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http://quantuminstitute.yale.edu/

Introduction to Circuit QED

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Army Research Office
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AFOSR AIR FORCE OFFICE OF SCIENTIFIC RESEARCH

Lecture notes on circuit QED (150 pages)
2011 Les Houches Summer School

https://girvin.sites.yale.edu/lectures

Lecture series on quantum error correction and fault tolerance

arXiv:2111.08894: Introduction to Quantum Error Correction and Fault Tolerance

Videos of above lectures: https://girvin.sites.yale.edu/lectures

OUTLINE:

Lecture 1: Introduction to Circuit QED

- Quantum LC Oscillators
- Josephson Junctions & Transmon Qubits
- Qubits coupled to microwave cavities

Lecture 2: Gates for Programmable Quantum Simulations of Bosons and Spins

- FQHE for Microwave Light
- Z2 Lattice Gauge Theories

Lecture 3: Gaussian Boson Sampling for Vibrational Spectroscopy of Small Molecules

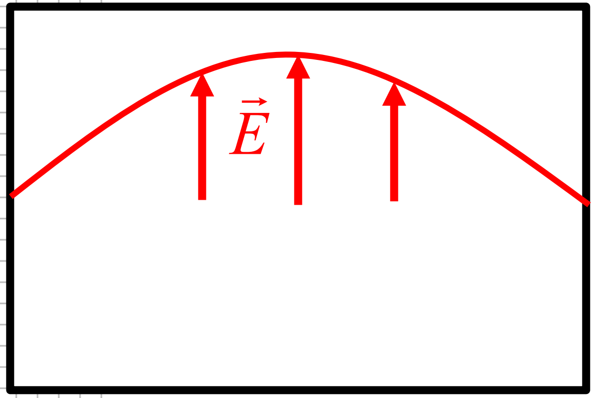
[If time: Experimental Bosonic QEC Beyond Breakeven]

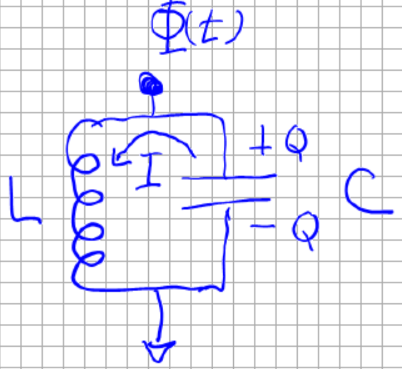
Introduction to Circuit QED

Artificial atoms and microwave photons

How to be a quantum electrical engineer

LC oscillator [Lumped element LC or single mode of a microwave cavity resonator]





Define generalized flux

$$\Phi(t) \equiv \int^{t} d\tau \, V(\tau)$$

$$\dot{\Phi} = V$$

Faraday induction (up to a minus sign)

electrostatic energy $\frac{1}{2} C \dot{\Phi}^2$

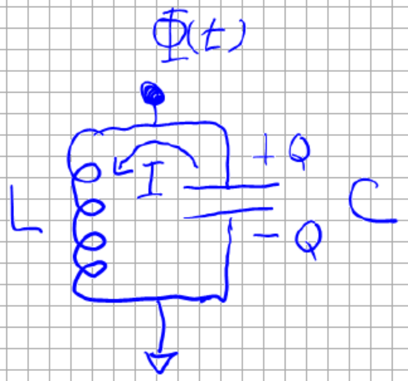
magnetic energy $\frac{1}{2} L I^2 = \frac{1}{2L} \Phi^2$ $(\Phi = IL)$

Lagrangian $\mathcal{L} = \frac{1}{2} C \dot{\Phi}^2 - \frac{1}{2L} \Phi^2$

$$\mathcal{L} = \frac{1}{2} C \dot{\Phi}^2 - \frac{1}{2L} \Phi^2$$

velocity

coordinate



momentum $Q \equiv \dfrac{\delta \mathcal{L}}{\delta \dot{\Phi}} = C \dot{\Phi} = CV$

charge Q is momentum canonically conjugate to flux.

Hamiltonian $H = Q \dot{\Phi} - \mathcal{L} = \dfrac{Q^2}{2C} + \dfrac{\Phi^2}{2L}$

harmonic oscillator with "mass" $m = C$

"spring constant" $k = 1/L$

resonance frequency $\omega_R = \sqrt{\dfrac{k}{m}} = \dfrac{1}{\sqrt{LC}}$

Hamilton eqn's of motion

$\dot{\Phi} = \dfrac{\partial H}{\partial Q} = \dfrac{Q}{C} = V$ ✓ Faraday induction

$\dot{Q} = -\dfrac{\partial H}{\partial \Phi} = -\dfrac{\Phi}{L} = -I$ ✓ charge conservation

$$\ddot{\Phi} = \frac{\dot{Q}}{C} = -\frac{1}{LC}\Phi$$

$$I = I_0 \sin(\omega_R t + \theta)$$

$$V = I_0 Z_R \cos(\omega_R t + \theta)$$

characteristic impedance

[Nothing to do with dissipation since current and voltage are 90 degrees out of phase.]

$$I = -\dot{Q} = -C\dot{V} = +\underbrace{\omega_R C Z_R}_{=1} I_0 \sin(\omega_R t + \theta)$$

$$\boxed{Z_R = \frac{1}{\omega_R C} = \sqrt{\frac{L}{C}}}$$

$Z_R \sim 50 - 500\,\Omega$ because impedance of free space $Z_0 = \sqrt{\frac{\mu_0}{\epsilon_0}} \approx 377\,\Omega$

quantum of impedance $Z_K = \frac{h}{e^2} \approx 25{,}812\ \Omega$

$$\alpha = \frac{e^2}{\hbar c}\frac{1}{[4\pi\epsilon_0]} \approx \frac{1}{137}$$

$$\boxed{Z_0 = 2\alpha Z_K}$$

Quantizing the oscillator

$$[\hat{Q},\hat{\Phi}] = -i\hbar \qquad \hat{\Phi} = \Phi_{ZPF}(a+a^{\dagger}) \qquad \hat{Q} = -i\,Q_{ZPF}(a-a^{\dagger})$$

$$[a,a^{\dagger}] = 1 \qquad Q_{ZPF}\,\Phi_{ZPF} = \frac{\hbar}{2}$$

virial thm $\langle 0|\frac{\hat{Q}^2}{2C}|0\rangle = \frac{1}{2}\left(\frac{1}{2}\hbar\omega_R\right) \Rightarrow Q_{ZPF} = \sqrt{\frac{\hbar}{2Z_R}}$

$$\langle 0|\frac{\hat{\Phi}^2}{2L}|0\rangle = \frac{1}{2}\left(\frac{1}{2}\hbar\omega_R\right) \Longrightarrow \Phi_{ZPF} = \sqrt{\frac{\hbar}{2}Z_R}$$

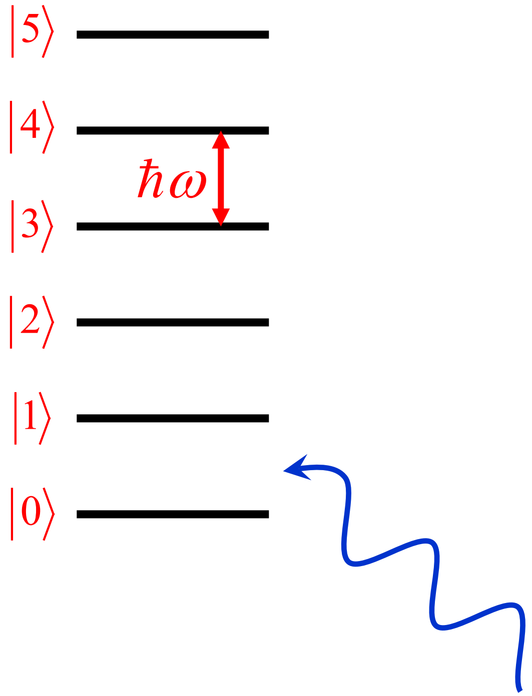
$Q_{ZPF}\,\Phi_{ZPF} = \frac{\hbar}{2}$ ✓ $\quad \psi(\Phi) = \langle\Phi|0\rangle$ is a minimum uncertainty packet

$$\frac{Q_{ZPF}}{e} = \sqrt{\frac{h}{4\pi e^2}\,\frac{1}{Z_R}} = \sqrt{\frac{Z_K}{4\pi Z_R}} \sim \sqrt{\frac{137}{4\pi}} \sim 3$$

Quantum Harmonic Oscillators have many important uses but:

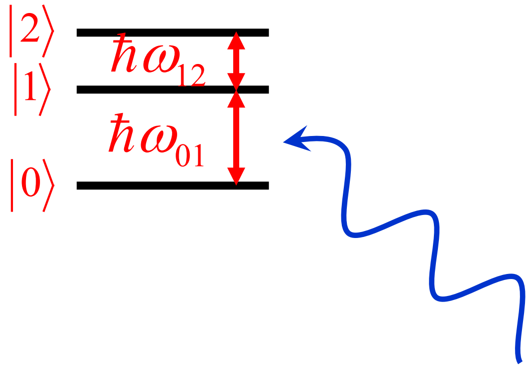
Their level spacing is uniform making them impossible to achieve full *quantum* control with *classical* signals.

$$H = \hbar\omega\, a^{\dagger}a$$



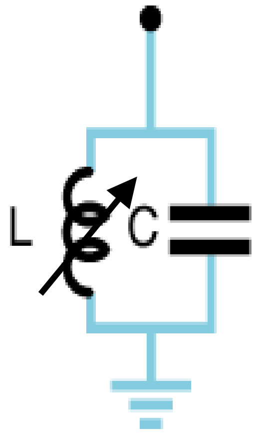
We need anharmonicity to make *qubits* and *ancilla controllers* for oscillators:

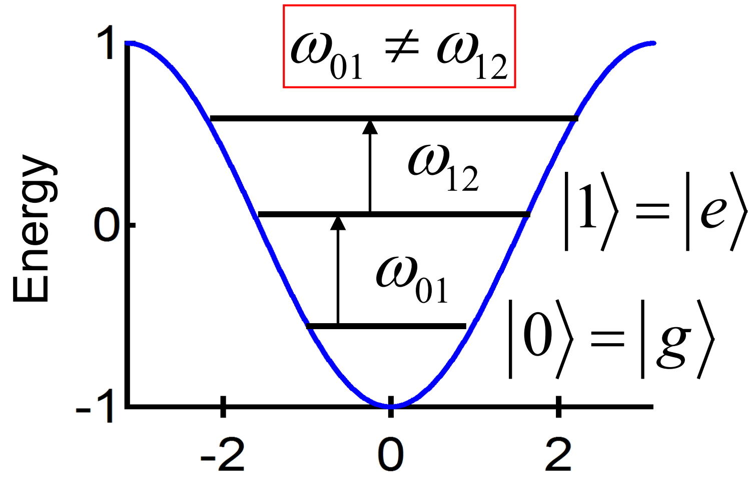
$$H = \hbar\left[\omega\, a^{\dagger}a - \frac{K}{2} a^{\dagger}a^{\dagger}aa\right]$$



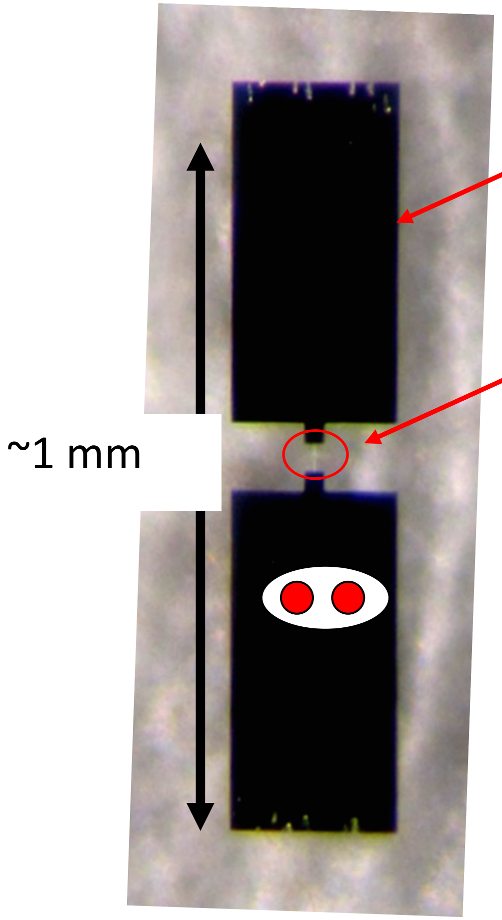
$$\omega_{12} - \omega_{01} = K$$

Joseph tunnel junctions act as non-linear inductors to produce anharmonic oscillators and qubits



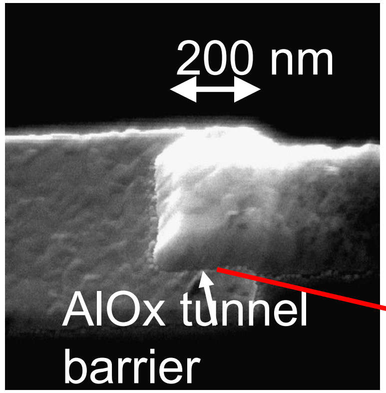


$$\omega_{01} \sim 5-10\ \mathbf{GHz}$$

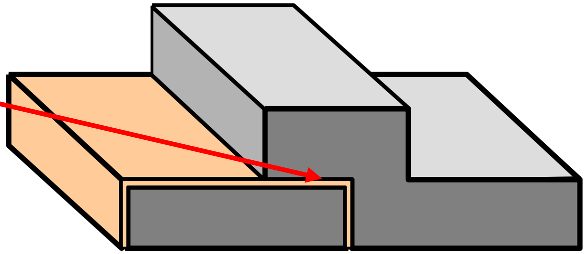


Antenna pads are capacitor plates

Josephson tunnel junction



‘transistor of quantum computing’



‘Circuit QED:’

- -microwave photons inside superconducting circuits
- -artificial atoms (Josephson junction qubits)

Ultra-strong photon-‘atom’ coupling:

- -non-linear quantum optics at the single photon level

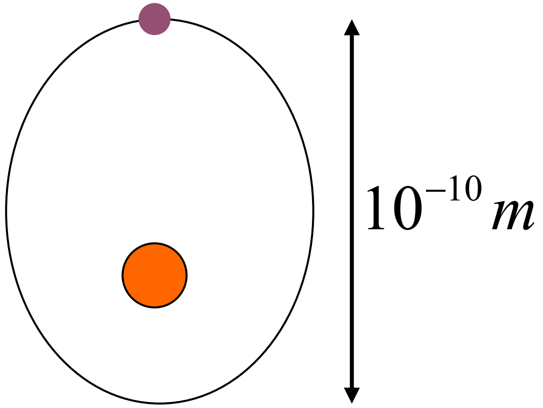
Hydrogen atom

$f_{1S-2P} \approx 2.46\times10^{15}\,\mathrm{Hz}$

$\tau_{2P} \approx 1.6\mathrm{ns}$

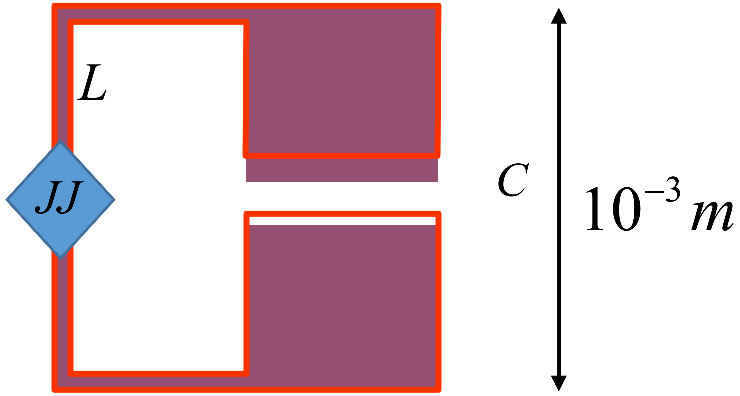
$\mathrm{Q}/2\pi \approx 4\times10^{6}$

$\mathrm{dipole} \sim 1\ \mathrm{Debye}$



(Not to scale!)

Superconducting oscillator/qubit



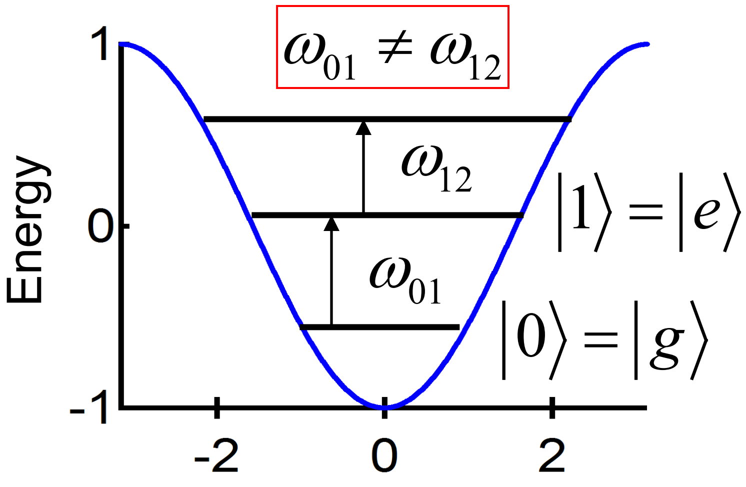
$f_{01} \approx 7\times10^{9}\,\mathrm{Hz}$

$\tau_{2P} \approx 300\,\mu\mathrm{s}$

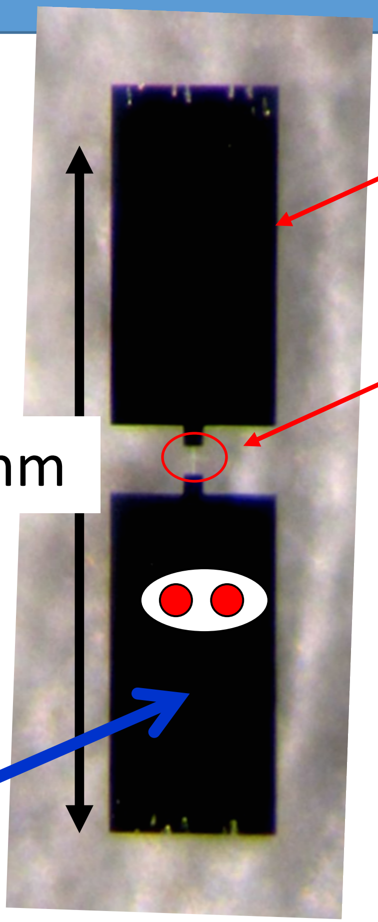
$\mathrm{Q}/2\pi \approx 2\times10^{6}$

$\mathrm{dipole} \sim 3\times10^{7}\ \mathrm{Debye}$

‘Transmon’ Qubit

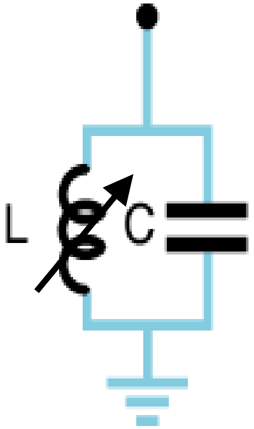


$\omega_{01} \sim 5-10\,\mathbf{GHz}$



Antenna pads are capacitor plates

Josephson tunnel junction



10^{12} mobile electrons

Superconductivity gaps out single-particle excitations

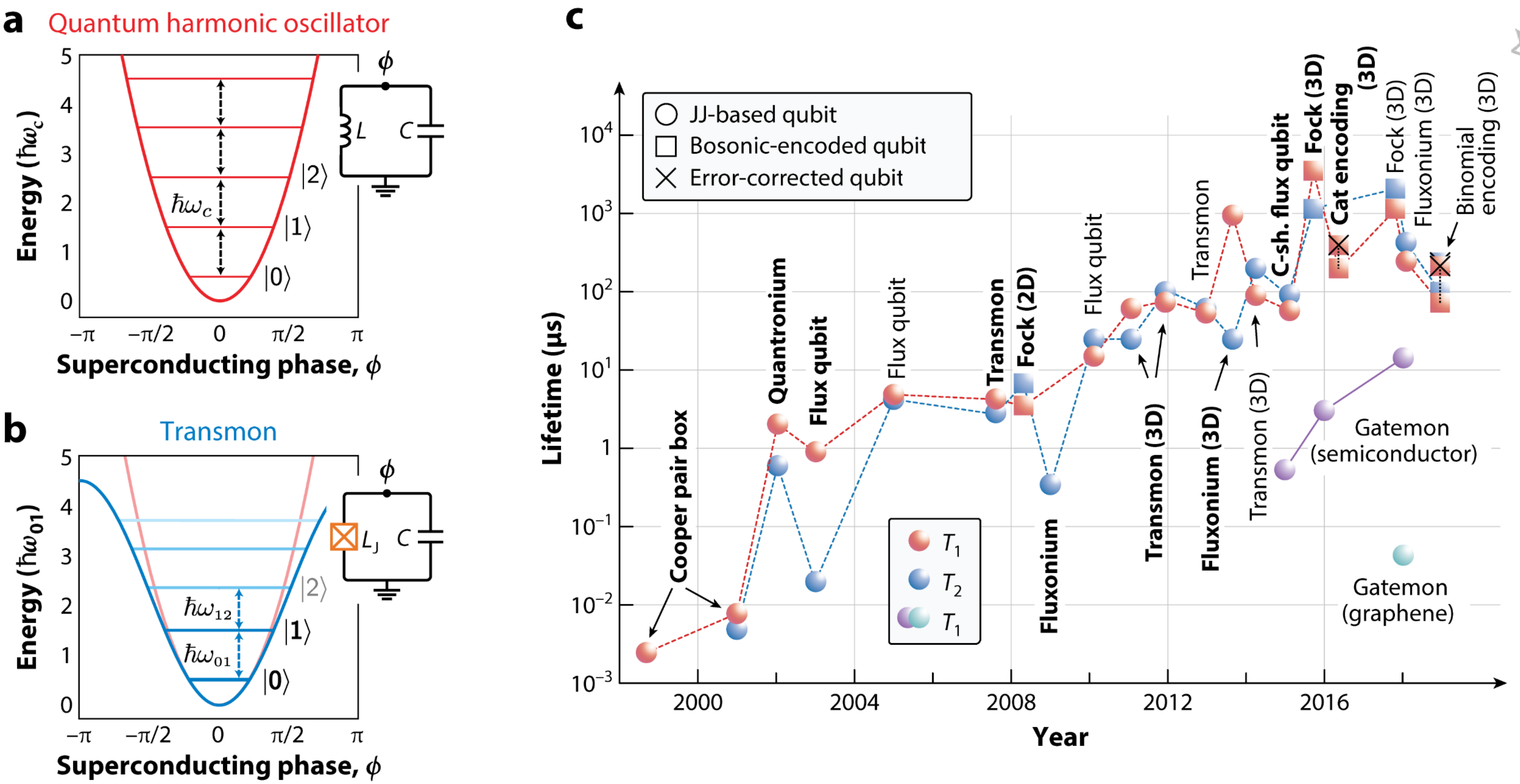
Quantized energy level spectrum is *simpler* than hydrogen

Quality factor $Q = \omega T_1$ *comparable to* that of hydrogen 1s-2p

Enormous transition dipole moment, atom has its own antenna:
ultra-strong coupling to microwave photons

“Circuit QED”

Remarkable Progress in Coherence Times



Kjaergaard M, et al. 2020.
Annu. Rev. Condens. Matter Phys. 11:369–95

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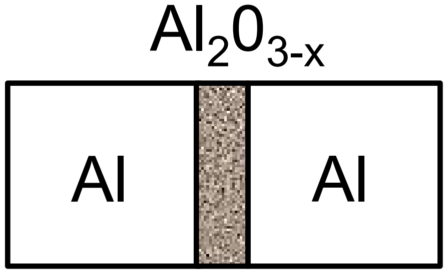
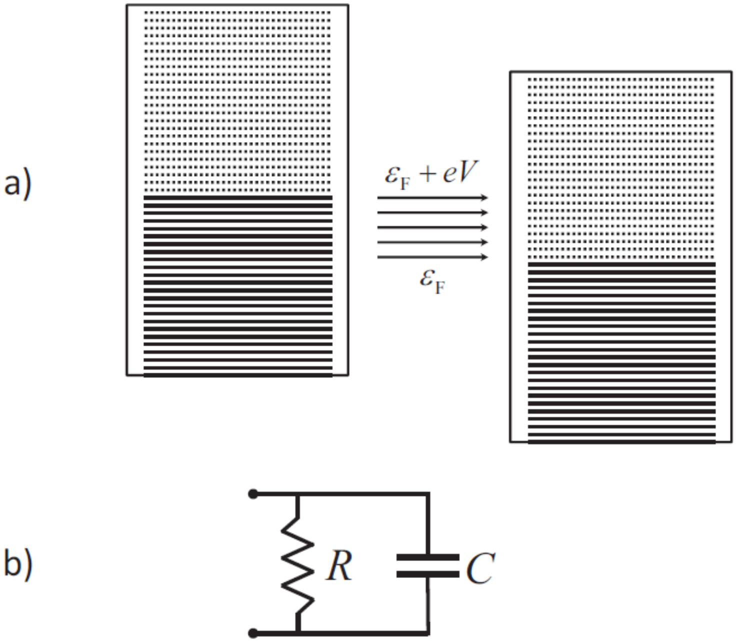
Annu. Rev. Condens. Matter Phys. 11:369-95

A simple approach to the Josephson Effect

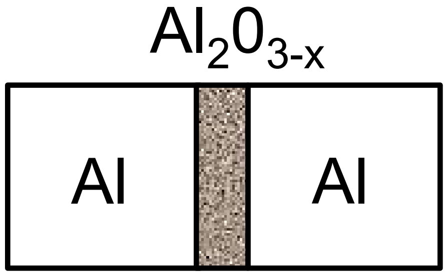
~~Josephson~~ Normal Tunnel Junctions



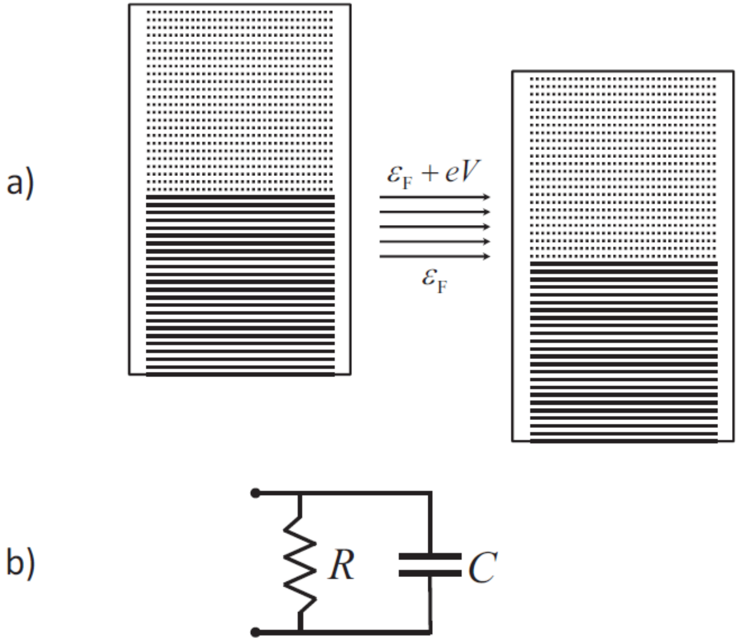
Normal tunnel junction



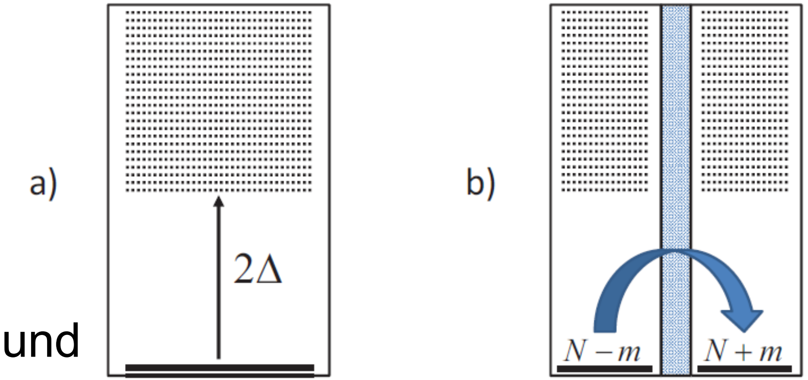
Josephson Tunnel Junctions



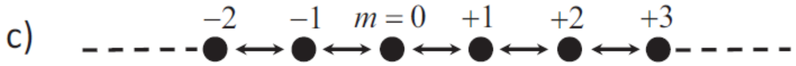
Normal tunnel junction



Superconducting tunnel junction



Unique ground state for N pairs on an island

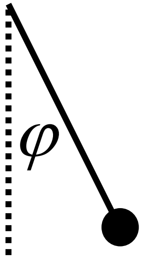


Total number m of Cooper pairs that have tunneled uniquely determines the non-degenerate low-energy quantum state of a pair of islands.

Josephson Tunnel Junctions

$$|\psi\rangle = \sum_{m=-\infty}^{+\infty} \psi_m \,|m\rangle$$

Exactly the same Hilbert space
(not necessarily the Hamiltonian) as a
quantum rotor (integer *angular momentum*)



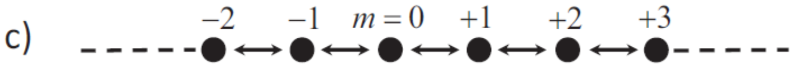
angular momentum basis $|m\rangle$

position basis $\quad |\varphi\rangle = \sum_m e^{im\varphi}\,|m\rangle$

angular position $-\pi < \varphi < +\pi$

$\psi_m(\varphi) = \langle \varphi \,|\, m\rangle = e^{im\varphi}$

integer $m \Leftrightarrow \varphi$ compact



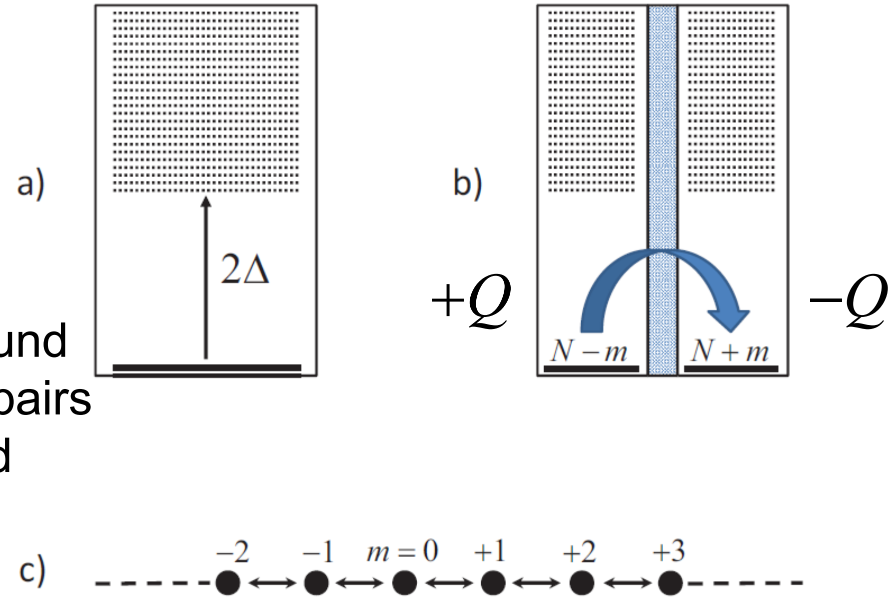
Total number of Cooper pairs that have tunneled uniquely determines the low-energy quantum state

Josephson Tunnel Junction as a capacitor
(N.B. ignoring offset charge, see my Les Houches notes)

$$Q = (2e)m$$

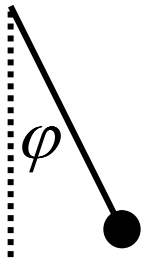
$$U = \frac{Q^2}{2C} = 4\frac{e^2}{2C}m^2 = 4E_{\mathrm{c}}\,m^2$$

Superconducting tunnel junction



Unique ground state for N pairs on an island

Quantum Rotor



$$T = \frac{L^2}{2I} = -\frac{1}{2I}\frac{d^2}{d\varphi^2}$$

$$T\left|m\right\rangle = \frac{m^2}{2I}\left|m\right\rangle$$

Charging energy looks like rotor K.E.

Total number of Cooper pairs that have tunneled uniquely determines the low-energy quantum state of a pair of islands.

Cooper Pair Tunneling (Josephson Effect)

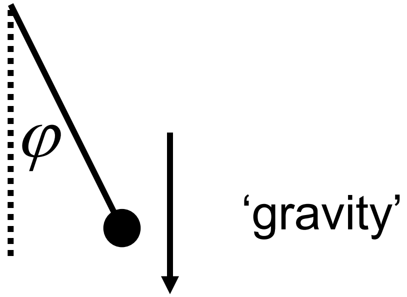
$$H_{\mathrm{J}} = -\frac{E_{\mathrm{J}}}{2}\sum_{m}\left\{|m+1\rangle\langle m| + |m\rangle\langle m+1|\right\}$$

[tight-binding hopping matrix element that changes position by ±1]

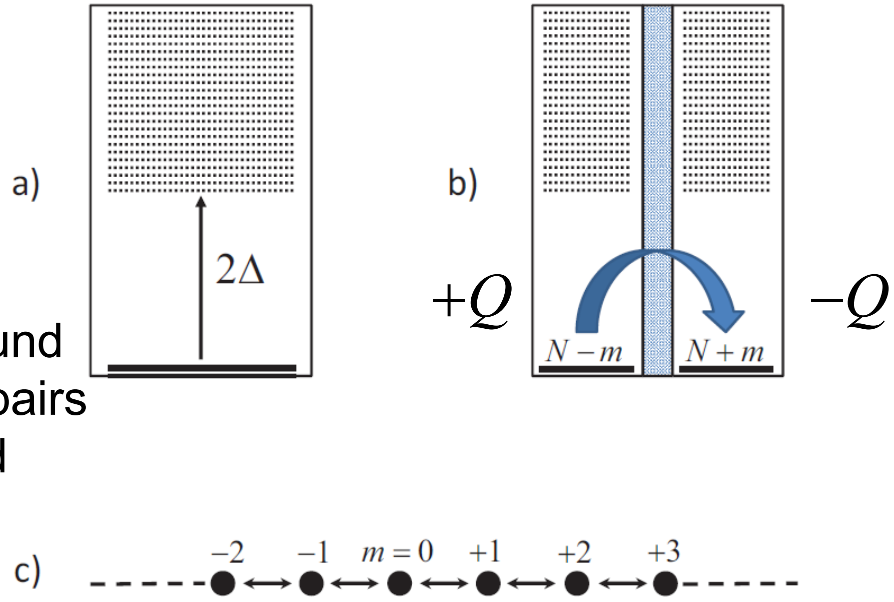
$$H_{\mathrm{J}} = -E_{\mathrm{J}}\cos\varphi$$

[gravitational potential producing a torque that changes the angular momentum by ±1]

Quantum Rotor



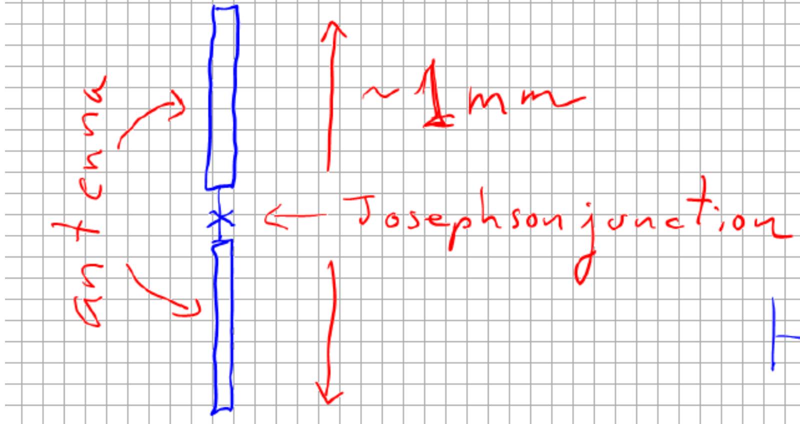
Superconducting tunnel junction



Unique ground state for N pairs on an island

Total number of Cooper pairs that have tunneled uniquely determines the low-energy quantum state of a pair of islands.

"Transmon" qubit



dipole moment

$\sim Q_{ZPF} \times 1\,mm$

$$\boxed{\hat{Q} = (2e)m}$$

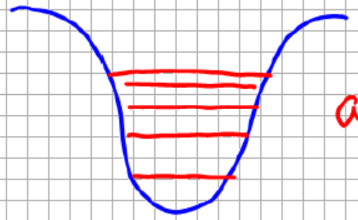
$$H = \frac{\hat{Q}^2}{2C_\Sigma} - E_J \cos\underbrace{\left(\frac{2e}{\hbar}\hat{\Phi}\right)}_{}$$

$C_\Sigma \equiv C_J + C_{geometric}$

$\varphi \equiv$ SC order parameter phase

$\hbar\dot{\varphi} = 2eV = 2e\dot{\Phi}$

Josephson relation



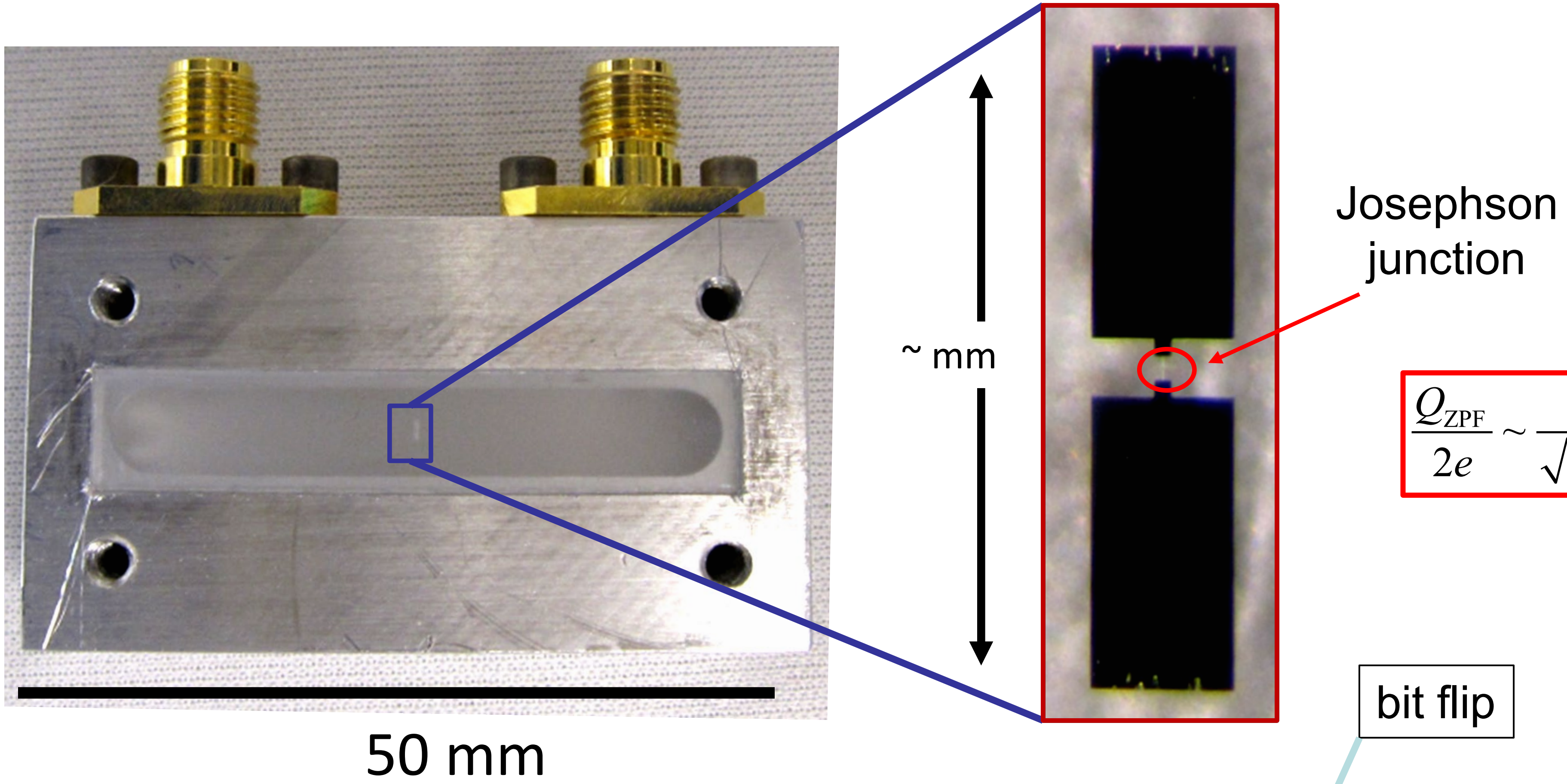
Subtlety: $\hat{Q} = (2e)\,\hat{n}$ is discrete <u>not</u> continuous.

For $E_J \gg E_c \equiv \frac{e^2}{2C_\Sigma}$ $\quad \langle\varphi^2\rangle \ll 2\pi$ so can expand the cosine and safely ignore the subtleties.

Typically $\frac{E_J}{E_c} \sim 10^2$

[$\hat{n}$ is angular momentum conjugate to angle φ.]

Transmon Qubit in 3D Cavity



$$\frac{Q_{\mathrm{ZPF}}}{2e} \sim \frac{1}{\sqrt{16\pi\alpha}} \sim 1-3$$

bit flip

$$V_{\mathrm{dipole}} = g\sigma^x(a + a^\dagger)$$

$$g = \frac{\vec{d} \bullet \vec{E}_{rms}}{h}$$

$$\left|\vec{d}\right| = 2e \times 1\ \mathrm{mm} \approx 10^7 \mathrm{Debye!!}$$

Huge dipole moment: strong coupling

g ~ 100 MHz

End of Lecture 1 Part 1 (60 minutes)

Lecture 1 Part 2

Black-Box Quantization (SC qubits coupled to resonators)

Phys. Rev. Lett. **108**, 240502 (2012)

Basic idea:

Poles of impedance $Z(\omega)$ yield normal mode frequencies

Residue of each pole determines characteristic impedance Z_R of each resonance

This is all that is needed to quantize an electrical oscillator system.

Reminder: "Transmon" Qubit [without a cavity]

$$H = \frac{\hat{Q}^2}{2C_\Sigma} - E_J \cos\left(\frac{2e}{\hbar}\hat{\Phi}\right) = H_0 + H_1(\hat{\Phi})$$

$$H_0 = \frac{\hat{Q}^2}{2C_\Sigma} + \underbrace{\frac{E_J}{2}\left(\frac{2e}{\hbar}\right)^2}_{\frac{1}{2L_J}} \hat{\Phi}^2$$

Josephson plasma oscillation ω_J

"Josephson inductance"

[Expand cosine to second order]

$$\hbar\omega_J = \frac{1}{\sqrt{L_J C_\Sigma}} \equiv \sqrt{8E_J E_c}$$

$$E_c \equiv \frac{e^2}{2C_\Sigma}$$

$$\frac{\omega_J}{2\pi} \sim 3-10\,\text{GHz}$$

$$H_1(\hat{\Phi}) = E_J\left\{-\frac{1}{4!}\left(\frac{2e}{\hbar}\hat{\Phi}\right)^4 + \cancel{\frac{1}{6!}\left(\frac{2e}{\hbar}\hat{\Phi}\right)^6} - \cdots\cdots\right\}$$

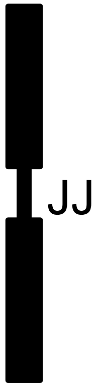
[Rest of the cosine]

$$\hat{\Phi} = \Phi_{ZPF}(a + a^\dagger)$$

$$\Phi_{ZPF} = \sqrt{\frac{\hbar}{2}Z_R}$$

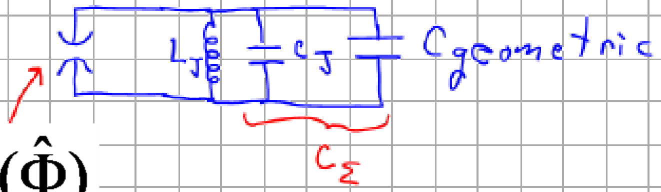
$$Z_R = \sqrt{\frac{L_J}{C_\Sigma}}$$

'transmon' qubit



$$\text{RWA}: H_1(\hat{\Phi}) \simeq -\frac{E_c}{2}\left[a^\dagger a^\dagger a a + 2a^\dagger a\right]$$

[RWA = rotating wave approximation]

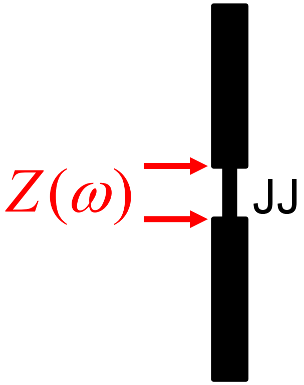


$H_1(\hat{\Phi})$

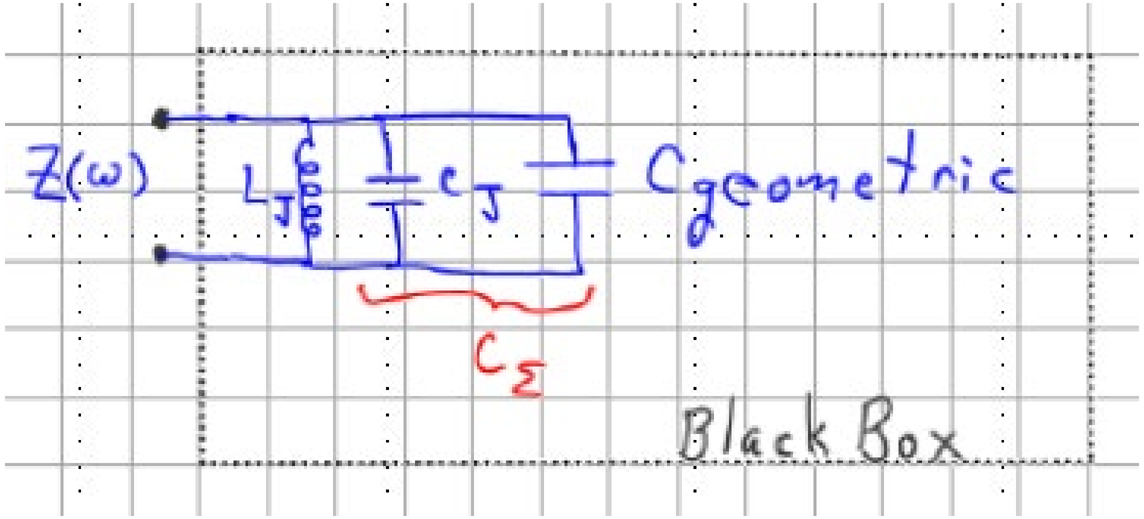
[Equivalent circuit, not literal geometry]

Black-box quantization procedure

1. Get quadratic part of Hamiltonian by replacing JJ by inductor L_J
2. Define a port at the location of the JJ where impedance $Z(\omega)$ is computed
3. Use finite-element Maxwell solver to compute $Z(\omega)$ from given geometry and measured value of E_J.

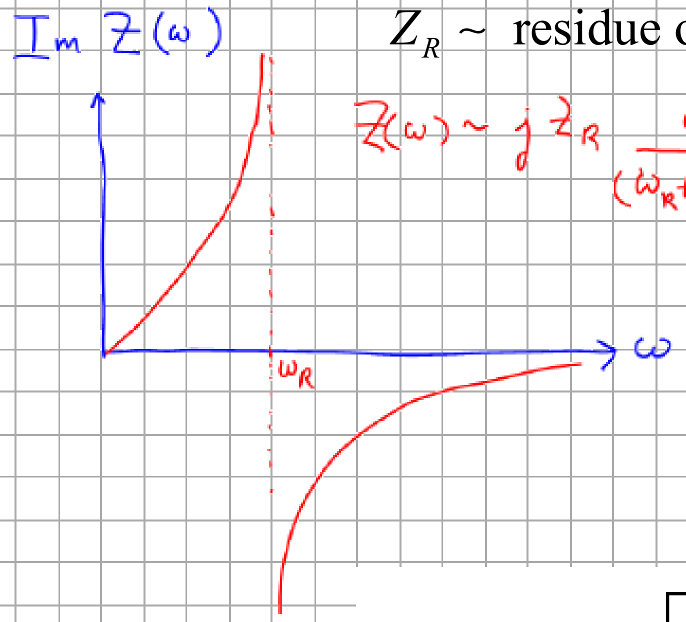


Physical layout



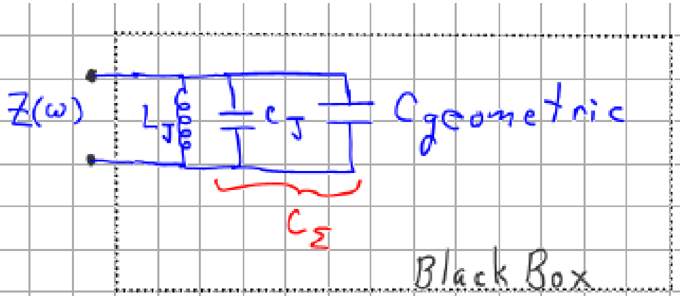
Equivalent lumped-element electrical circuit

Blackbox Quantization



Z_R ~ residue of pole

$$Z(\omega) \sim j Z_R \frac{\omega_R^2}{(\omega_R+\omega)(\omega_R-\omega)}$$



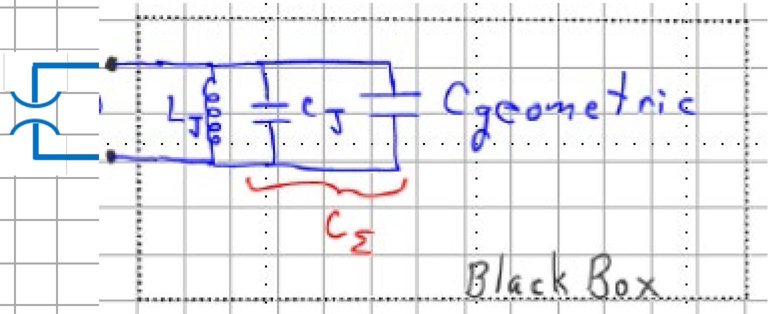
[Equivalent circuit for quadratic part of H, not literal geometry]

pole of Z(ω) determines mode frequency. Residue determines $Z_R = \sqrt{\frac{L}{C}}$

Quantum flux at the port

$$\hat{\Phi} = \Phi_{ZPF}(a+a^\dagger)$$

$$\Phi_{ZPF} = \sqrt{\frac{\hbar}{2} Z_R}$$



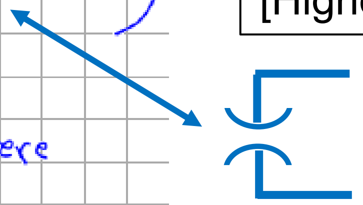
$$H_0 = \hbar\omega_R\left[a^\dagger a + \frac{1}{2}\right]$$

$$H_1 \approx -\frac{E_C}{2}\left[a^\dagger a^\dagger aa + 2a^\dagger a\right]$$

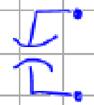
RWA

$$-\frac{E_C}{2}\left(a^\dagger a^\dagger a a + 2 a^\dagger a\right)$$

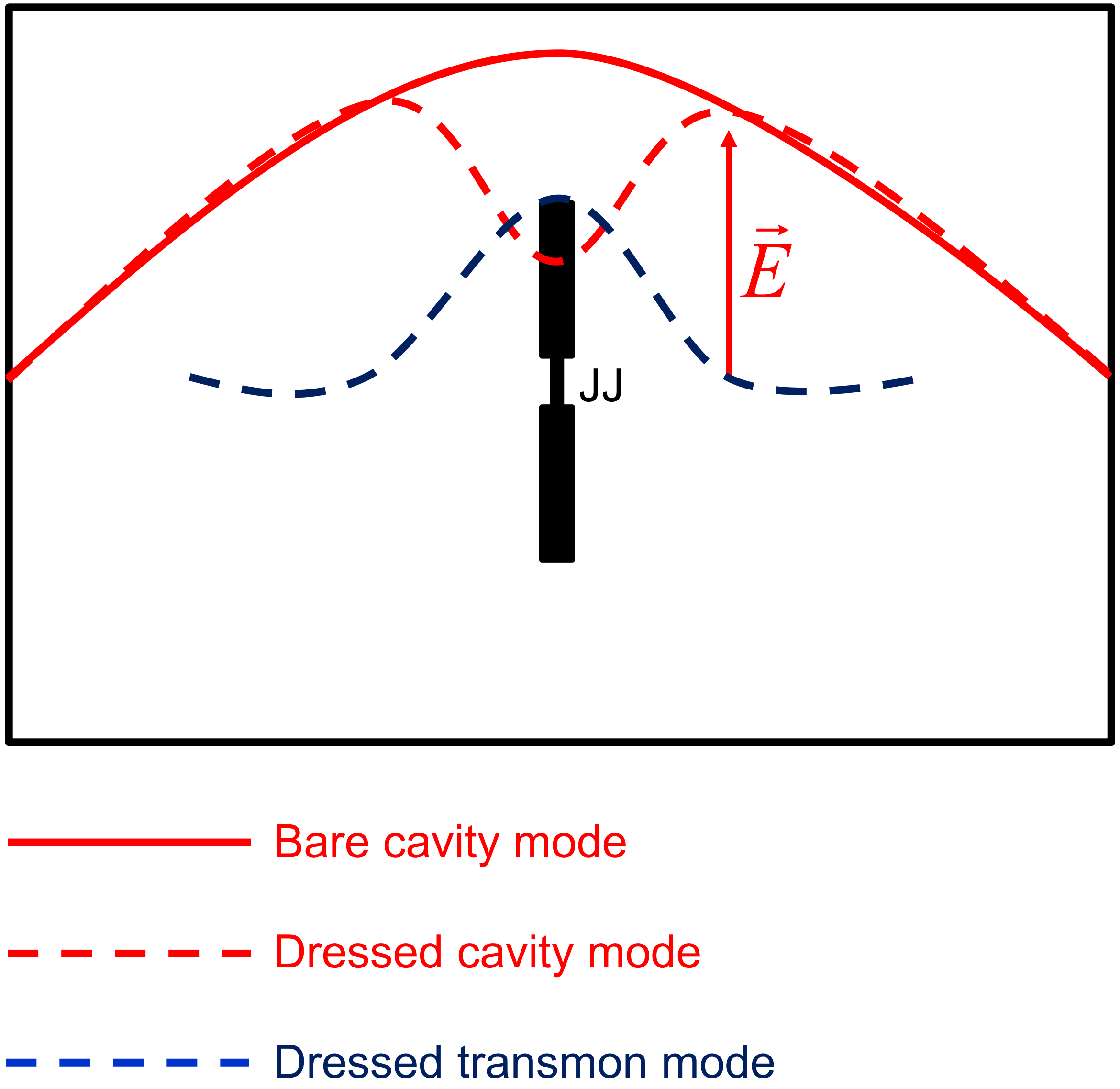
[Higher-order part of H]



$$H_1 \equiv -E_J \cos\left(\frac{2e}{\hbar}\Phi\right) - \frac{1}{2L_J}\Phi^2$$

Pole and residue of Z(ω) (measured at the point where the non-linearity  is connected) are all that are needed to quantize the full Hamiltonian in the basis of the dressed normal modes.

BBQ (black-box quantization) of a transmon coupled to a cavity



Basic Idea:

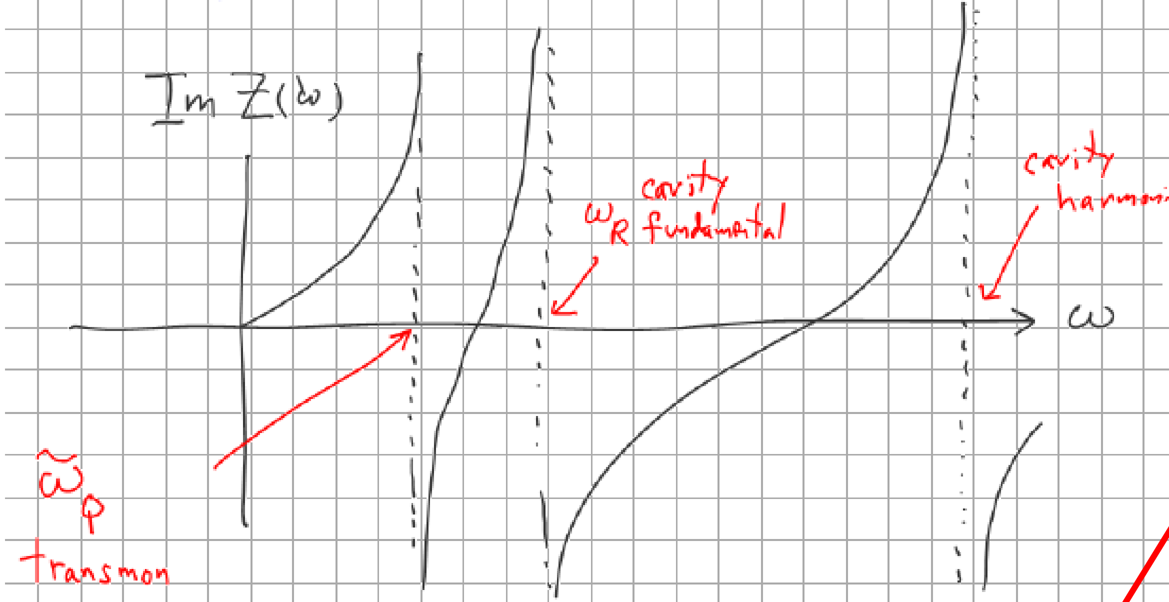
Solve Maxwell equations for normal modes of strongly coupled (quadratic approximation to) transmon and cavity modes.

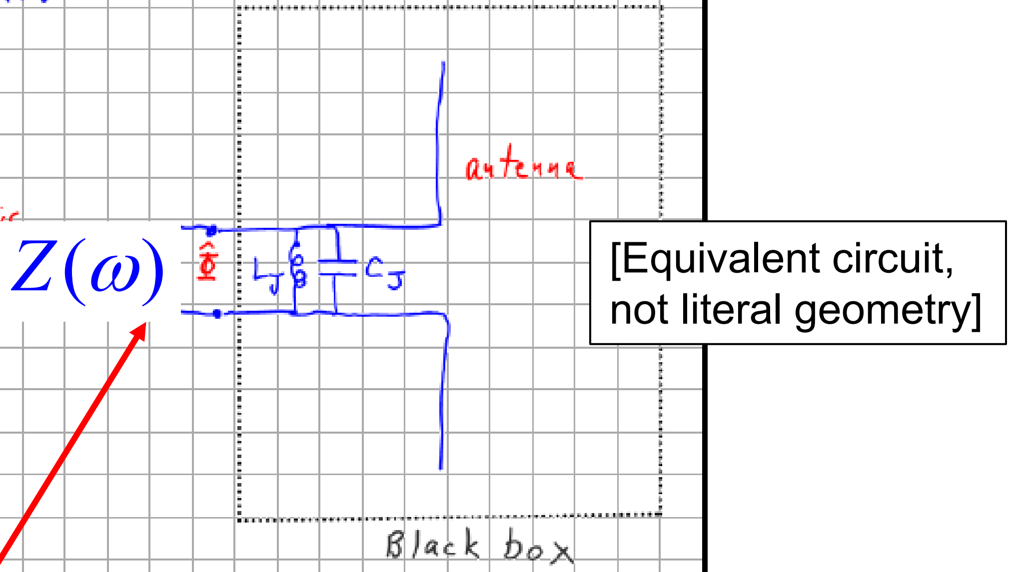
Quantize the normal modes.

Use as an efficient basis in which to express the non-linear part of the transmon Hamiltonian.

Use poles and residues of $Z(\omega)$ to express flux across JJ as $\hat{\Phi} = \Phi_{\text{ZPF}}[a^{\dagger} + a]$

BBQ of a transmon coupled to a resonator





[Equivalent circuit, not literal geometry]

$$\boxed{\hat{\Phi} = \sum_k \Phi^{(k)}_{ZPF}\left(a_k + a_k^{\dagger}\right)}$$

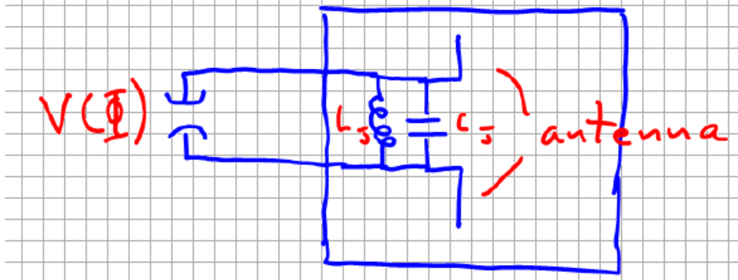
$$\Phi^{(k)}_{ZPF} = \sqrt{\frac{\hbar}{2} Z^{(k)}}$$

impedance of k^{th} mode @ location of $\frac{\perp}{\top}$

$$H_0 = \sum_k \hbar \omega_k a_k^{\dagger} a_k$$

$$H_1 \equiv -E_J \cos\left[\frac{2e}{\hbar}\Phi\right] - \frac{1}{2L_J}\Phi^2$$

Transmon coupled to resonator (BBQ)



$$\hat{\Phi} = \sum_k \Phi^{(k)}_{ZPF}\left(a_k + a_k^\dagger\right)$$

Diagonalize quadratic H_0 ; Express H_1 in that basis

$$H_1 = -\sum_\ell \delta\omega_\ell A_\ell^\dagger A_\ell + \sum_{jk} \chi_{jk}\, \hat{n}_j\, \hat{n}_k$$

Most anharmonic mode identified as qubit (say $j=0$)

χ_{00} big χ_{0k} next biggest $\chi_{\ell k}$ smallest $\ell, k \neq 0$

$\hat{n}_0 = 0, 1 \longleftrightarrow \sigma^z = \pm 1$ two-level approximation

$$\chi_{0k}\, \hat{n}_0\, \hat{n}_k \longrightarrow \frac{\chi_{0k}}{2}\, \sigma^z\, \hat{n}_k$$

cross Kerr ⟶ Jaynes-Cummings in dispersive limit

mode frequency depends on qubit state

Strong Dispersive Limit

$$H = \omega_{\mathrm{r}} a^{\dagger} a + \frac{\omega_{\mathrm{q}}}{2} \sigma^{z} + g\sigma^{x}[a + a^{\dagger}] + H_{\mathrm{damping}} \qquad \text{[Rabi]}$$

$$H = \omega_{\mathrm{r}} a^{\dagger} a + \frac{\omega_{\mathrm{q}}}{2} \sigma^{z} + g[a\sigma^{+} + a^{\dagger}\sigma^{-}] + H_{\mathrm{damping}} \qquad \text{[Jaynes-Cummings]}$$

$$\boxed{H = \omega_{\mathrm{r}} a^{\dagger} a + \frac{\omega_{\mathrm{q}}}{2} \sigma^{z} + \chi\sigma^{z} a^{\dagger} a + H_{\mathrm{damping}} \qquad \text{[Dispersive]}}$$

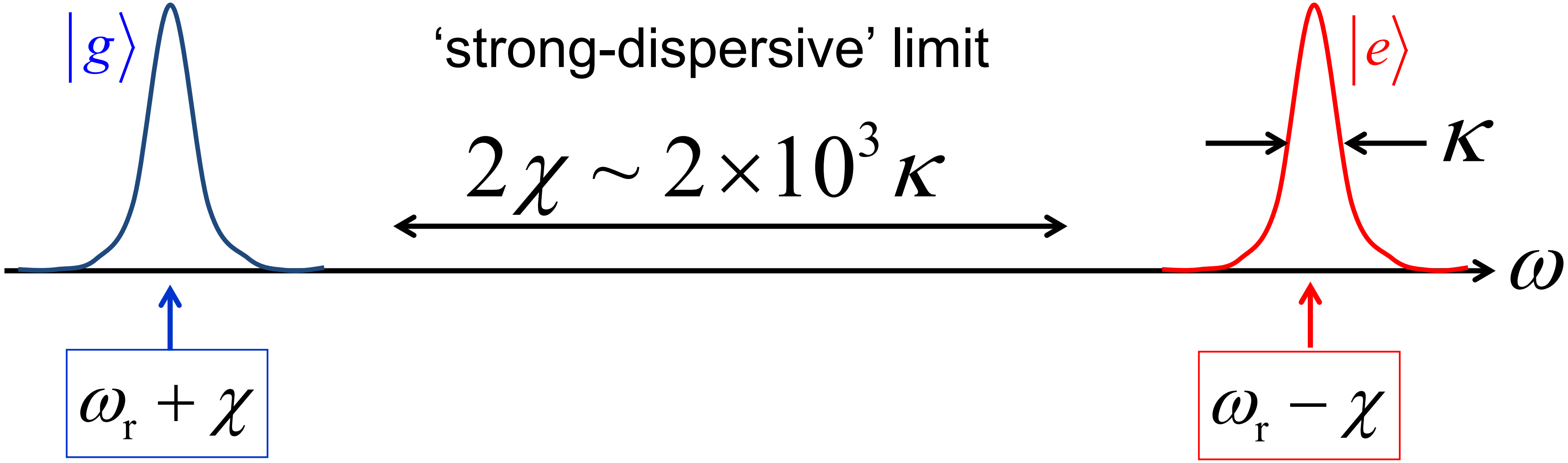
Strong Dispersive Hamiltonian

$$\boxed{H = \omega_{\mathrm{r}} a^{\dagger} a + \frac{\omega_{\mathrm{q}}}{2}\sigma^{z} + \chi\sigma^{z} a^{\dagger} a + H_{\mathrm{damping}}}$$

$$\boxed{\chi >> \kappa, \Gamma}$$

resonator qubit dispersive coupling

$$\boxed{\text{cavity frequency} = \omega_{\mathrm{r}} + \chi\sigma^{z}}$$



$|g\rangle$

'strong-dispersive' limit

$|e\rangle$

$2\chi \sim 2\times10^{3}\kappa$

κ

ω

$\boxed{\omega_{\mathrm{r}} + \chi}$

$\boxed{\omega_{\mathrm{r}} - \chi}$

Photon ‘Number Splitting’ in the Strong Dispersive Limit

Using strong-dispersive coupling to measure the photon number distribution in a cavity

Strong Dispersive Hamiltonian

$$\boxed{H = \omega_{\mathrm{r}} a^{\dagger} a + \frac{\omega_{\mathrm{q}}}{2} \sigma^{z} + \chi \sigma^{z} a^{\dagger} a + H_{\mathrm{damping}}} \qquad \boxed{\chi >> \kappa, \Gamma}$$

resonator

qubit

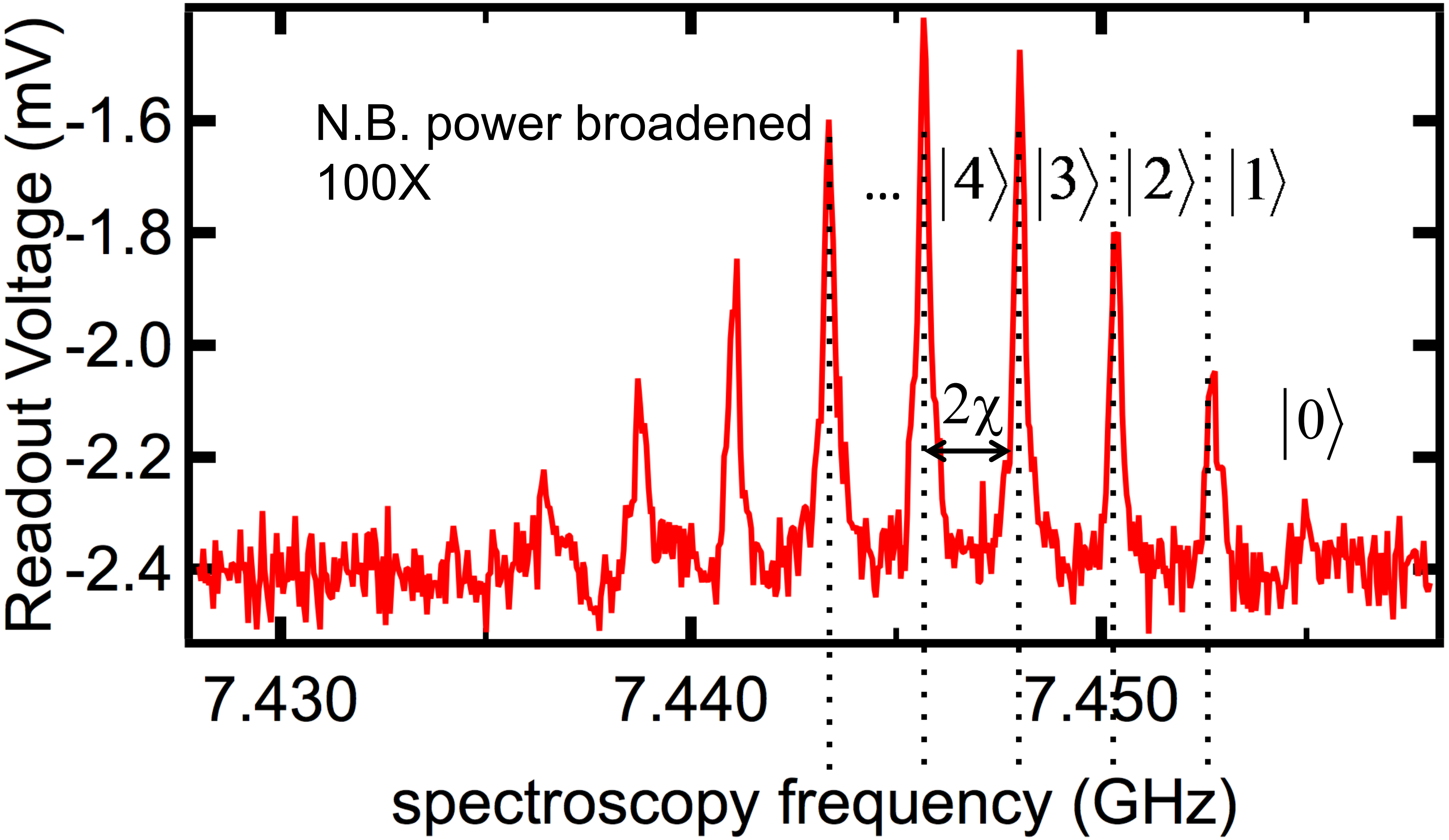
dispersive coupling

Reinterpretation of same Hamiltonian: Quantized Light Shift of Qubit Transition Frequency

$$\boxed{H = \omega_{\mathrm{r}} a^{\dagger} a + \frac{1}{2} \sigma^{z} \left[\omega_{\mathrm{q}} + 2 \chi a^{\dagger} a \right] + H_{\mathrm{damping}}}$$

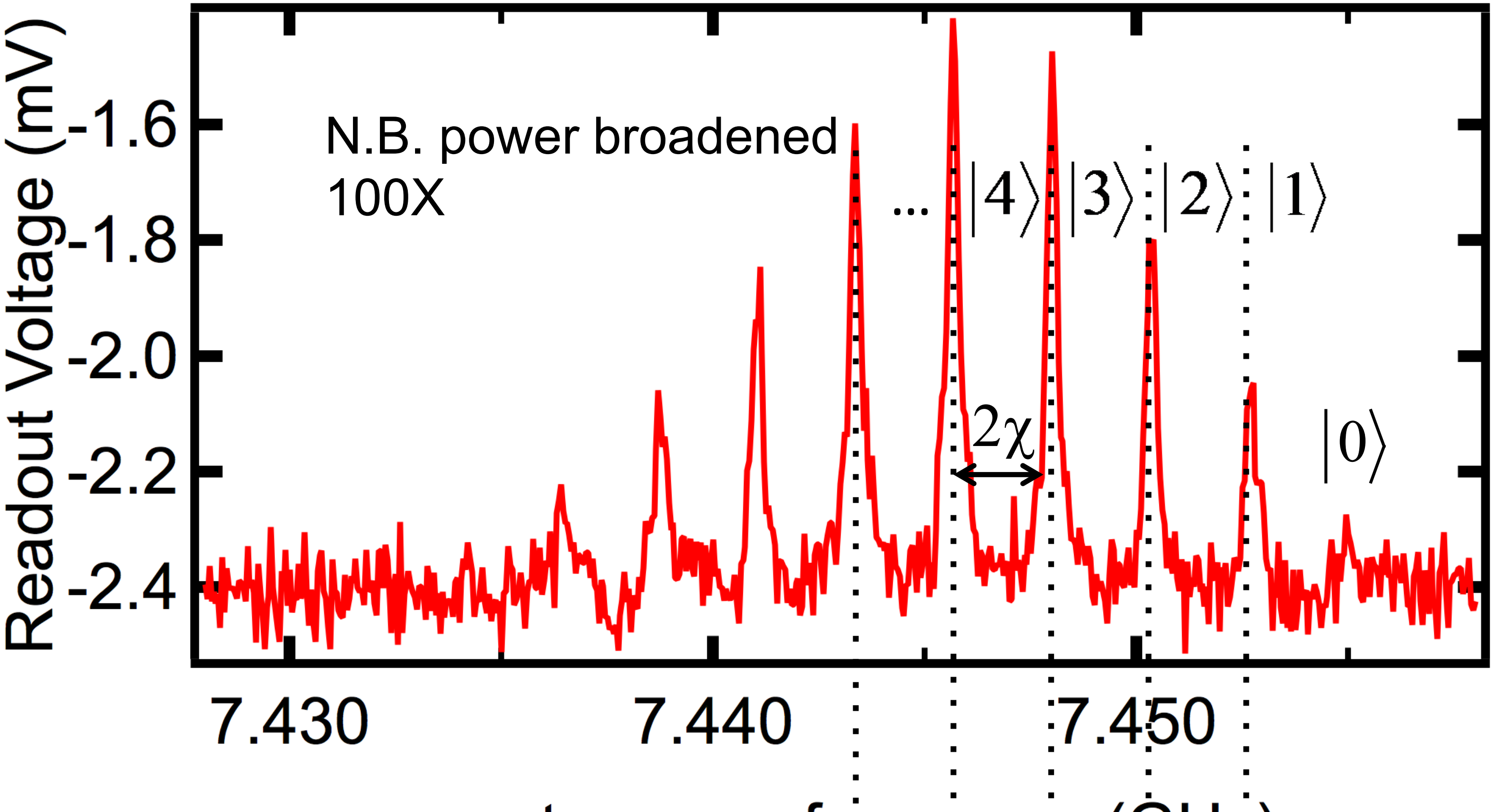
- quantized light shift of qubit frequency (coherent microwave state)

$$\boxed{\frac{\omega_q + 2\chi a^\dagger a}{2}\sigma^z}$$



- quantized light shift of qubit frequency (coherent microwave state)

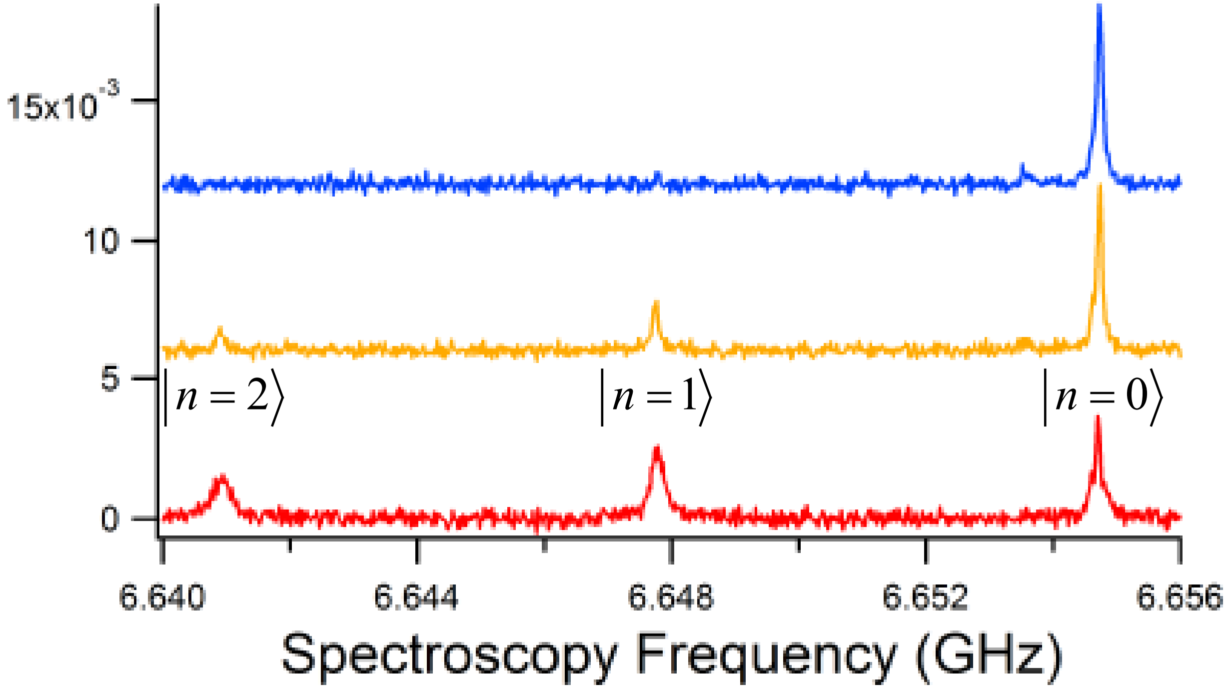
$$\boxed{\frac{\omega_q + 2\chi a^\dagger a}{2}\sigma^z}$$



Microwaves are particles!

- quantized light shift of qubit frequency
  (coherent microwave state)

$$\boxed{\frac{\omega_{\mathrm{q}} + 2\chi a^{\dagger} a}{2}\sigma^{z}}$$



New low-noise way to do axion dark matter detection by QND photon counting
Zheng et al. arXiv:1607.02529 → A. Chou: PRL **126**, 141302 (2021)