Topological band structures in AMO physics

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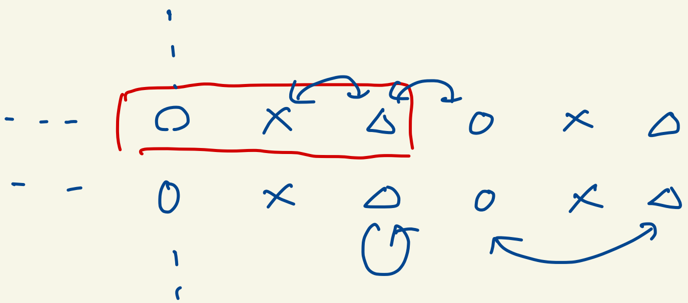
1. AMO platforms to realize band structures
2. Basics of topological band structures
3. Realizations and manifestations of topological band structures in AMO systems

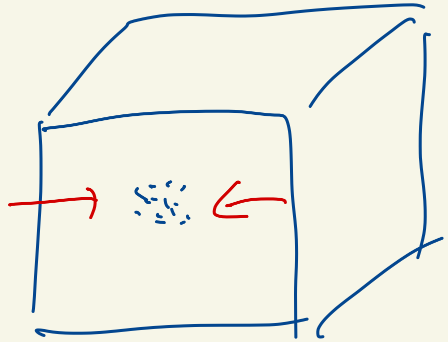


$$i \frac{\partial}{\partial t} \vec{\psi} = H \vec{\psi}$$

$\hbar = 1$

$$H \vec{\psi} = E \vec{\psi}$$

1-1 Ultracold Atoms



$\vec{E} \Longrightarrow$ atom $\quad \propto \langle E^2(\vec{r}, t) \rangle_t$

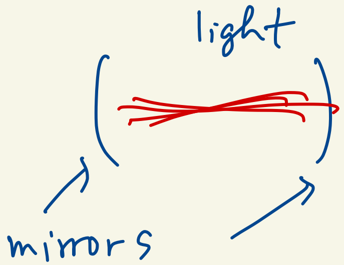
time average

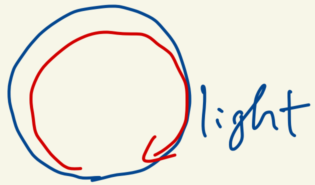
$$E(\vec{x}, t) = E_0 \cos(qx - \omega t) + E_0 \cos(qx + \omega t)$$

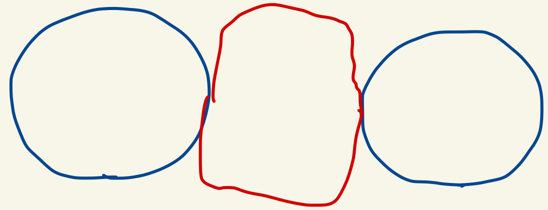
$$= 2E_0 \cos(qx) \cos(\omega t)$$

$$\langle E^2(\vec{x}, t) \rangle_t = 4 E_0^2 \cos^2(qx) \underbrace{\langle \cos^2(\omega t) \rangle_t}_{= \frac{1}{2}} = 2 E_0^2 \cos^2(qx)$$

1-2. Optical cavities, resonators





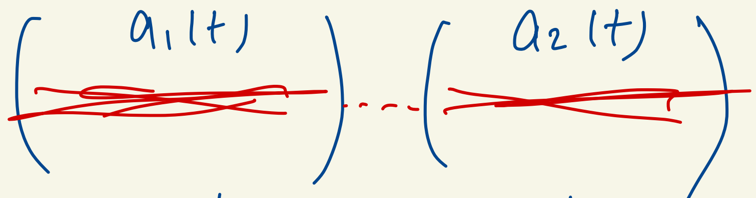


Assume that a mode with frequency ω is present inside a cavity

Electric field amplitude $a(t) \propto e^{-i\omega t}$

In terms of a differential equation, $i\dfrac{da(t)}{dt} = \omega a(t)$

Now I consider two resonators



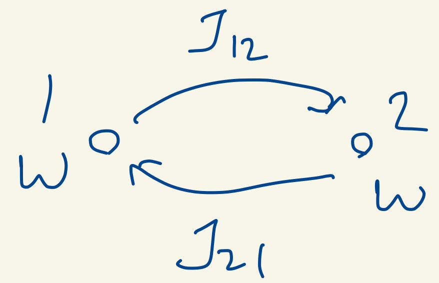
If the two resonators are independent If there is a weak coupling

$$i\frac{da_1(t)}{dt} = \omega a_1(t) + J_{12}a_2(t)$$

$$i\frac{da_2(t)}{dt} = \omega a_2(t) + J_{21}a_1(t)$$

$$\Rightarrow \quad i\frac{d}{dt}\begin{pmatrix} a_1(t) \\ a_2(t) \end{pmatrix} = \underbrace{\begin{pmatrix} w & J_{12} \\ J_{21} & w \end{pmatrix}}_{H} \underbrace{\begin{pmatrix} a_1(t) \\ a_2(t) \end{pmatrix}}_{\vec{\psi}(t)}$$

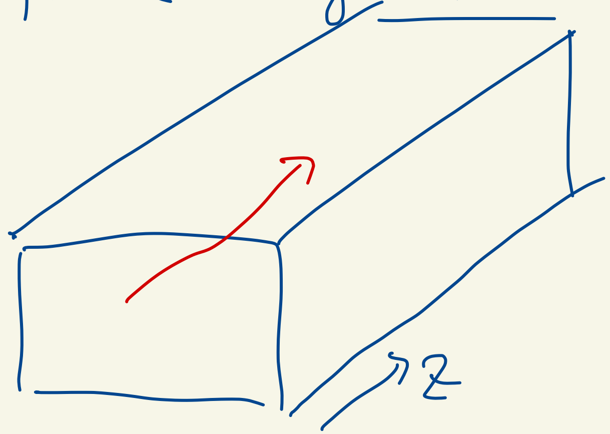
… Schrödinger eq. for 2-site lattice model



Assuming $|a_1|^2 + |a_2|^2$ conserves,

$J_{12} = J_{21}^* \Rightarrow H$ is Hermitian

1-3. Optical waveguides



Maxwell's equation

$$\begin{cases} \vec{\nabla}\cdot\vec{B} = 0 \\ \vec{\nabla}\cdot(\varepsilon\vec{E}) = 0 \leftarrow \\ \vec{\nabla}\times\vec{E} + \dfrac{\partial\vec{B}}{\partial t} = 0 \\ \vec{\nabla}\times(\mu^{-1}\vec{B}) - \dfrac{\partial(\varepsilon\vec{E})}{\partial t} = 0 \end{cases}$$

$$\varepsilon = \varepsilon(\vec{r})\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}, \quad \mu = \mu_0\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}$$

I look for a solution $\vec{E}(\vec{r},t) = \vec{E}(\vec{r})e^{-i\omega t}$, $\vec{B}(\vec{r},t) = \vec{B}(\vec{r})e^{-i\omega t}$

$$\Rightarrow \vec{\nabla}\times\vec{E}(\vec{r}) - i\omega\vec{B}(\vec{r}) = 0, \quad \vec{\nabla}\times\vec{B}(\vec{r}) + i\omega\mu_0\varepsilon(\vec{r})\vec{E}(\vec{r}) = 0$$

$$\vec{B}(\vec{r}) = \frac{1}{i\omega}\vec{\nabla}\times\vec{E}(\vec{r})$$

$$\Rightarrow \vec{\nabla}\times(\vec{\nabla}\times\vec{E}(\vec{r})) = \omega^2\underbrace{\mu_0\varepsilon(\vec{r})}_{= n(\vec{r})^2/c^2}\vec{E}(\vec{r})$$

$$\parallel$$

$$\vec{\nabla}\underbrace{(\vec{\nabla}\cdot\vec{E}(\vec{r}))}_{=0} - \nabla^2\vec{E}(\vec{r})$$

$$\Rightarrow -\nabla^2\vec{E}(\vec{r}) = \frac{\omega^2 n(\vec{r})^2}{c^2}\vec{E}(\vec{r})$$

Assume $\vec{E}(\vec{r}) = \underbrace{E(\vec{r})}\,\underbrace{e^{ik_0z}}\,\underbrace{\hat{e}_x}$

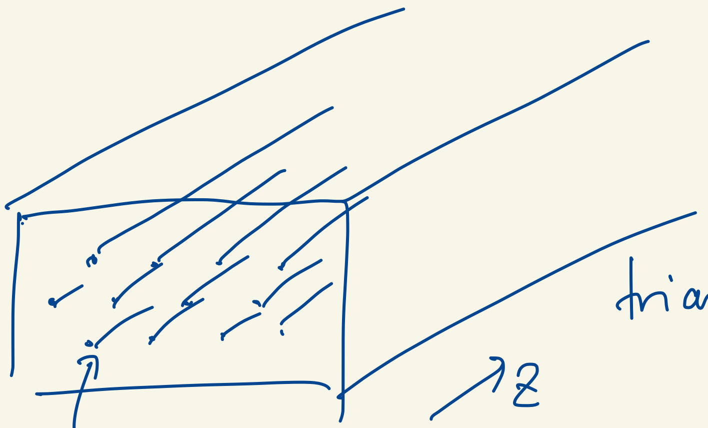
slowly varying envelope function

$$\Rightarrow \left[-\frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2} \underbrace{\cancel{- \frac{\partial^2}{\partial z^2}}} \underbrace{- 2 i k_0 \frac{\partial}{\partial z}} + k_0^2 - \frac{\omega^2}{c^2} n(\vec{r})^2 \right] E(\vec{r}) = 0$$

$$\Rightarrow \quad i \frac{\partial}{\partial z} E(\vec{r}) = \frac{1}{2k_0} \left[-\frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2} \right] E(\vec{r}) + \underbrace{\frac{k_0^2 - \omega^2 n(\vec{r})^2 / c}{2k_0}}_{\text{scalar potential } V(\vec{r})} E(\vec{r})$$

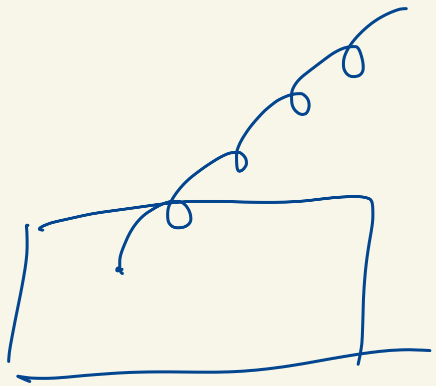
time

mass

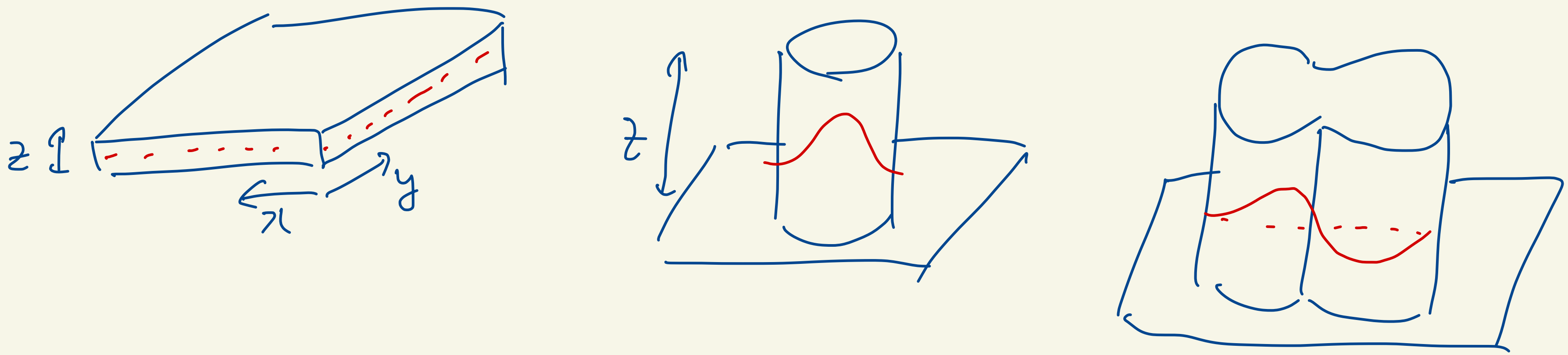


triangular lattice

deep maximum of $n(\vec{r})$ ... waveguide



1-4. Exciton-polariton lattice



1-5. Comparison of the platforms

many-body sense

Ultracold gas	Quantum	Ground, thermal	real, momentum
Exciton-polariton		steady state	real, momentum
Optical cavities, resonators		steady state	real
Waveguide arrays	Classical	Real-time dynamics	real

2. Basics of topological band structures

Some may say:

Quantum states $|\alpha\rangle$ and $|\beta\rangle$ are in the same topological phase

if there exists a family of Hamiltonians $H(\lambda)$, $0 \leq \lambda \leq 1$

and $|\alpha\rangle$ is the ground state of $H(0)$

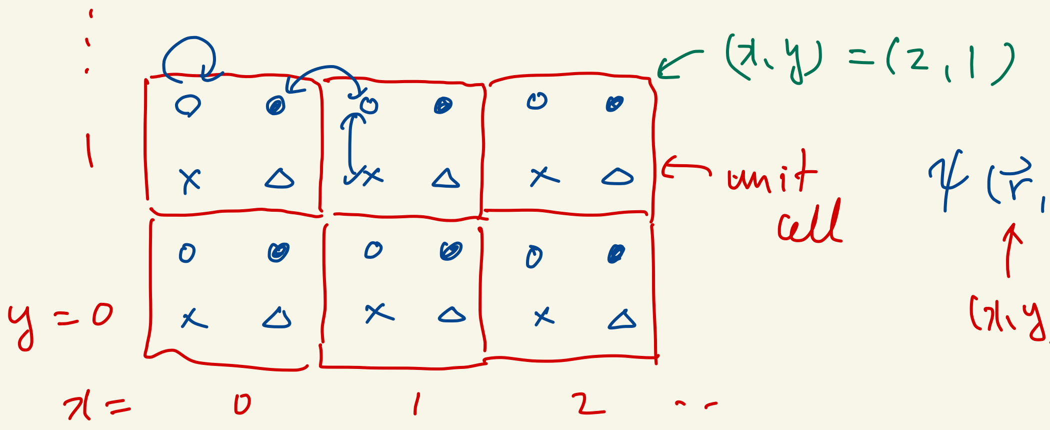
$|\beta\rangle$ — " — $H(1)$

and $H(\lambda)$ is always gapped in $0 \leq \lambda \leq 1$

2nd lecture : Topological band structures

2-1. Bloch's theorem

We focus on lattice models



$\psi(\vec{r}, i)$ — internal. $i = 1, 2, 3, 4$ (= ○, ●, ×, △)

$\vec{r} = (x,y)$

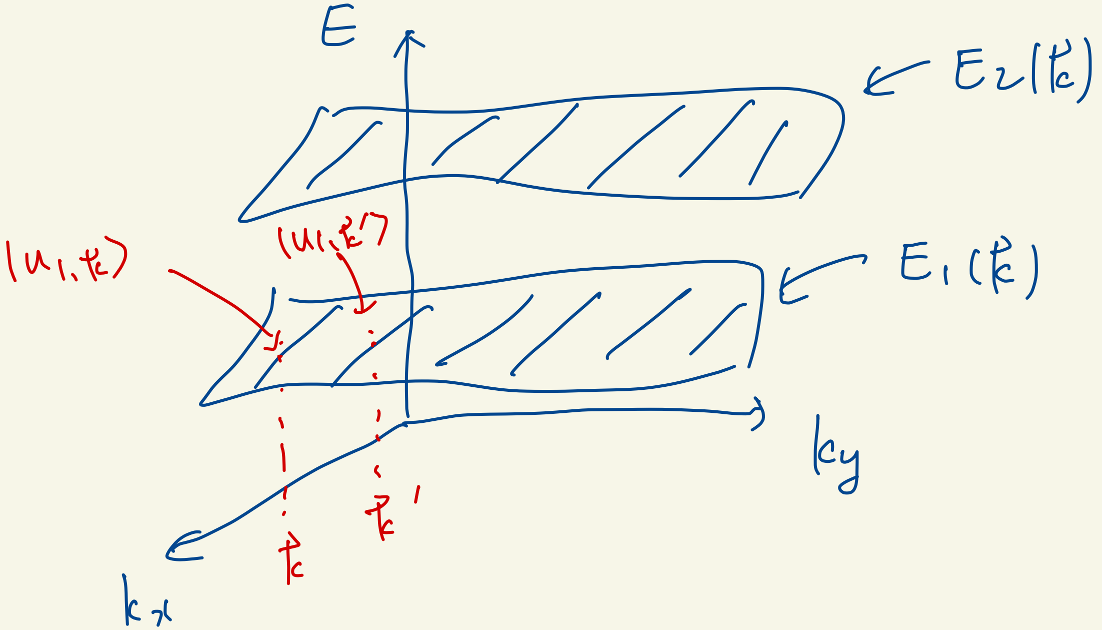
$N = 4$

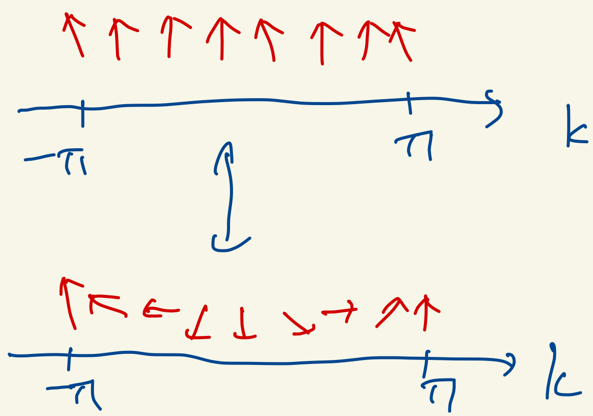
Bloch's theorem : $H\psi_{n,\vec{k}}(\vec{r},i) = E_n(\vec{k})\,\psi_{n,\vec{k}}(\vec{r},i)$ $\quad n = 1, 2, \cdots, N$

$\vec{k} \in$ Brillouin zone

$$\psi_{n,\vec{k}}(\vec{r},i) = e^{i\vec{k}\cdot\vec{r}}\,\underbrace{u_{n,\vec{k}}(i)}_{\text{Bloch state}}$$

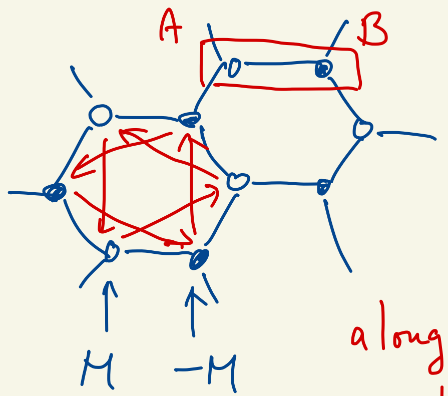
$$\longrightarrow \quad |u_{n,\vec{k}}\rangle = \begin{pmatrix} u_{n,\vec{k}}(1) \\ u_{n,\vec{k}}(2) \\ \vdots \\ u_{n,\vec{k}}(N) \end{pmatrix}$$





2-2. Chern insulator

2-2-1. Haldane model



along ↑ $e^{+i\phi}$

opposite to ↑ $e^{-i\phi}$

$N = 2$

position $\vec{r}'$, & sublattice A

position $\vec{r}$, & sublattice B

$$H = t_1 \sum_{\langle \vec{r},\vec{r}' \rangle} \left[a_{\vec{r}}^\dagger b_{\vec{r}'} + H.c. \right] + M \sum_{\vec{r}} \left[a_{\vec{r}}^\dagger a_{\vec{r}} - b_{\vec{r}}^\dagger b_{\vec{r}} \right]$$

n.n.

$$+ t_2 \sum_{\langle\langle \vec{r},\vec{r}' \rangle\rangle} \left(e^{\pm i\phi} a_{\vec{r}'}^\dagger a_{\vec{r}} + e^{\pm i\phi} b_{\vec{r}'}^\dagger b_{\vec{r}} \right)$$

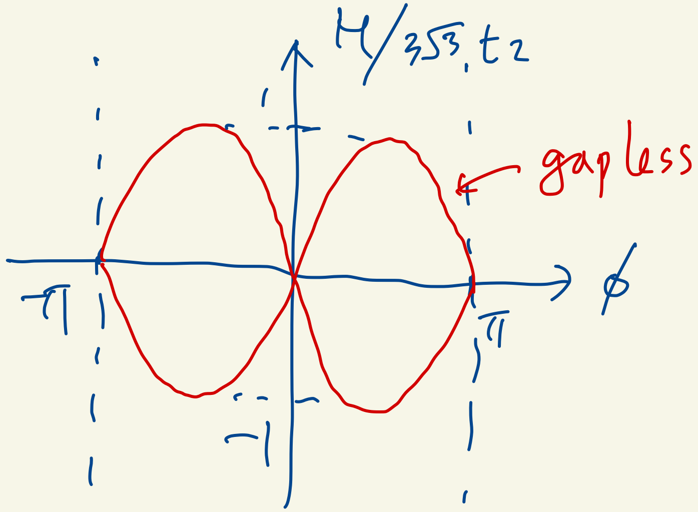
n.n.n

$$H_{\vec{k}} = \begin{pmatrix} M + 2t_2 \sum_j \cos(\phi - \vec{k}\cdot\vec{R}_j') & t_1 \sum_j e^{i\vec{k}\cdot(\vec{R}_j - \hat{e}_x)} \\ t_1 \sum_j e^{-i\vec{k}\cdot(\vec{R}_j - \hat{e}_x)} & -M + 2t_2 \sum_j \cos(\phi + \vec{k}\cdot\vec{R}_j') \end{pmatrix}$$

$$\left\{ \begin{array}{l} \hat{e}_x = (1, 0) \\ \vec{R}_j : \text{n.n. vectors from A site} \\ \vec{R}_j' : \text{n.n.n vectors from A site} \end{array} \right.$$

$$H_{\vec{k}} \, | u_\pm(\vec{k}) \rangle = E_\pm(\vec{k}) \, | u_\pm(\vec{k}) \rangle$$

Gapless when $M = \pm 3\sqrt{3}\, t_2 \sin\phi$



look at the "shape" of $|u_-(\vec{k})\rangle$

$$\parallel$$

$$\begin{pmatrix} * \\ * \end{pmatrix}$$

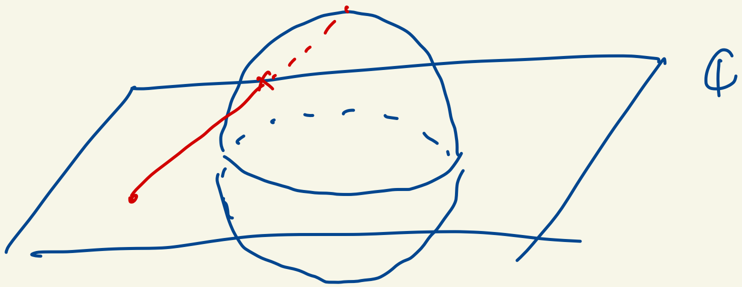
$|u_-(\mathbf{k})\rangle \in \mathbb{C}^2$

$$= \begin{pmatrix} z_1(\mathbf{k}) \\ z_2(\mathbf{k}) \end{pmatrix}$$

Multiplication by a constant factor doesn't change anything $\longrightarrow$ a state is an element of $\mathbb{C}P^1$

$z(\mathbf{k}) = z_1(\mathbf{k}) / z_2(\mathbf{k})$ if $z_2(\mathbf{k}) = 0$, I say $z(\mathbf{k}) =$ "∞"

$\in \mathbb{C} \cup \{\infty\}$



$|u_-(\mathbf{k})\rangle \longleftrightarrow$ a point on a sphere S^2

We can integrate local measure of how "curved" $|u_-(\mathbf{k})\rangle$ are to obtain the topological number

Berry curvature : $\Omega(\vec{k}) = i\left[\left\langle \frac{\partial u_-(\vec{k})}{\partial k_x} \middle| \frac{\partial u_-(\vec{k})}{\partial k_y} \right\rangle - \left\langle \frac{\partial u_-(\vec{k})}{\partial k_y} \middle| \frac{\partial u_-(\vec{k})}{\partial k_x} \right\rangle\right]$

Here, $\left| \frac{\partial u_-(\vec{k})}{\partial k_i} \right\rangle = \frac{\partial}{\partial k_i} | u_-(\vec{k}) \rangle$

I am assuming $\langle u_-(\vec{k}) | u_-(\vec{k}) \rangle = 1$

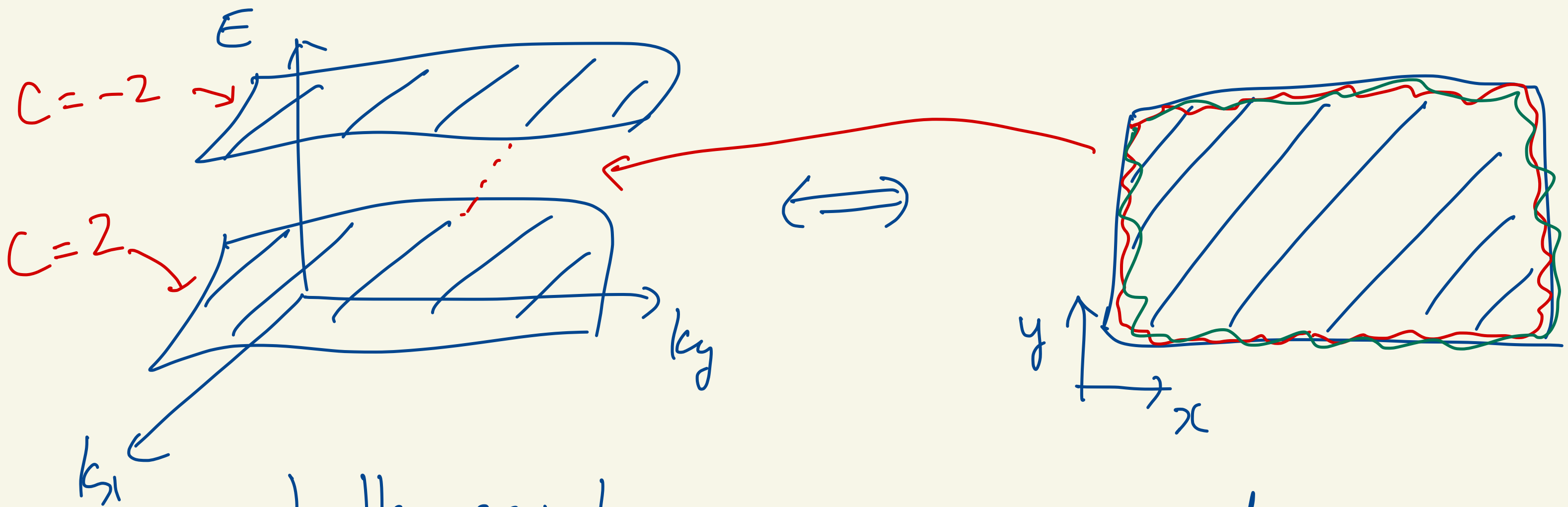
Integrating $\Omega(\vec{k})$, we get the Chern number

$$C = \frac{1}{2\pi}\int_{B.Z.} d^2k\, \Omega(\vec{k}) \leftarrow \text{always an integer.}$$

In order to obtain $C \neq 0$, time reversal symmetry should be broken

If time reversal symmetry exists, $\Omega(-\vec{k}) = -\Omega(\vec{k}) \Rightarrow C = 0$

Bulk-edge correspondence



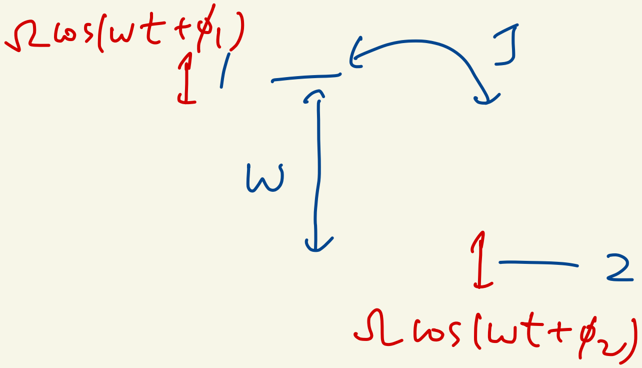
bulk property ⟷ edge

3rd Lecture : Realizations and manifestations of topological band structures in AMO physics and beyond.

3-1. Modulation assisted tunneling

$$H(t+T) = H(t)$$

$$H_{\text{eff}} = \frac{1}{T}\int_0^T H(t)\,dt \qquad \omega = \frac{2\pi}{T}$$



$$H(t) = \begin{pmatrix} \omega/2 & J \\ J & -\omega/2 \end{pmatrix} + \Omega \begin{pmatrix} \cos(\omega t+\phi_1) & 0 \\ 0 & \cos(\omega t+\phi_2) \end{pmatrix}$$

Go to a basis by a unitary transformation $U(t)$

$$H(t) \longrightarrow \hat{H}(t) = i\frac{dU^+}{dt}U + U^+ H(t)\, U$$

If we choose $\quad U = \exp\left[-i\int_0^t dt' \,[\text{diagonal term of } H(t')]\right]$

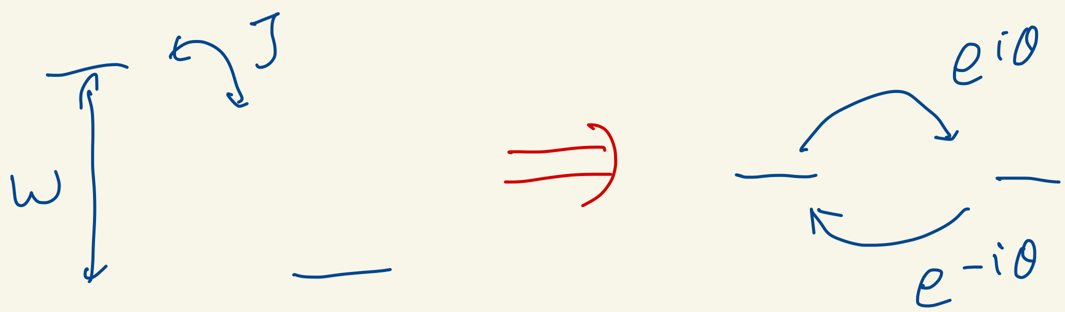
$$\hat{H}(t) = U^\dagger \begin{pmatrix} 0 & J \\ J & 0 \end{pmatrix} U = J \begin{bmatrix} 0 & \exp\left[i\omega t + i\frac{2\Omega}{\omega}\sin\frac{\phi_1-\phi_2}{2}\cos\left(\omega t + \frac{\phi_1+\phi_2}{2}\right)\right] \\ c.c. & 0 \end{bmatrix}$$

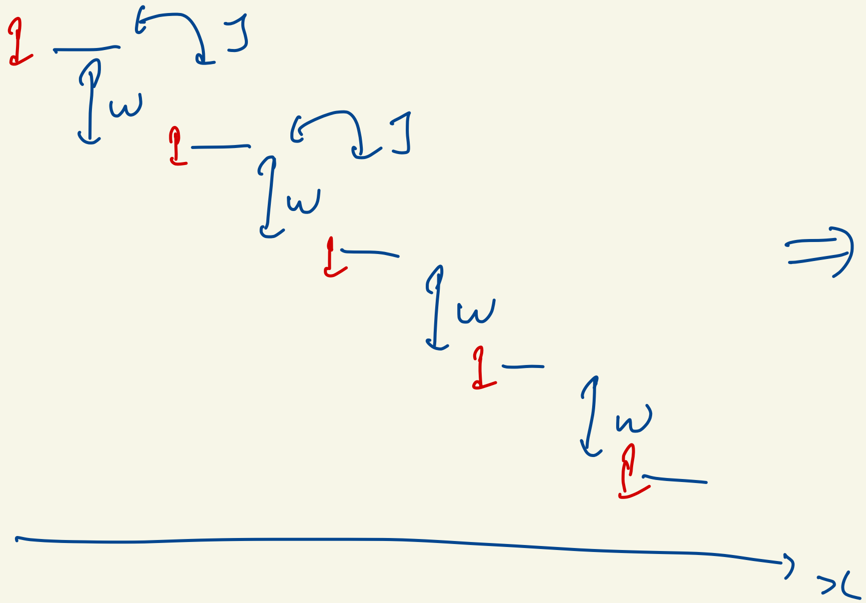
$$\Rightarrow \quad H_{eff} = \frac{\omega}{2\pi}\int_0^{\frac{2\pi}{\omega}} dt\, \hat{H}(t) = J\, \mathcal{J}_1\left[\frac{2\Omega}{\omega}\sin\frac{\phi_1-\phi_2}{2}\right] \begin{pmatrix} 0 & e^{-i(\phi_1+\phi_2-\pi)/2} \\ e^{i(\phi_1+\phi_2-\pi)/2} & 0 \end{pmatrix}$$

$$\mathcal{J}_n(x) = \frac{1}{2\pi i^n}\int_0^{2\pi} dt\, e^{i(nt + x\cos t)}$$

Choosing $\phi_1 = \theta + \pi$, $\phi_2 = \theta$, we obtain

$$H_{eff} = J\, \mathcal{J}_1\left[\frac{2\Omega}{\omega}\right] \begin{pmatrix} 0 & e^{-i\theta} \\ e^{i\theta} & 0 \end{pmatrix}$$

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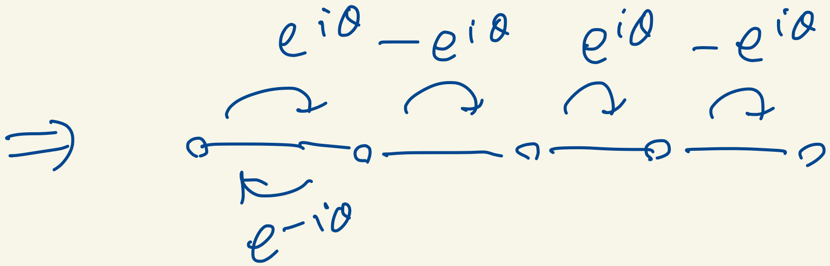
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$e^{i\theta}$
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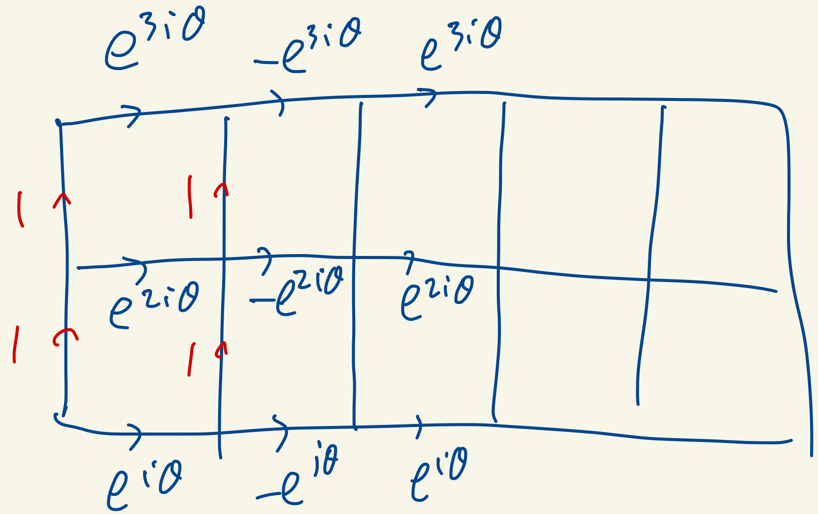
$e^{-i\theta}$


$e^{3i\theta}$
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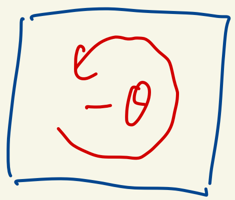
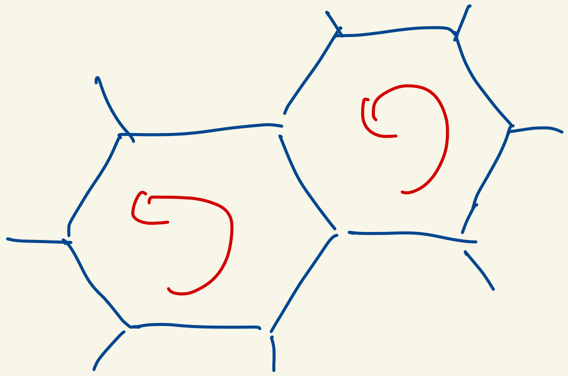

$-\theta$

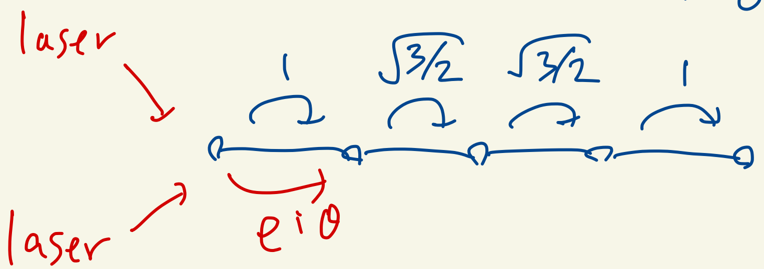
Haldane model



3-2. Synthetic dimension.

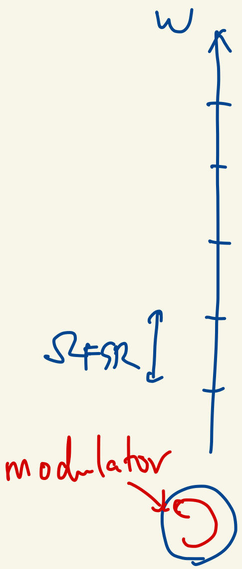
Trivial example. Spin 2 particle in $\vec{B} = (B, 0, 0)$

$$H_{\text{zeeman}} \propto B\hat{S}_x = B\begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & \sqrt{3/2} & 0 & 0 \\ 0 & \sqrt{3/2} & 0 & \sqrt{3/2} & 0 \\ 0 & 0 & \sqrt{3/2} & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$



Frequency synthetic dimension

resonator



Hamiltonian is $H_0 = \sum_n (w_0 + n\,\Omega_{FSR})\, a_n^\dagger a_n$

$$H(t) = H_0 + \Omega \sum_n \left[e^{i(\Omega_{FSR} t + \phi)} a_{n+1}^\dagger a_n + H.c. \right]$$

I go to a rotating frame

$$U = \exp\left[-i \sum_n (w_0 + n\,\Omega_{FSR})\, t\, a_n^\dagger a_n \right]$$

$$H(t) \to \tilde{H}(t) = U^\dagger\, \Omega \sum_n \left[e^{i(\Omega_{FSR} t + \phi)} a_{n+1}^\dagger a_n + H.c. \right] U$$

$$= \Omega \sum_n \left(e^{i\phi} a_{n+1}^\dagger a_n + H.c. \right)$$