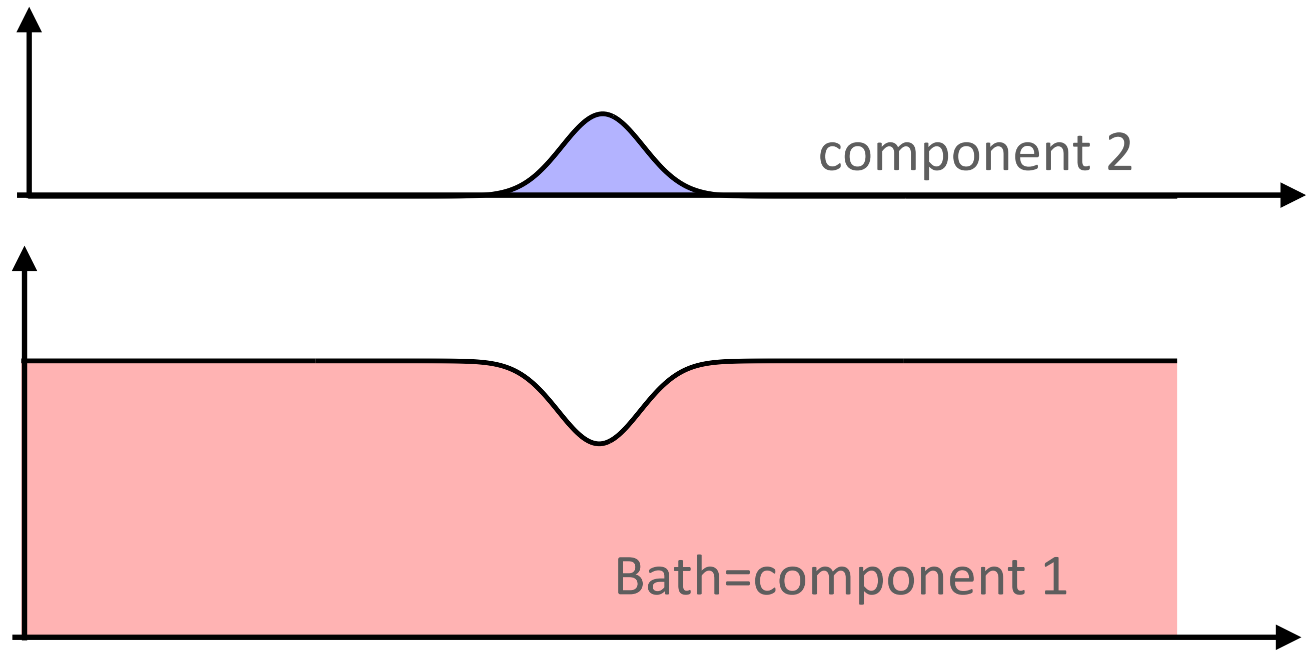
B. Bakkali-Hassani, C. Maury, S. Stringari, S. Nascimbene, J. Dalibard, J. Beugnon, arXiv

See also P. Cheiney, C. R. Cabrera, J. Sanz, B. Naylor, L. Tanzi, and L. Tarruell,
PRL **120**, 135301 (2018) for a transition 1D to 3D in the beyond-mean-field context

4. The excitation spectrum of the soliton (and its daughter states)

# Non-local corrections

Back to the Gross-Pitaevskii equation for the minority component




$$\mu_2 \psi_2(\boldsymbol{r}) = \frac{\hbar^2}{m}\left[-\frac{1}{2}\nabla^2 + g_{22}n_2(\boldsymbol{r}) + \int U_{\text{med}}(\boldsymbol{r}-\boldsymbol{r}')\, n_2(\boldsymbol{r}')\, \mathrm{d}^2 r'\right]\psi_2(\boldsymbol{r})$$

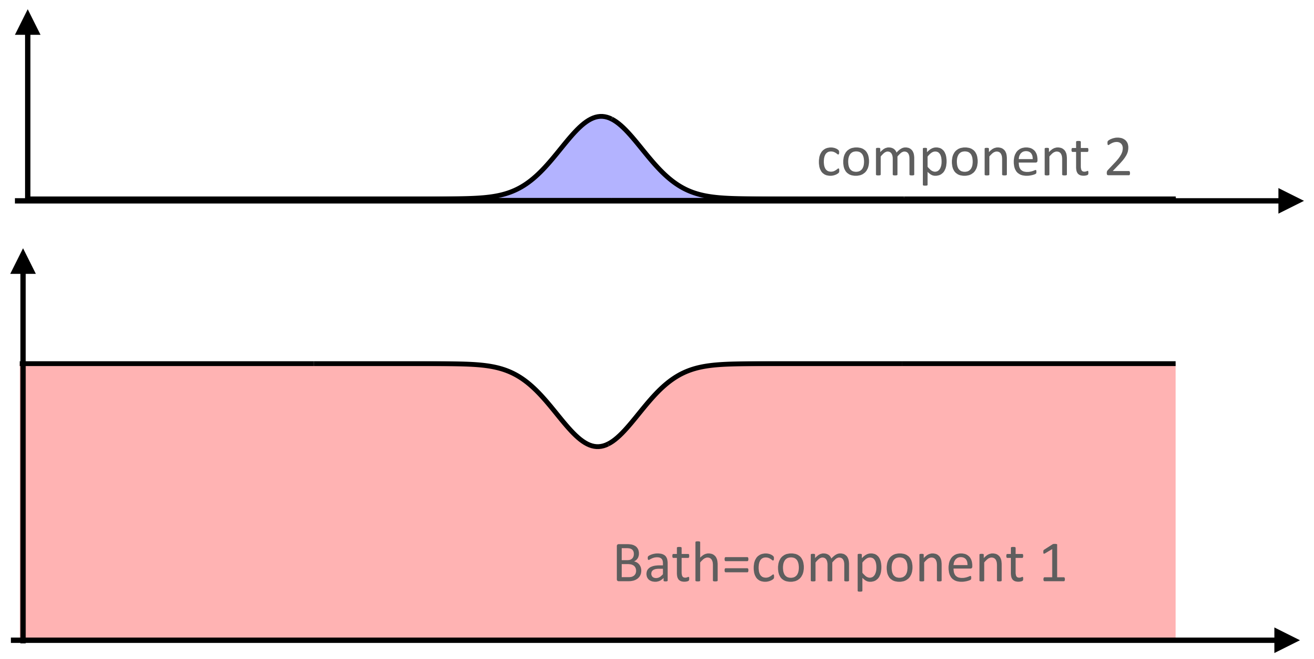
Taylor expansion of $U_{\text{med}}(\boldsymbol{r}-\boldsymbol{r}')$ :

$$\mu_2\, \partial_t \psi_2(\boldsymbol{r}) = \frac{\hbar^2}{m}\left[-\frac{1}{2}\nabla^2 + \underbrace{\left(g_{22} - \frac{g_{12}^2}{g_{22}}\right)}_{g_{\text{eff}}} n_2(\boldsymbol{r}) \underbrace{- \beta\, \nabla^2 n_2(\boldsymbol{r})}_{\text{Non-local correction that breaks scale invariance (stabilizing role !)}} + \dots\right]\psi_2(\boldsymbol{r})$$

$$\beta = \left(\frac{g_{12}}{g_{11}}\right)^2 \frac{1}{4n_\infty}$$

Equation that has been studied in detail in the context of optics (Rosanov et al., 2002)

# Stabilization of the Townes soliton




**Component 2 with $N_2$ atoms and a size $\ell$ : $n_2 \sim N_2/\ell^2$**

Consider the equation including the first non-local correction

$$\mu_2 \psi_2(\boldsymbol{r}) = \frac{\hbar^2}{m}\left[-\frac{1}{2}\nabla^2 \; - \; |g_{\rm eff}|\, n_2(\boldsymbol{r}) \; - \; \beta\, \nabla^2 n_2(\boldsymbol{r})\right] \psi_2(\boldsymbol{r})$$

$$\beta \sim 1/n_\infty$$

Energy/particle for component 2 for a Townes-like profile:

$$\frac{E}{N_2} \sim \frac{\alpha}{\ell^2} - \frac{|g_{\rm eff}|\, N_2}{\ell^2} + \beta \frac{N_2}{\ell^4}$$

$$\frac{E}{N_2} \sim \frac{|g_{\rm eff}|\,(N_{\rm Townes} - N_2)}{\ell^2} + \beta \frac{N_2}{\ell^4}$$

stable if and only if
$|g_{\rm eff}|\, N_2 = 5.80$

Expand for $N_2 < N_{\rm Townes}$ ⟵ ⟶ Collapse for $N_2 > N_{\rm Townes} = 5.80/|g_{\rm eff}|$

prevents the collapse
when $|g_{\rm eff}|\, N_2 > 5.80$

$$\ell^2 \sim \frac{\beta}{|g_{\rm eff}|}\, \frac{N_2}{N_2 - N_{\rm Townes}} \qquad n_2 \sim n_\infty \frac{N_2 - N_{\rm Townes}}{N_{\rm Townes}}$$

**Valid only for a small depletion $n_2 \ll n_\infty$, i.e. if $N_2 - N_{\rm Townes} \ll N_{\rm Townes}$**

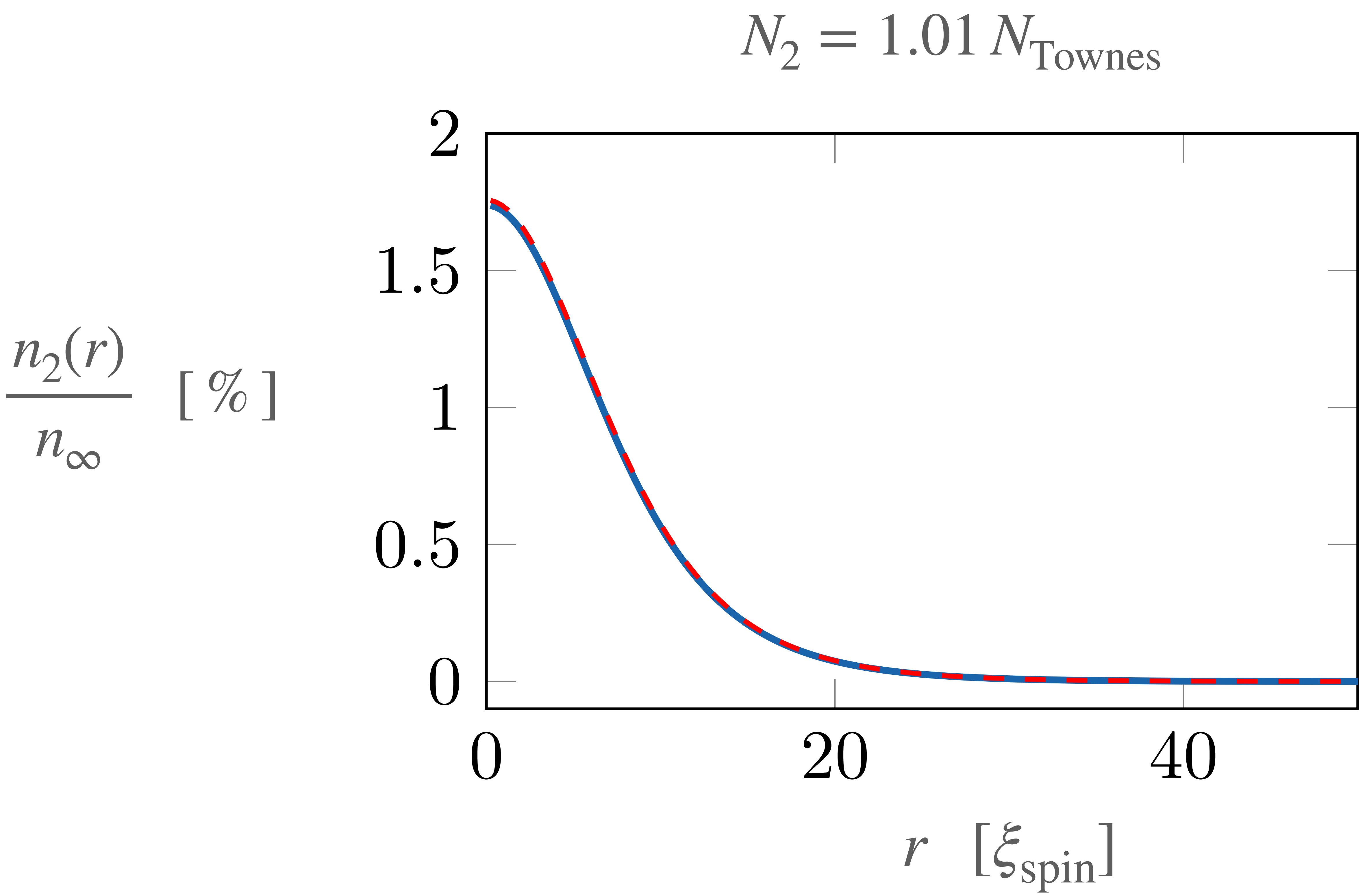
# Check of the validity of this “simple” non-local equation

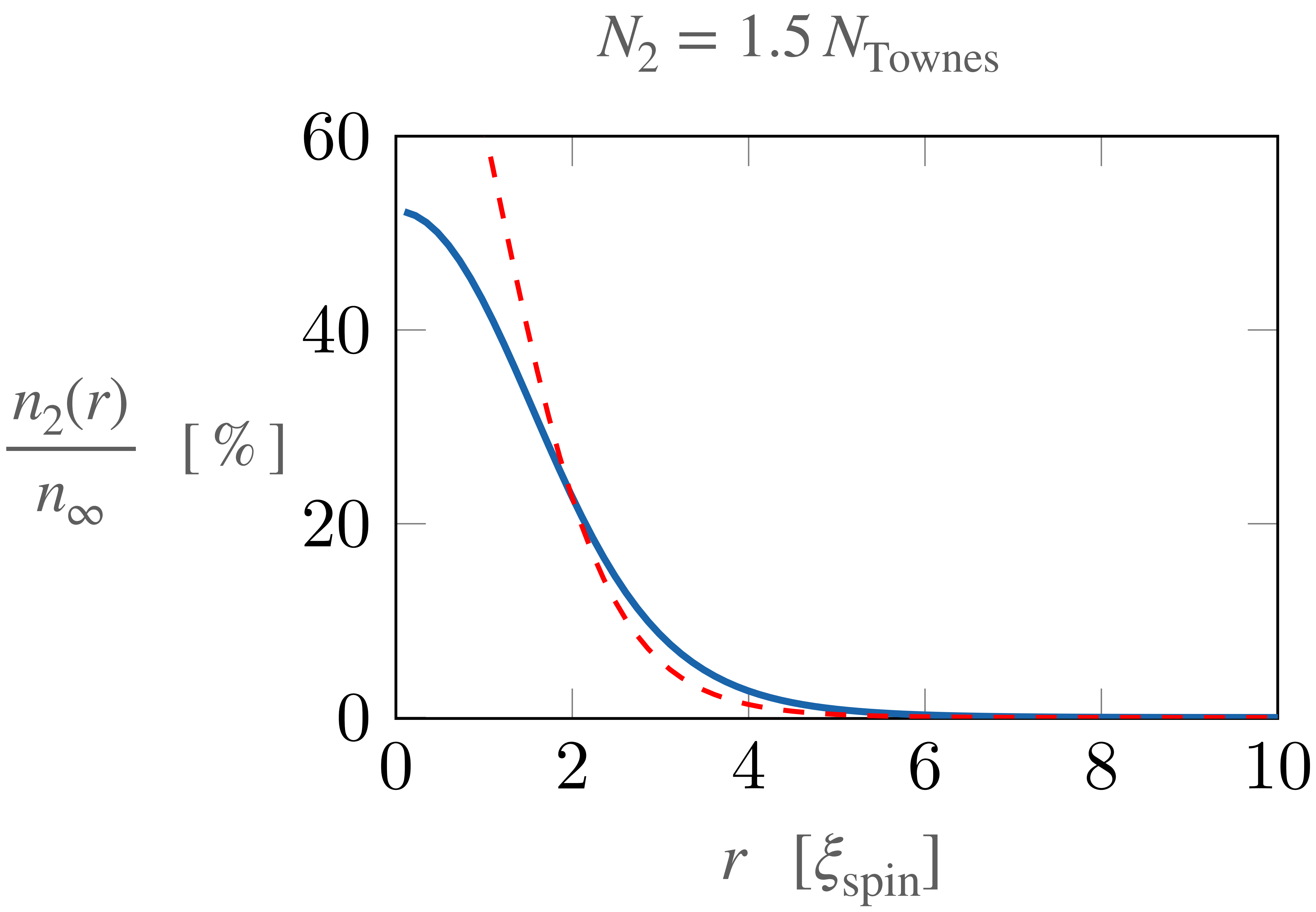
$g_{11} = g_{22}$

$g_{12} = 1.01\, g_{11}$

Solution of $\mu_2 \psi_2(\boldsymbol{r}) = \frac{\hbar^2}{m}\left[-\frac{1}{2}\nabla^2 - |g_{\rm eff}|\, n_2(\boldsymbol{r}) - \beta\, \nabla^2 n_2(\boldsymbol{r})\right]\psi_2(\boldsymbol{r})$

vs. solution of the two coupled GP equations

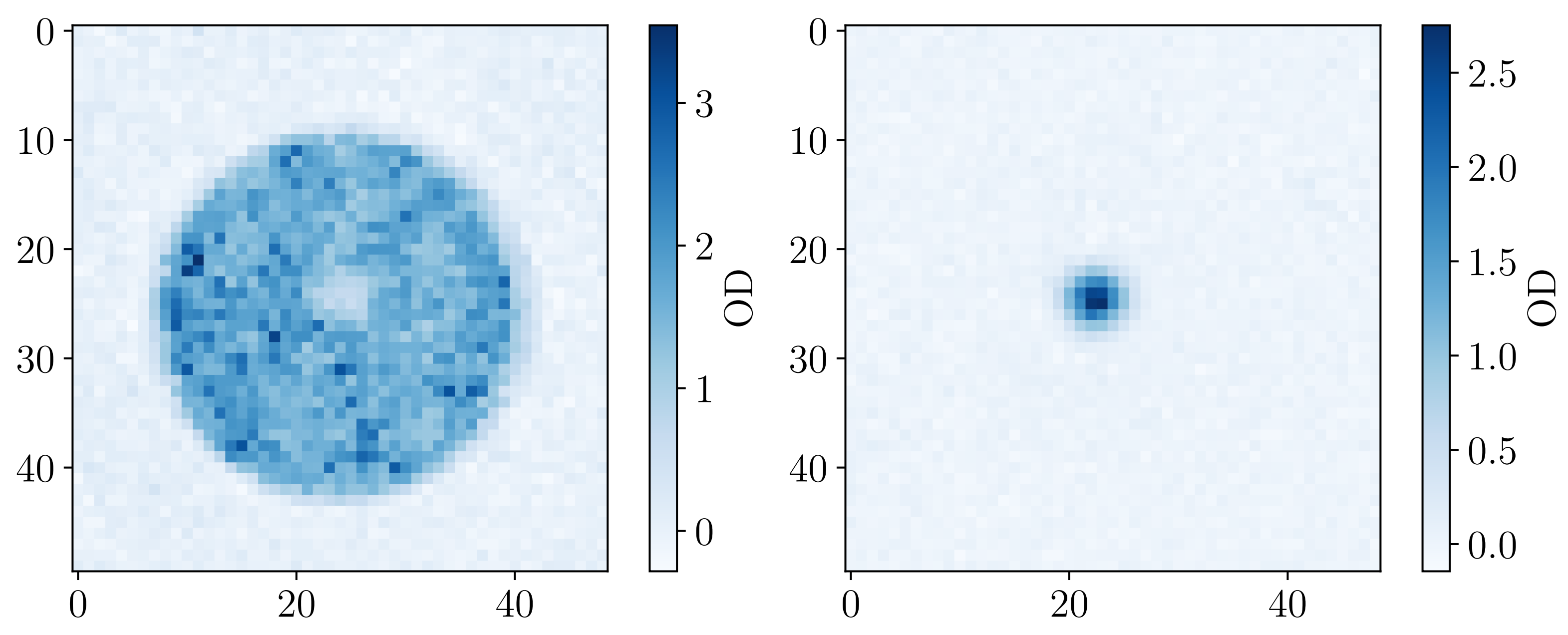


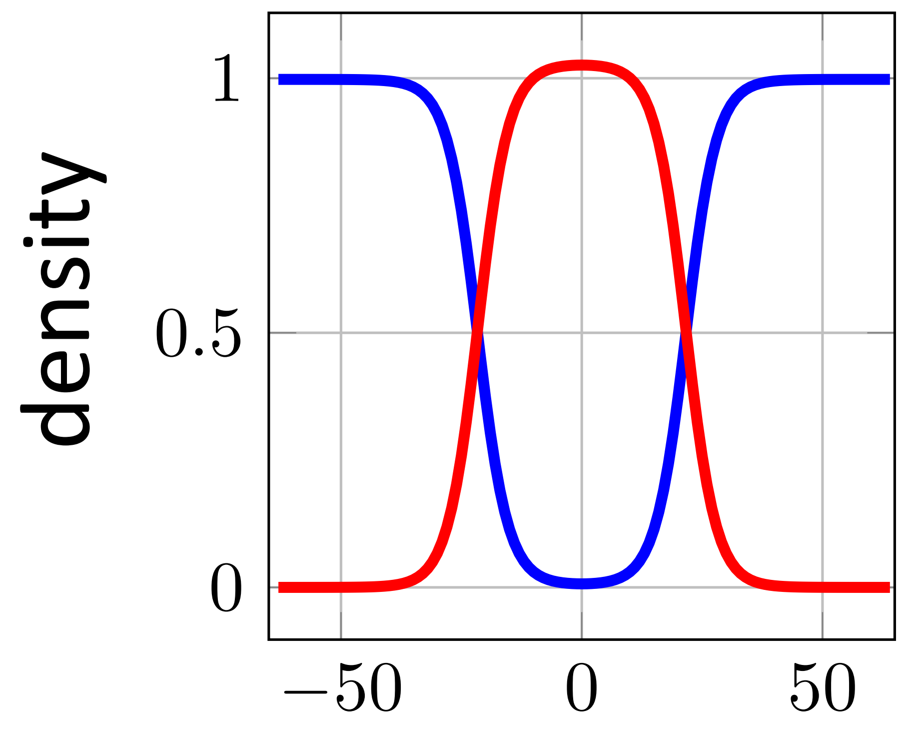

$\xi_{\rm spin} = 1/\sqrt{2\,|g_{\rm eff}|\, n_\infty}$

# Beyond the weak depletion regime

Formation of quasi-pure domains of the minority component




*F*=1 *F*=2




Take advantage the similarity of all coupling constants: $g_{11} \approx g_{12} \approx g_{22}$ (SU(2) symmetry)

⟶ $n_1 + n_2 = n + \delta n$ where $\delta n$ is treated as a small parameter

⟶ Spin healing length $\xi_{\mathrm{spin}} = \dfrac{1}{\sqrt{2 g_{\mathrm{eff}} n_\infty}}$ $\gg$ Bath healing length $\xi = \dfrac{1}{\sqrt{2 g_{11} n_\infty}}$

$$\mu_2 \psi_2(\boldsymbol{r}) = \frac{\hbar^2}{m}\left[-\frac{1}{2}\nabla^2 + g_{\mathrm{eff}} n_2(\boldsymbol{r}) + \frac{1}{2}\frac{\nabla^2\sqrt{n_\infty - n_2(\boldsymbol{r})}}{\sqrt{n_\infty - n_2(\boldsymbol{r})}}\right]\psi_2(\boldsymbol{r})$$

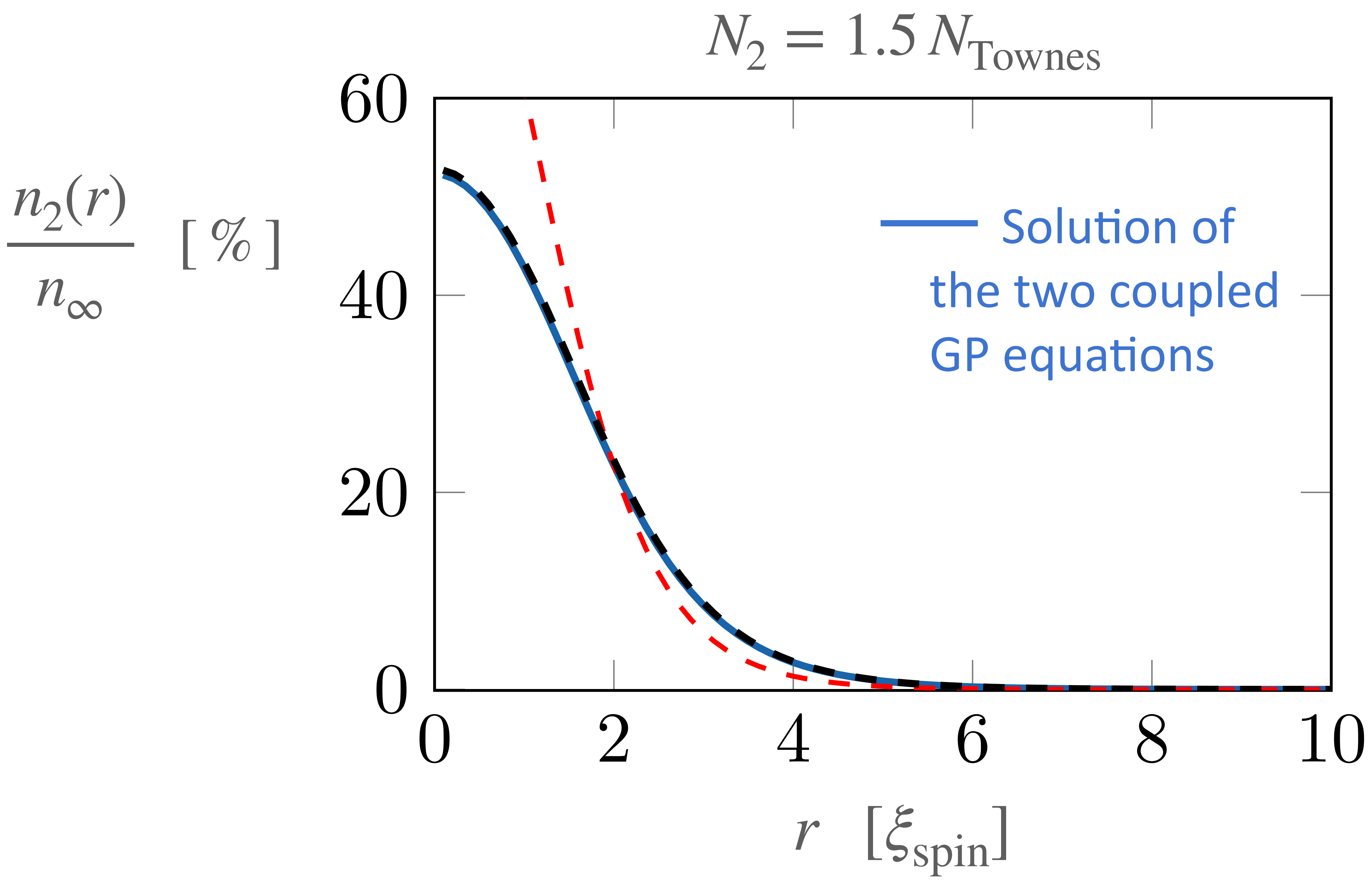
# Validity of single component equations

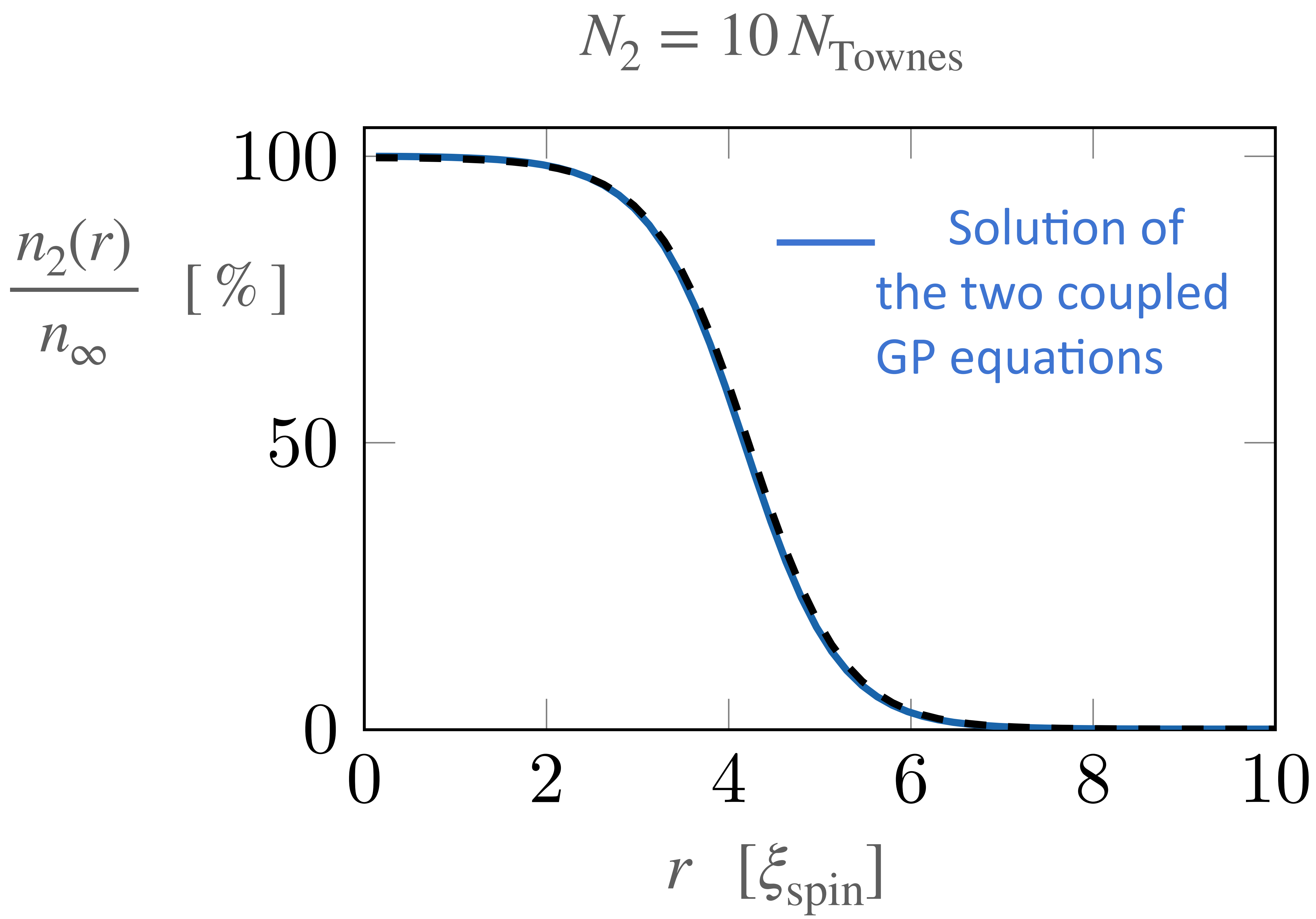
$g_{11} = g_{22}$

$g_{12} = 1.01\, g_{11}$

Weak depletion limit: $\mu_2 \psi_2(\boldsymbol{r}) = \frac{\hbar^2}{m}\left[-\frac{1}{2}\nabla^2 - |g_{\text{eff}}|\, n_2(\boldsymbol{r}) - \beta \nabla^2 n_2(\boldsymbol{r})\right]\psi_2(\boldsymbol{r})$ $\beta = \left(\frac{g_{12}}{g_{11}}\right)^2 \frac{1}{4n_\infty}$

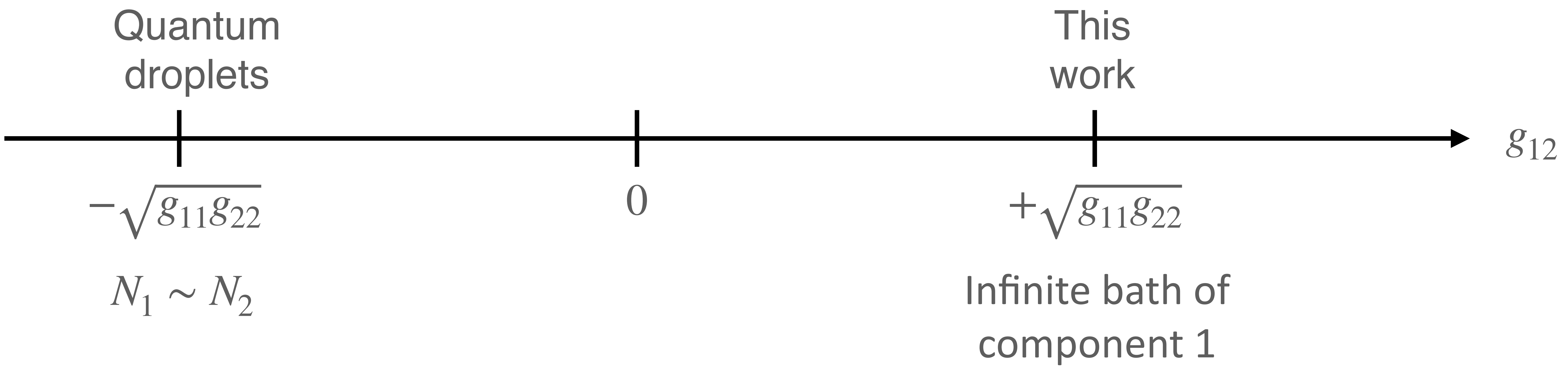
Exploiting $\sim$SU(2) symmetry: $\mu_2 \psi_2(\boldsymbol{r}) = \frac{\hbar^2}{m}\left[-\frac{1}{2}\nabla^2 + g_{\text{eff}} n_2(\boldsymbol{r}) + \frac{1}{2}\frac{\nabla^2\sqrt{n_\infty - n_2(\boldsymbol{r})}}{\sqrt{n_\infty - n_2(\boldsymbol{r})}}\right]\psi_2(\boldsymbol{r})$



# Comparison with quantum droplets

D. S. Petrov, PRL **115**, 155302 (2015)




| Energy/particle | Kinetic | Mean-field (contact) | Stabilizing term | |
|---|---|---|---|---|
| Quantum droplet (3D) | $1/\ell^2$ | $-\lvert\delta g\rvert\, N/\ell^3$ | $\bar{g}^{5/2}N^{3/2}/\ell^{9/2}$ | beyond mean-field |
| This work (2D) (low depletion case) | $1/\ell^2$ | $-\lvert g_{\text{eff}}\rvert\, N/\ell^2$ | $\beta\, N/\ell^4$ | non-local potential |

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**4. The excitation spectrum of the soliton and of its daughter states**

B. Bakkali-Hassani, C. Maury, S. Stringari, S. Nascimbene, J. Dalibard, J. Beugnon, arXiv 2207.06939

Strong connection with the results of D. S. Petrov, PRL **115**, 155302 (2015)
in the context of beyond-mean-field Quantum Droplets

Quantum droplets in low dimension: Petrov & Astrakharchik, PRL **117**, 100401 (2016).
Stürmer et al., PRA **103**, 053302 (2021)

# Bogoliubov analysis

Two-component system $\psi_j(\boldsymbol{r},t) = \left[\psi_j^{\mathrm{stat}}(r) + \alpha_j(\boldsymbol{r},t) + \mathrm{i}\beta_j(\boldsymbol{r},t)\right] \mathrm{e}^{-\mathrm{i}\mu_j t}$ with $\psi_j^{\mathrm{stat}}, \alpha_j, \beta_j$ real $j = 1{,}2$

Bogoliubov linear system

$$\partial_t \begin{pmatrix} \alpha_1 \\ \beta_1 \\ \alpha_2 \\ \beta_2 \end{pmatrix} = \begin{pmatrix} 0 & \hat{L}_0^{(1)} & 0 & 0 \\ -\hat{L}_1^{(1)} & 0 & -\hat{L}_{12} & 0 \\ 0 & 0 & 0 & \hat{L}_0^{(2)} \\ -\hat{L}_{12} & 0 & -\hat{L}_1^{(2)} & 0 \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \beta_1 \\ \alpha_2 \\ \beta_2 \end{pmatrix}$$

$$\hat{L}_0^{(1)} = -\mu_1 - \frac{1}{2}\nabla^2 + g_{11}n_1^{\mathrm{stat}} + g_{12}n_2^{\mathrm{stat}}$$

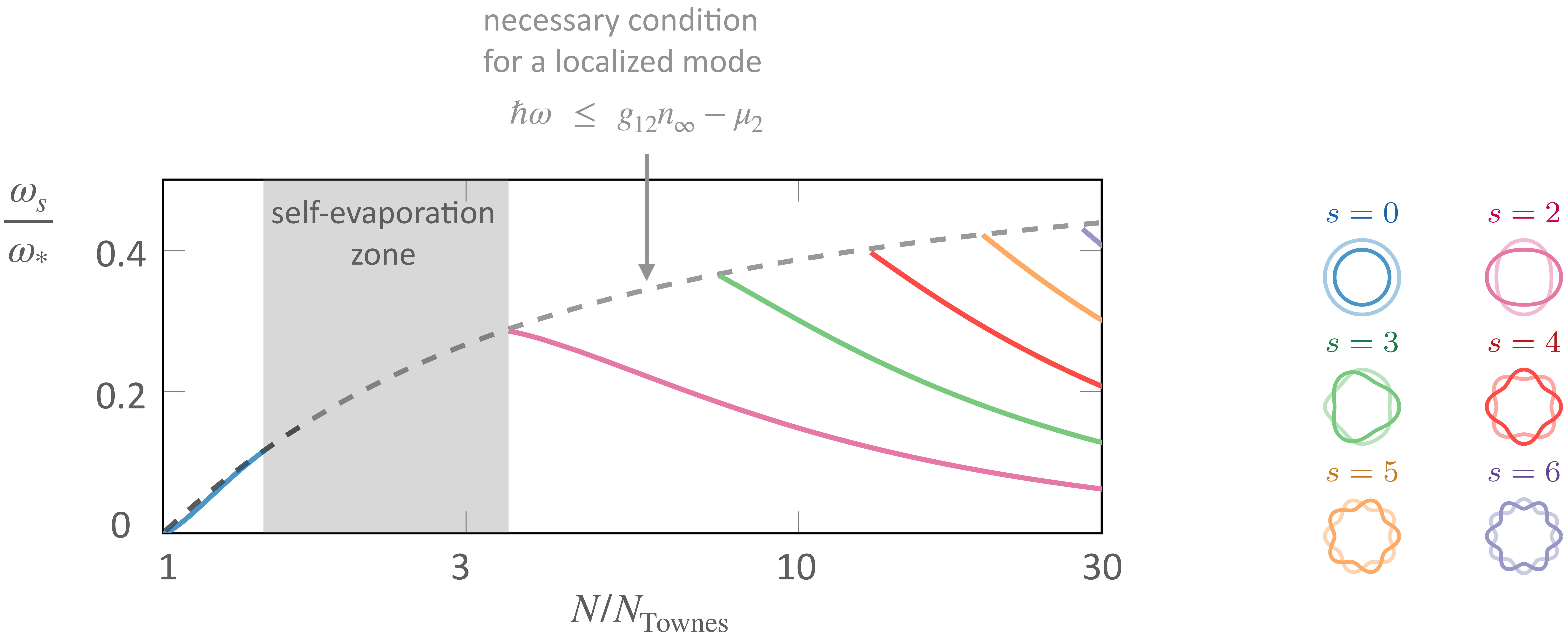
$$\hat{L}_1^{(1)} = -\mu_1 - \frac{1}{2}\nabla^2 + 3g_{11}n_1^{\mathrm{stat}} + g_{12}n_2^{\mathrm{stat}}$$

$$\hat{L}_{12} = 2g_{12}\sqrt{n_1^{\mathrm{stat}}n_2^{\mathrm{stat}}}$$

Numerical solution, keeping only the modes corresponding to localized $\alpha_j, \beta_j$ functions

# The result for the localized spectrum

Calculation performed for $g_{11} = g_{22} \equiv g$ and $g_{12} = 1.01\, g$




$$\omega_* = \left(\frac{g}{g_{12}}\right)^2 |g_{\mathrm{eff}}|\, n_\infty$$

Self-evaporation: Stringari & Vautherin, Physics Letters B **88**, 1 (1979)
Ferioli et al., PR Research **2**, 013269 (2020)
Fort & Modugno, Applied Sciences **11**, 866 (2021)

# The monopole mode $s = 0$

$g_{11} = g_{22} \equiv g$

Back to the single-component equation with its non-local correction

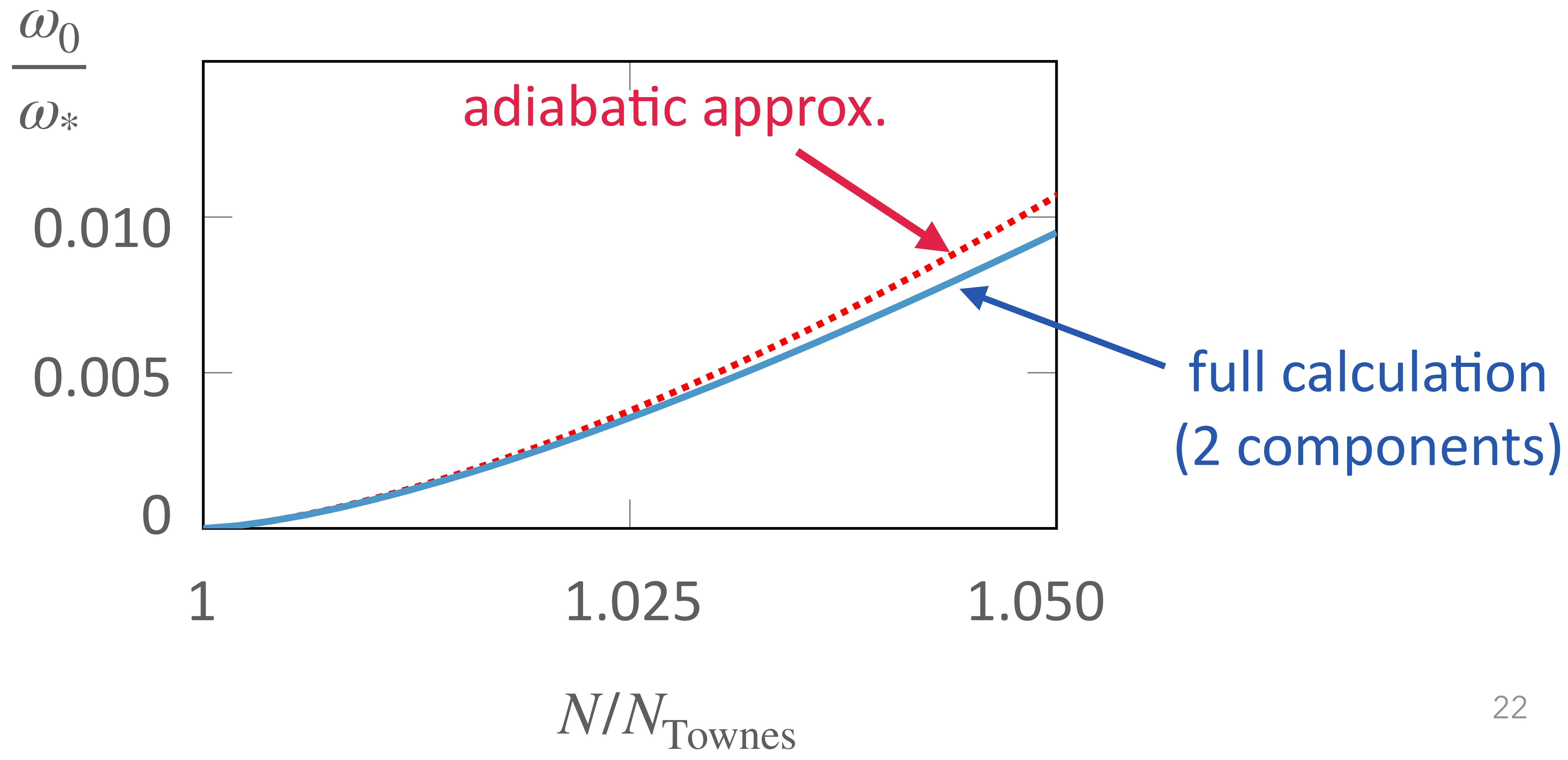
$$i\hbar\, \partial_t \psi_2(\boldsymbol{r}) = \frac{\hbar^2}{m}\left[-\frac{1}{2}\nabla^2 - |g_{\text{eff}}|\, n_2(\boldsymbol{r}) - \beta \nabla^2 n_2(\boldsymbol{r})\right] \psi_2(\boldsymbol{r})$$

*Time-dependent version assuming that the bath follows adiabatically the minority component*

Rosanov et al., 2002:

$$\omega_0 = 0.95\ \omega_* \left(\frac{N}{N_{\text{Townes}}} - 1\right)^{3/2}$$

$$\omega_* = \left(\frac{g}{g_{12}}\right)^2 |g_{\text{eff}}|\, n_\infty$$



# The monopole mode $s = 0$ : sum rule

Pitaevskii & Stringari, chap.12

$$m_p = \int \omega^p \, S(\omega) \, \mathrm{d}\omega$$

$S(\omega)$ : dynamic structure factor

$$S(\omega) = \sum_n \left| \langle n | F | 0 \rangle \right|^2 \delta(\omega - \omega_{n0})$$

$$F = \sum_j x_j^2 + y_j^2$$

The sum rule approach provides an upper bound for $\omega_0$ : $\quad \omega_0^2 \lesssim \dfrac{m_1}{m_{-1}}$

$m_1$ : energy-weighted sum rule

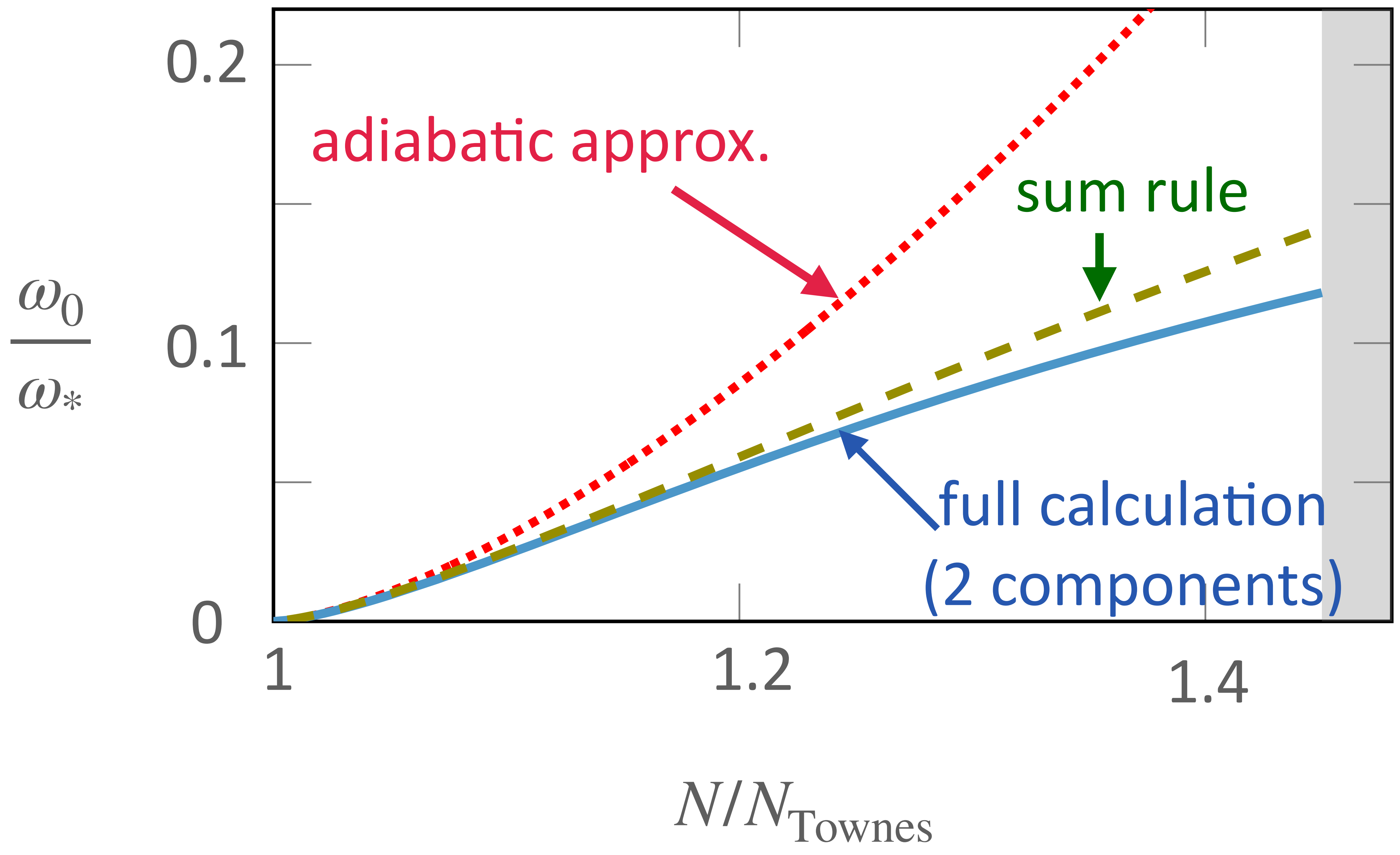
$$m_1 = -2\langle x^2 + y^2 \rangle$$

*Average taken by integrating the density of the minority component using the ground state wave function of the mixture.*

$m_{-1}$ : inverse energy-weighted sum rule

$$m_{-1} = -\frac{1}{2\lambda_0}\delta\langle x^2 + y^2 \rangle$$

*Static response of the system* $\delta\langle x^2 + y^2 \rangle$ *to a perturbation* $\lambda_0(x^2 + y^2)$ *for component 2*



# Surface waves

Petrov (2015) for quantum droplets

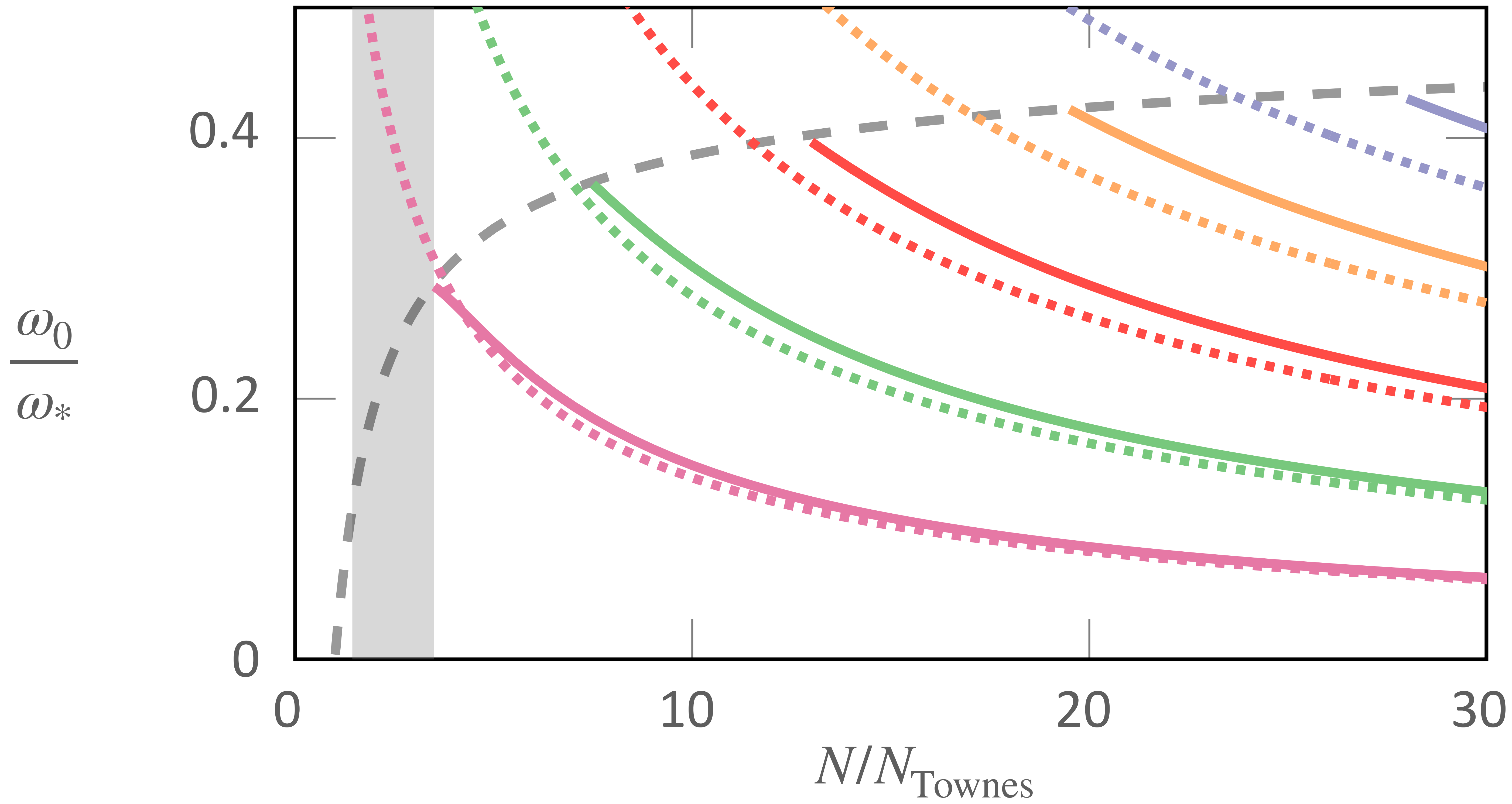
Interpretation in terms of surface tension (two-dimensional ripplons) by 2D incompressible hydrodynamics

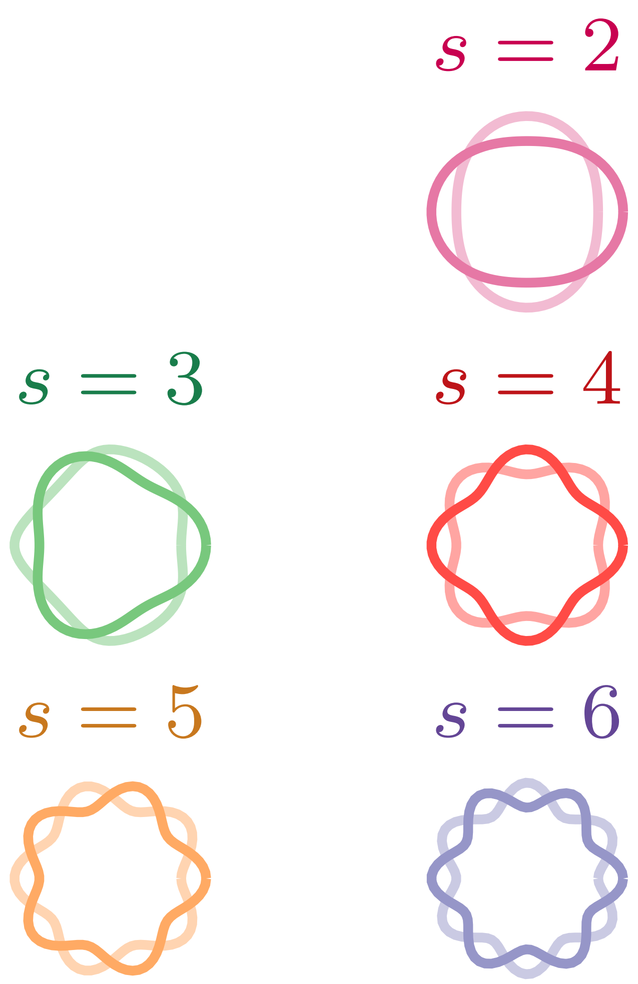
$$\omega_s = \sqrt{\frac{\mathcal{T}}{2n_\infty R^3} s(s-1)(s+1)}$$

Akulenko & Nesterov, 1998

$$\mathcal{T} \simeq \frac{1}{2}\sqrt{|g_{\rm eff}|}\; n_\infty^{3/2}$$

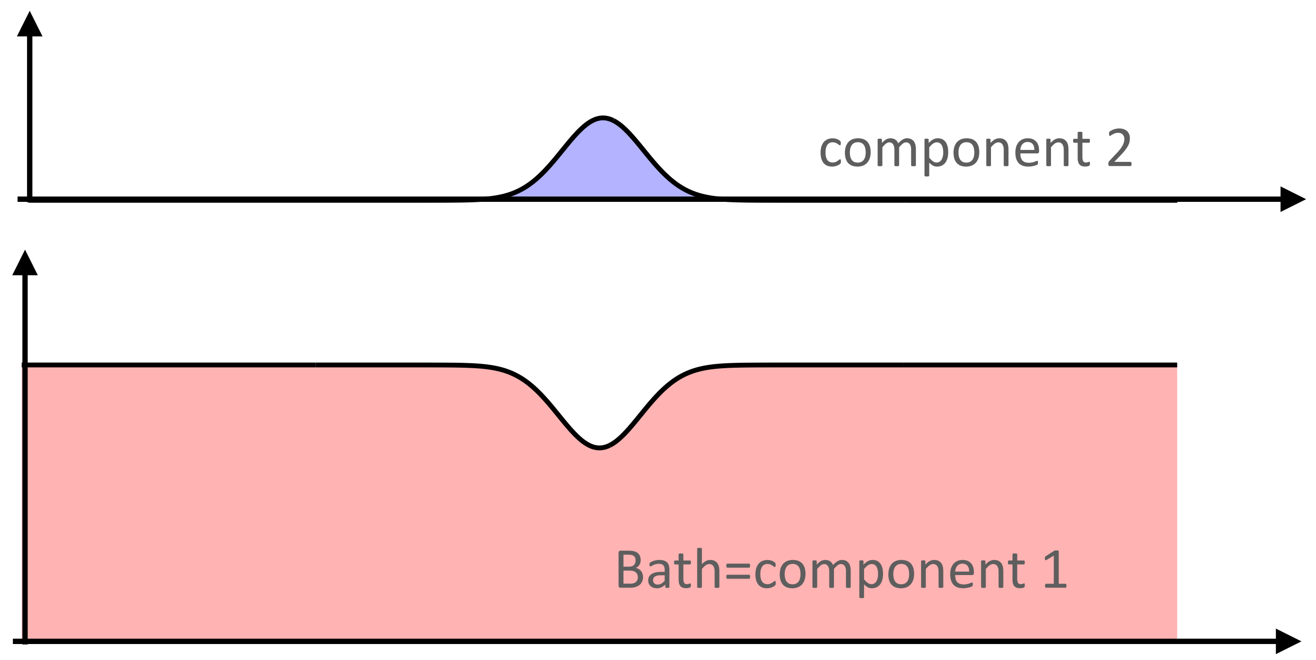
Ao & Chui, 1998 ; Barankov, 2002



# Concluding remarks

The Townes soliton problem: a paradigm to study the consequences of scale invariance… and the ways to break it




A non-linear physics problem:
addressable by mean-field Gross-Pitaevski equations, but leading to a phenomenology close to the quantum droplets stabilized by beyond-mean-field terms

Intriguing domain of parameters leading to self-evaporation: can be studied here at a very low density, where other loss mechanisms can be minimized

See also Fort & Modugno (2021) for $^{41}K$-$^{87}Rb$