Beyond mean-field effects in mixtures: few-body and many-body aspects

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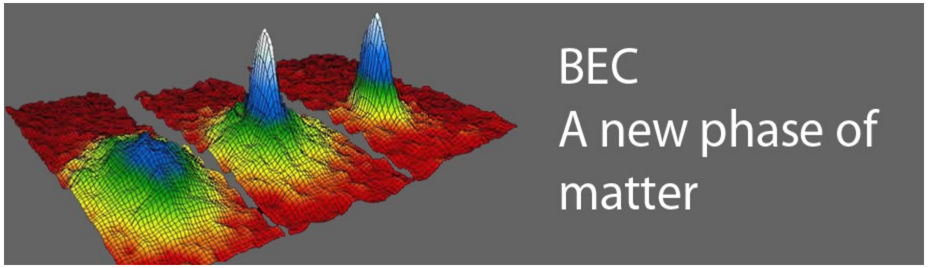
Laboratoire Physique Théorique et Modèles Statistiques (Orsay)

Lecture 1

Quantum droplets



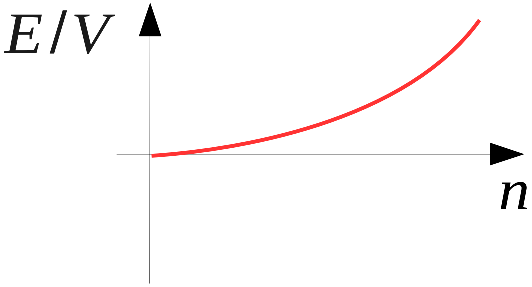
(JILA'95)

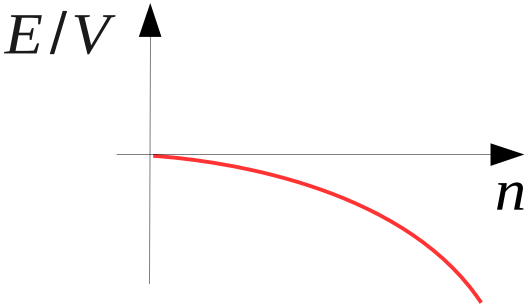
GAS requires trapping or collapses

$g_2>0$

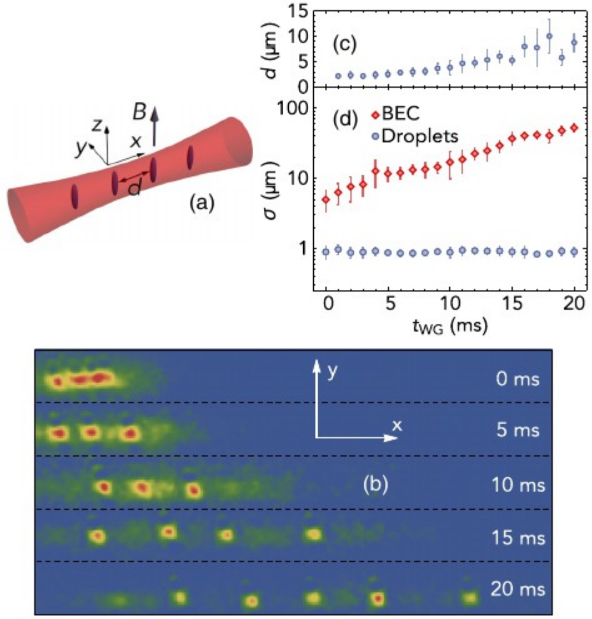
$$E/V=g_2 n^2/2$$

$g_2<0$

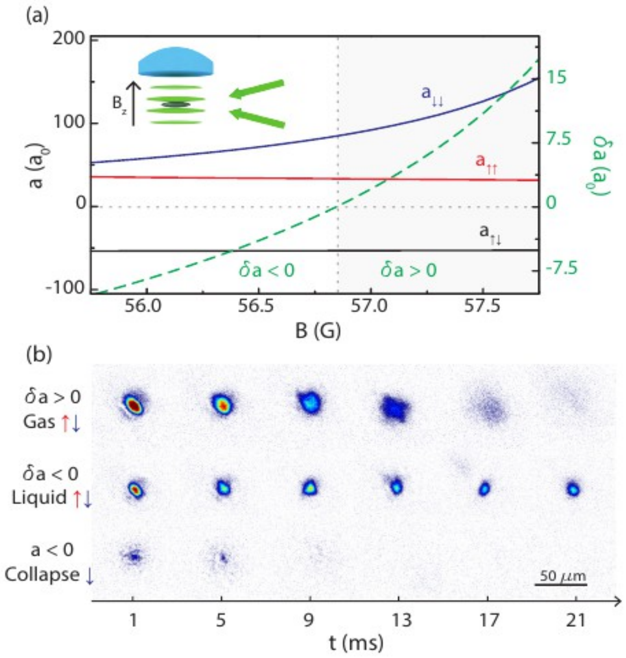




Dipolar and mixture droplets



(Ferrier-Barbut et al '16)



(Cabrera et al '18)

Observation of dipolar droplets and their properties (Stuttgart, Innsbruck, Pisa)

coherent arrays of droplets = 1D supersolid

2D supersolids

...lots of theory...

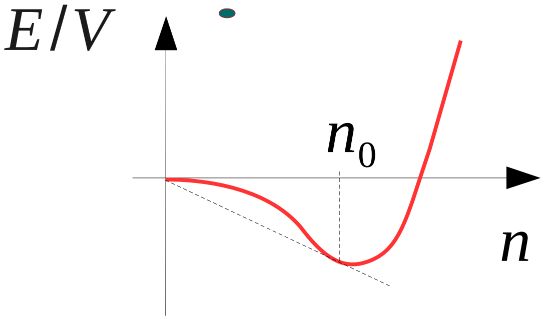
Observation of mixture droplets (Barcelona, Florence) and heteronuclear droplets (Florence, Hong-Kong)

``LHY'' gases (Aarhus)
BMF effects in driven mixtures (Palaiseau)

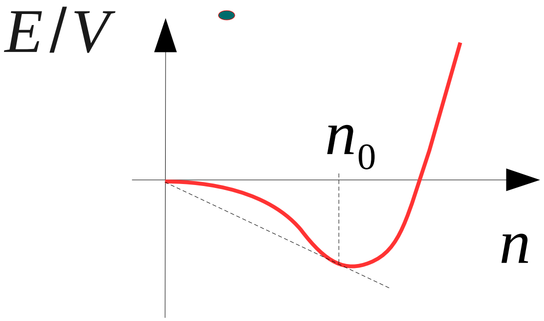
...lots of theory...

Regime of competing MF and BMF. We learned a lot about Bogoliubov theory and BMF effects!

Liquid?



Liquid?



``New'' terminology:

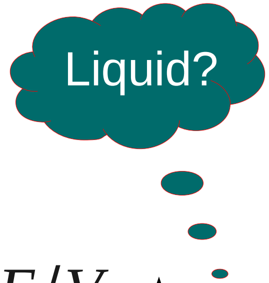
liquid ≠ fluid

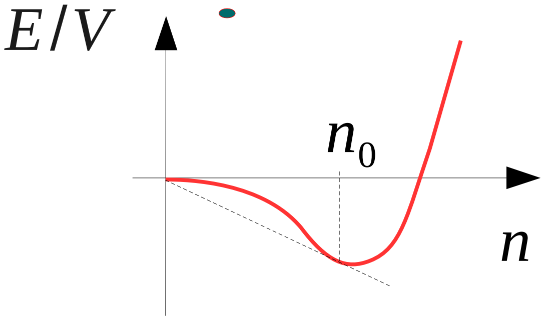
saturation density

particle-emission threshold

surface tension

surface modes, etc.





$$E/V \propto g_2 n^2/2 + g_\alpha n^\alpha/\alpha!, \quad \alpha > 2$$

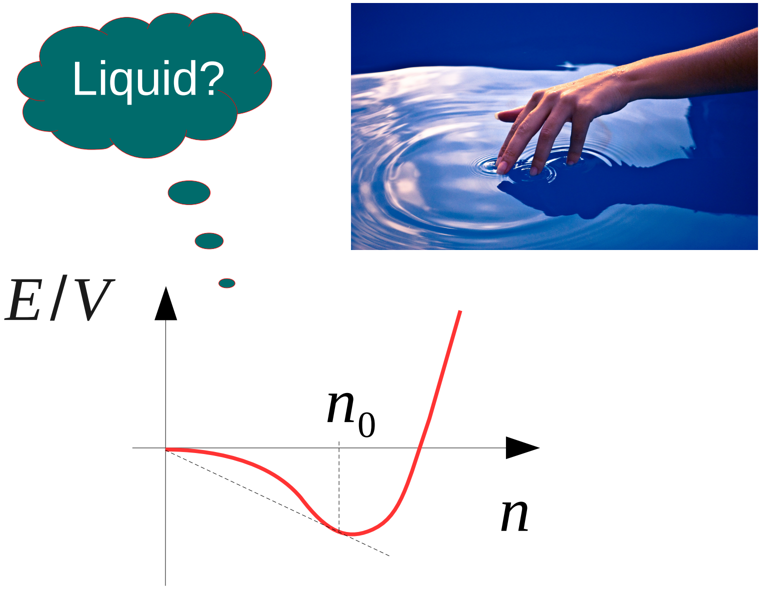
3-body $(\alpha = 3)$

Lee-Huang-Yang $(\alpha = 5/2)$

Resonant (Efimov)
3-body force
(Bulgac'02)

Non-resonant
3-body force
(DP'14)

Beyond-mean-field QUANTUM MECHANISM!
(DP'15)



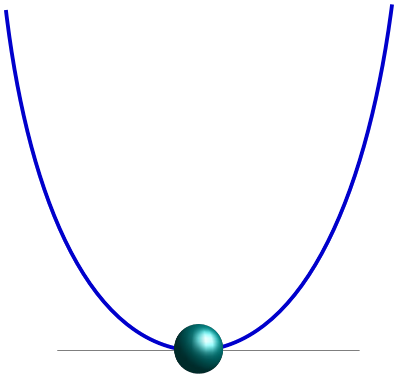
<u>dilute</u> liquid = stabilization against collapse at <u>low</u> densities

need <u>two</u> degrees of freedom:

1 soft, slow, “collapsing” + 1 stiff, fast, “stabilizing”

Quantum stabilization

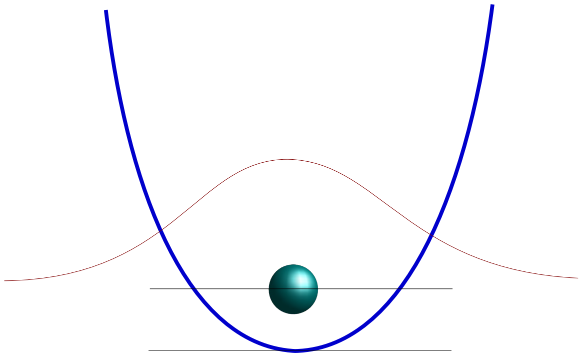
Classical



BEC analog:
Classical or mean-field limit =
Gross-Pitaevskii equation

Classical vacuum

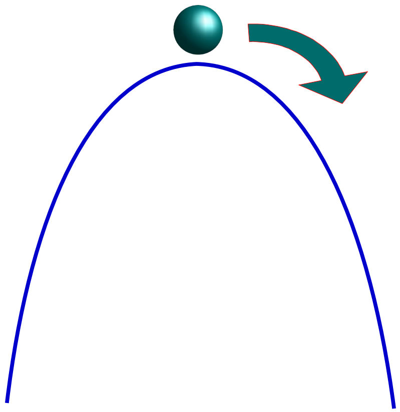
Quantum



BEC analog:
Mean field + Gaussian fluctuations
= GP+LHY

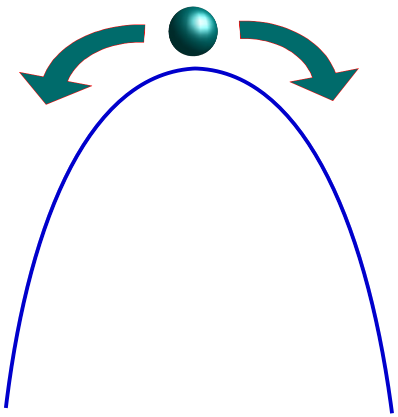
Bogoliubov vacuum

Classical



BEC analog:
collapse

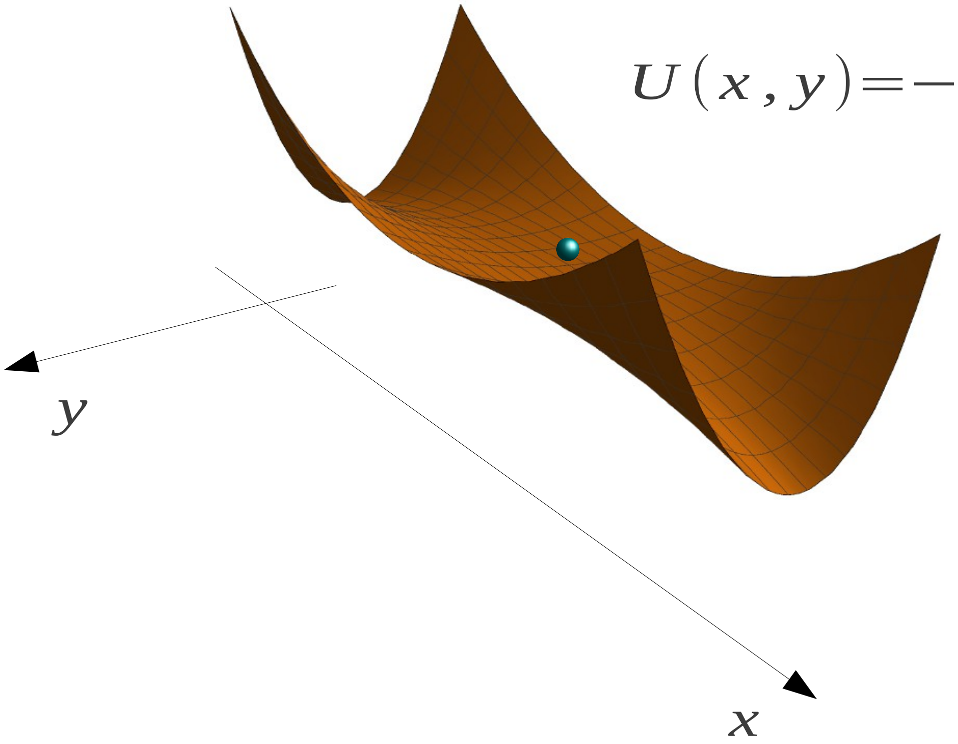
Quantum



BEC analog:
collapse :(

Can there be a classically unstable system,
yet stable when quantum mechanics is “switched on” ?

Quantum stabilization idea



$$U(x,y) = -\alpha x^2 + \frac{\omega^2(x)}{2} y^2$$

Stable for sufficiently fast growing $\omega(x)$

Classically unstable degree of freedom stabilized by quantum fluctuations in another degree of freedom!

BEC analog:

quantum droplet!

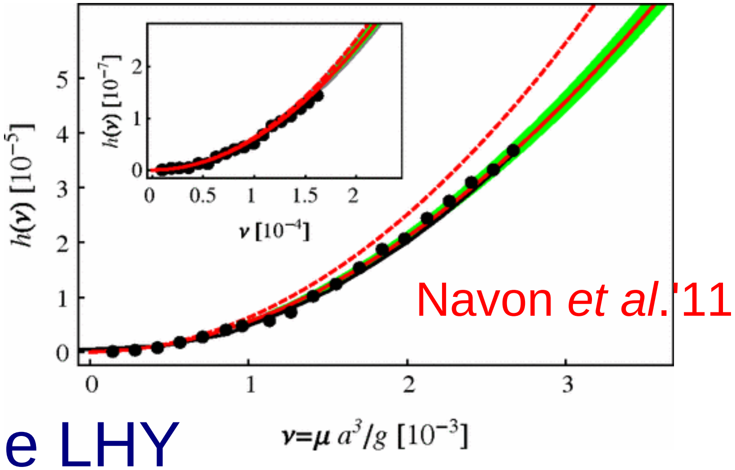
LHY mechanism

For spinless BEC: $$\frac{E}{\text{Volume}} = \frac{g_2 n^2}{2}\left(1 + \frac{128}{15}\sqrt{\frac{na^3}{\pi}} + \ldots\right)$$

Lee-Huang-Yang correction $\propto g_2^{5/2} n^{5/2}$

LHY correction is UNIVERSAL (depends only on the scattering length) and QUANTUM (zero-point energy of Bogoliubov phonons)!

Observed in ultracold gases where the scattering length is tunable by using Feshbach resonances (Innsbruck, MIT, ENS, JILA, Rice)



Navon *et al.*'11

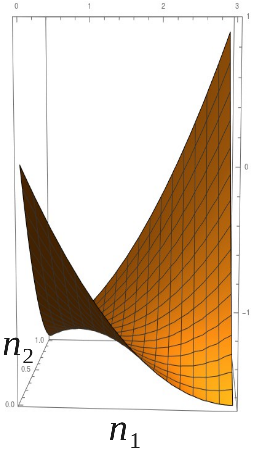
Unfortunately, the effect is perturbative and the LHY term is smaller than the mean-field one!

Bose-Bose mixture, mean field

Mean-field energy density: $$\frac{E_{MF}}{\text{Volume}}=\frac{g_{11}n_1^2+g_{22}n_2^2+2g_{12}n_1n_2}{2}$$

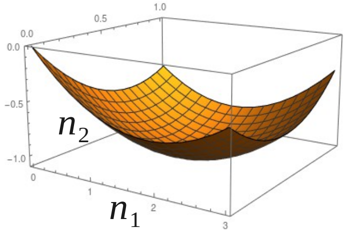
g_{12}

$g_{12}>\sqrt{g_{11}g_{22}}$ phase separation

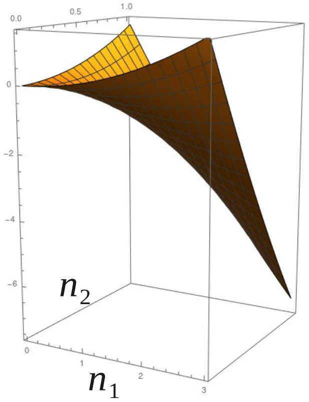


mean-field stability

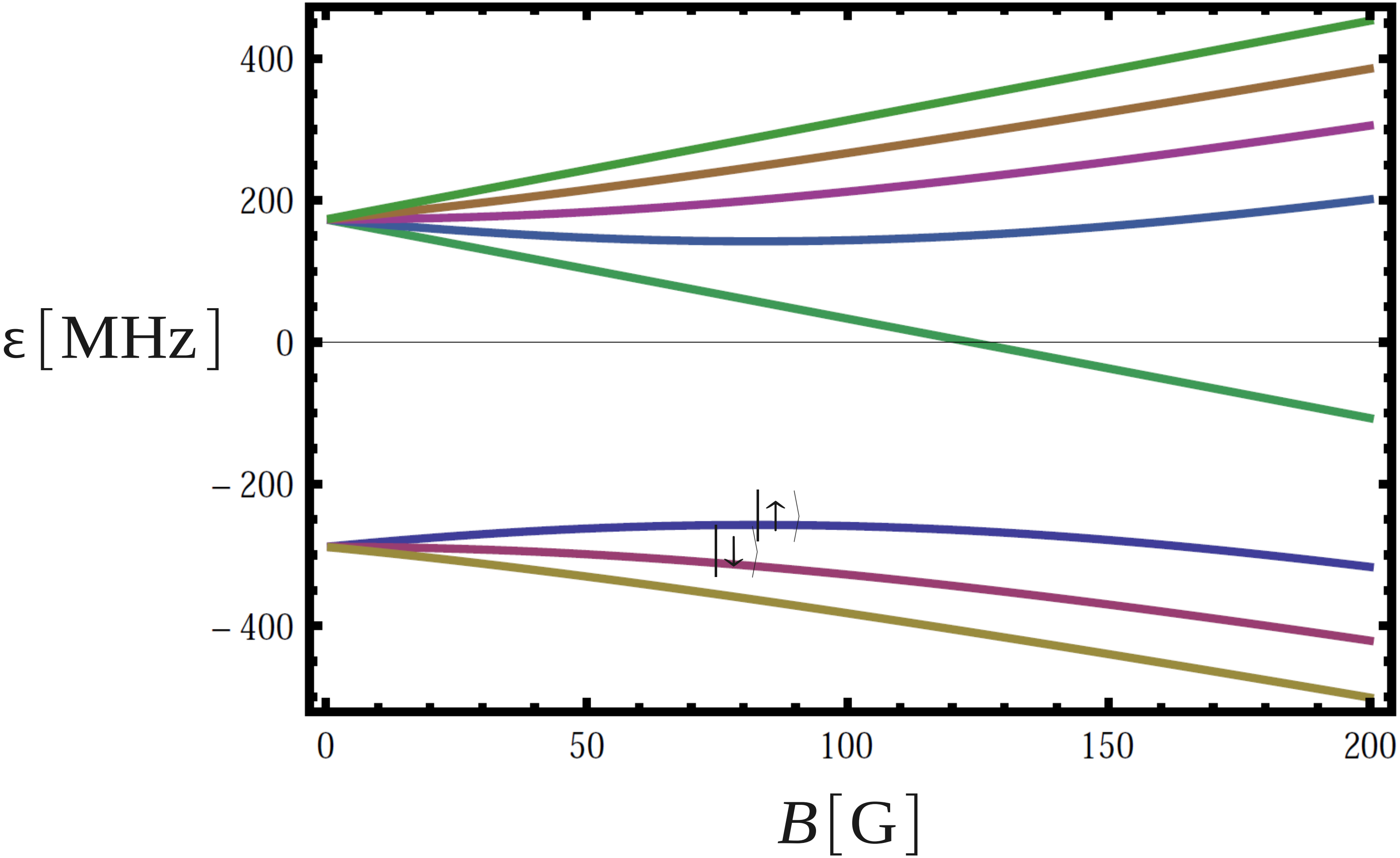
$g_{11}>0,\quad g_{22}>0,\quad \text{and}\quad g_{12}^2<g_{11}g_{22}$



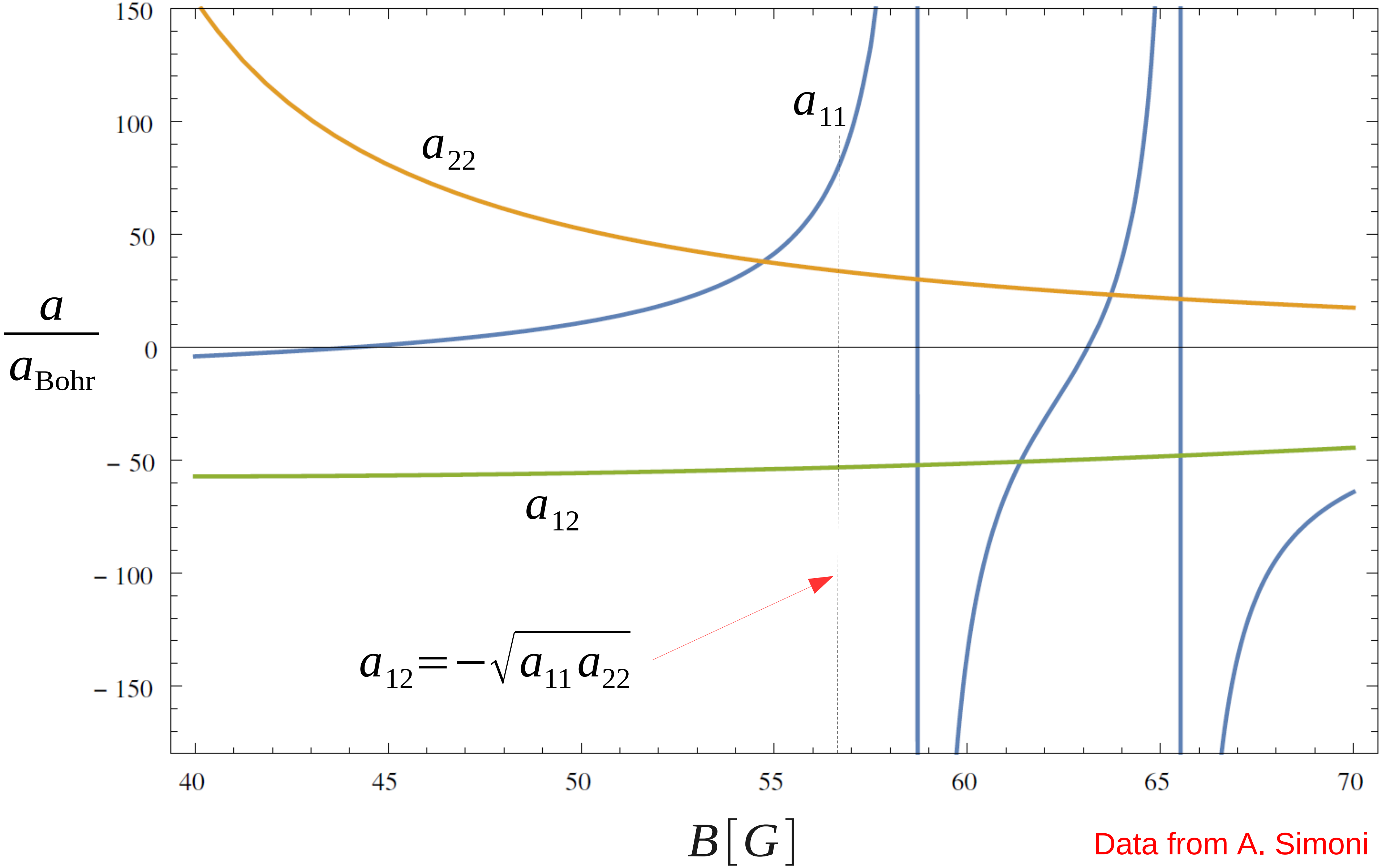
$g_{12}<-\sqrt{g_{11}g_{22}}$ collapse



^{39}K: |F=1,m$_F$=0> and |F=1,m$_F$=-1>



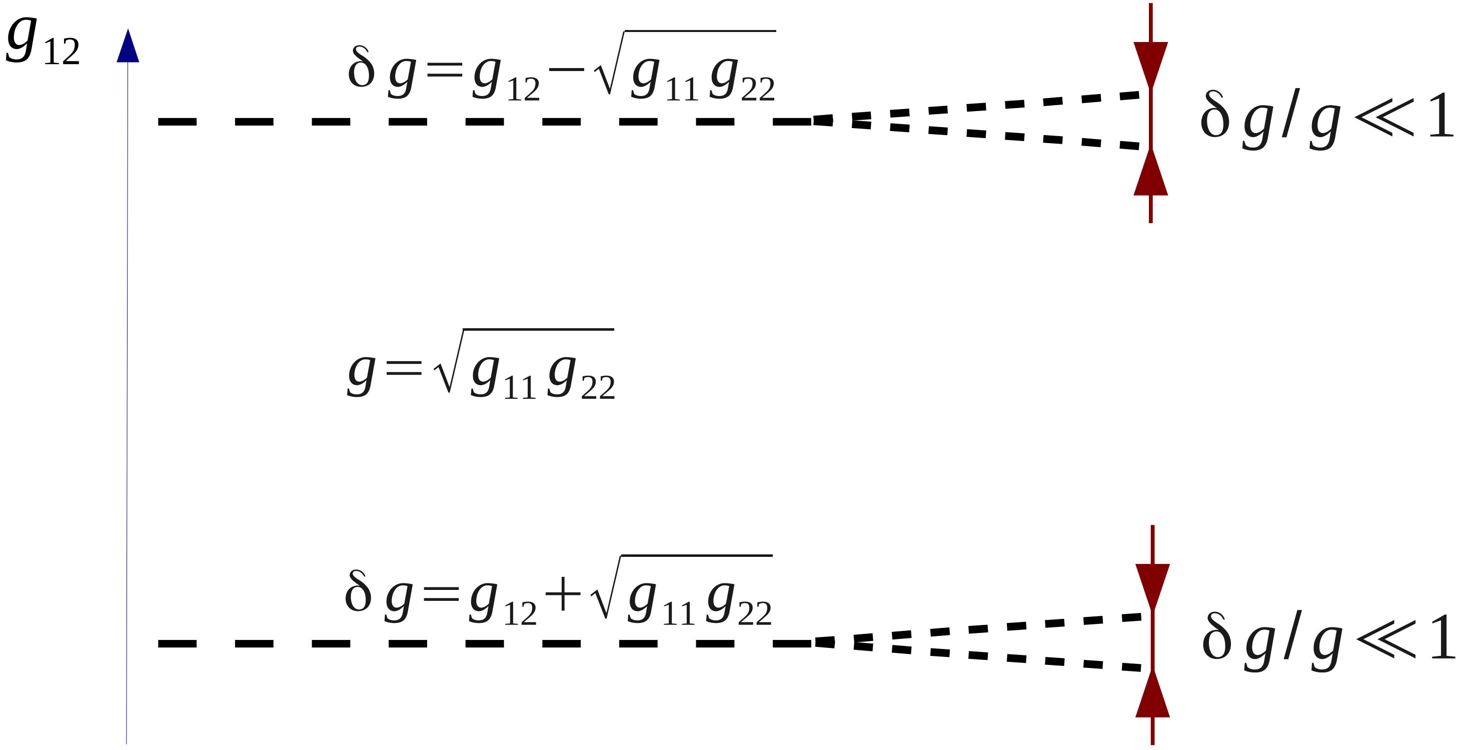
^{39}K: |F=1,m$_F$=0> and |F=1,m$_F$=-1>



Data from A. Simoni

Bose-Bose mixture, mean field

$$E_{MF} = \frac{1}{2}(n_1\, n_2)\begin{pmatrix} g_{11} & g_{12} \\ g_{12} & g_{22} \end{pmatrix}\begin{pmatrix} n_1 \\ n_2 \end{pmatrix} \sim g n_+^2 \pm \delta g\, n_-^2$$



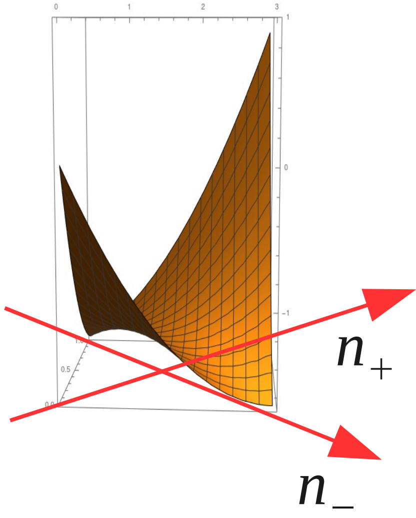
$\delta g = g_{12} - \sqrt{g_{11} g_{22}}$

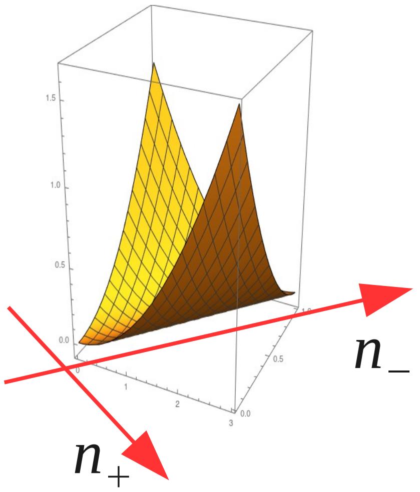
$\delta g / g \ll 1$

$g = \sqrt{g_{11} g_{22}}$

$\delta g = g_{12} + \sqrt{g_{11} g_{22}}$

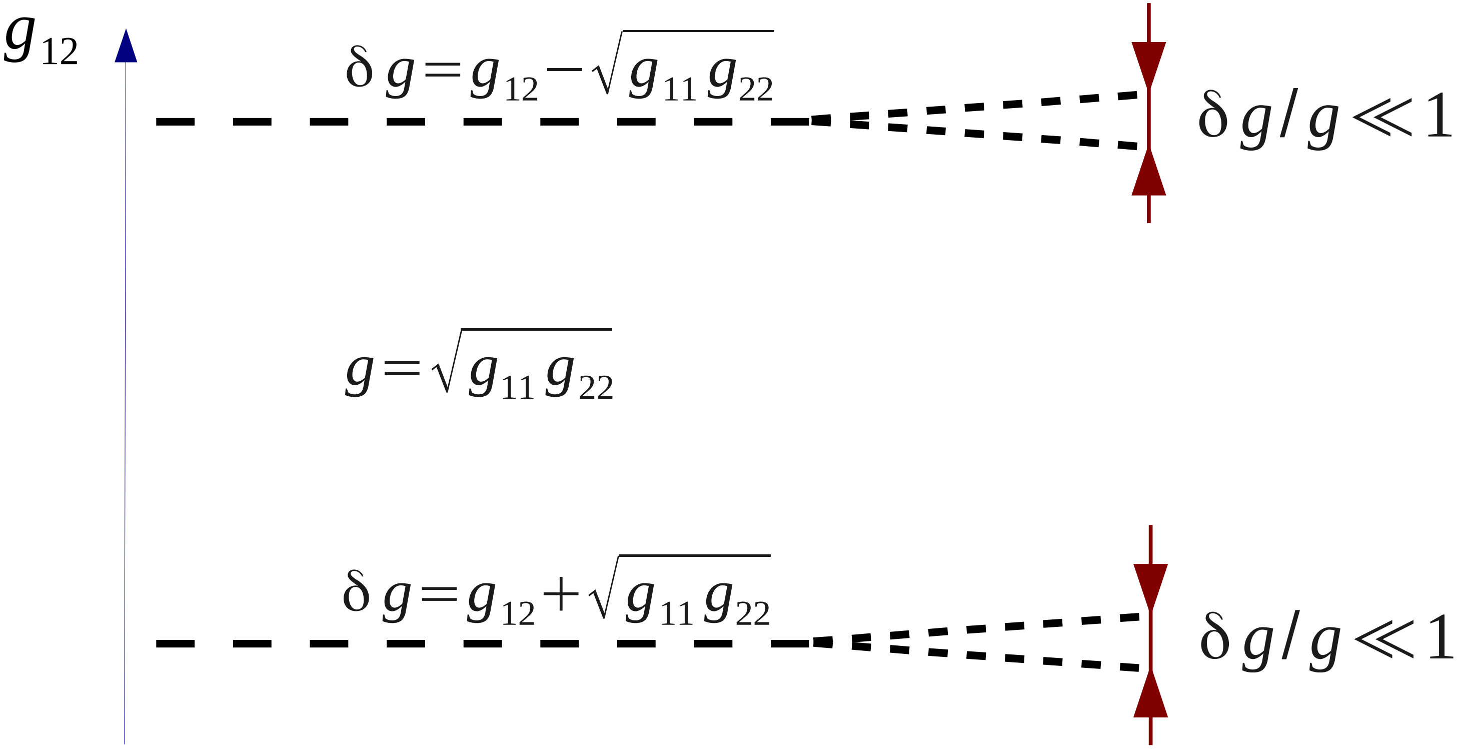
$\delta g / g \ll 1$





Bose-Bose mixture, mean field + LHY (Larsen'63)

$$E_{MF}+E_{LHY}=\frac{1}{2}(n_1\, n_2)\begin{pmatrix} g_{11} & g_{12}\\ g_{12} & g_{22}\end{pmatrix}\begin{pmatrix} n_1\\ n_2\end{pmatrix}+E_{LHY}\sim gn_+^2\pm\delta g\, n_-^2+\frac{8}{15\pi^2}(g_{11}n_1+g_{22}n_2)^{5/2}$$



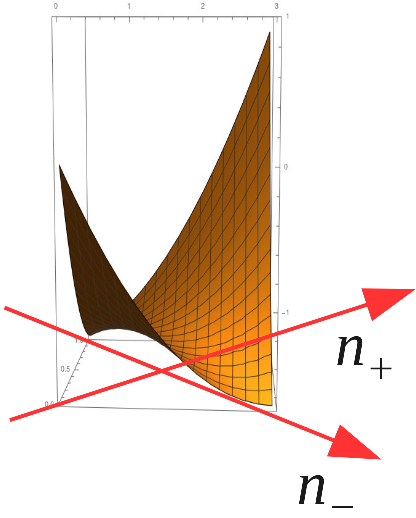
$\delta g=g_{12}-\sqrt{g_{11}g_{22}}$

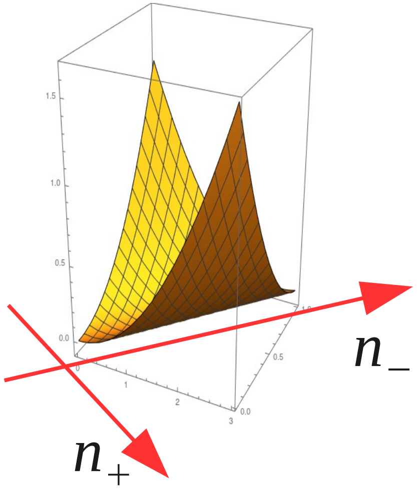
$\delta g/g\ll 1$

$g=\sqrt{g_{11}g_{22}}$

$\delta g=g_{12}+\sqrt{g_{11}g_{22}}$

$\delta g/g\ll 1$





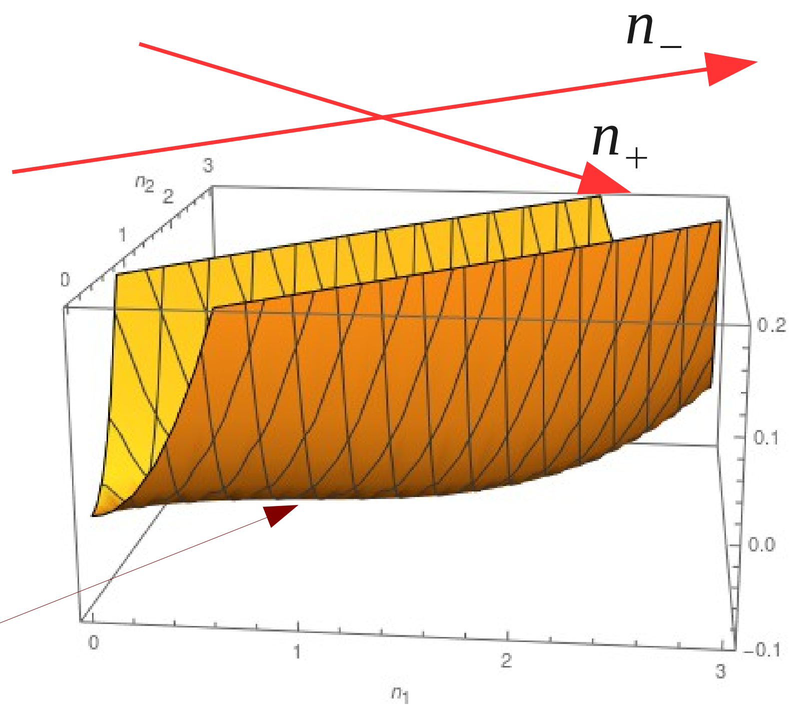
Quantum droplet

$$\delta g = g_{12} + \sqrt{g_{11} g_{22}} \ll \sqrt{g_{11} g_{22}} = g$$

$$n_+ \propto \sqrt{g_{11}}\, n_1 - \sqrt{g_{22}}\, n_2 \approx 0$$

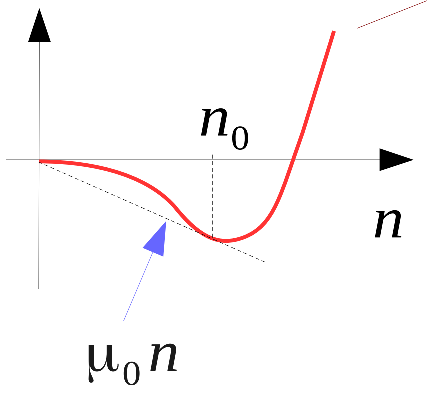
The mean-field term “locks” the ratio $\frac{n_2}{n_1} = \sqrt{\frac{g_{11}}{g_{22}}}$

$$n_- \propto \sqrt{g_{22}}\, n_1 + \sqrt{g_{11}}\, n_2 \propto n$$



$$\frac{E}{\text{volume}} = \delta g\, n^2 + \ldots (gn)^{5/2}$$

$\delta g < 0$



Equilibrium with vacuum.
Saturation density

$$n_0 \propto \frac{1}{a^3} \left(\frac{\delta g}{g} \right)^2$$

Note: $\delta g / g \sim \sqrt{n a^3}$

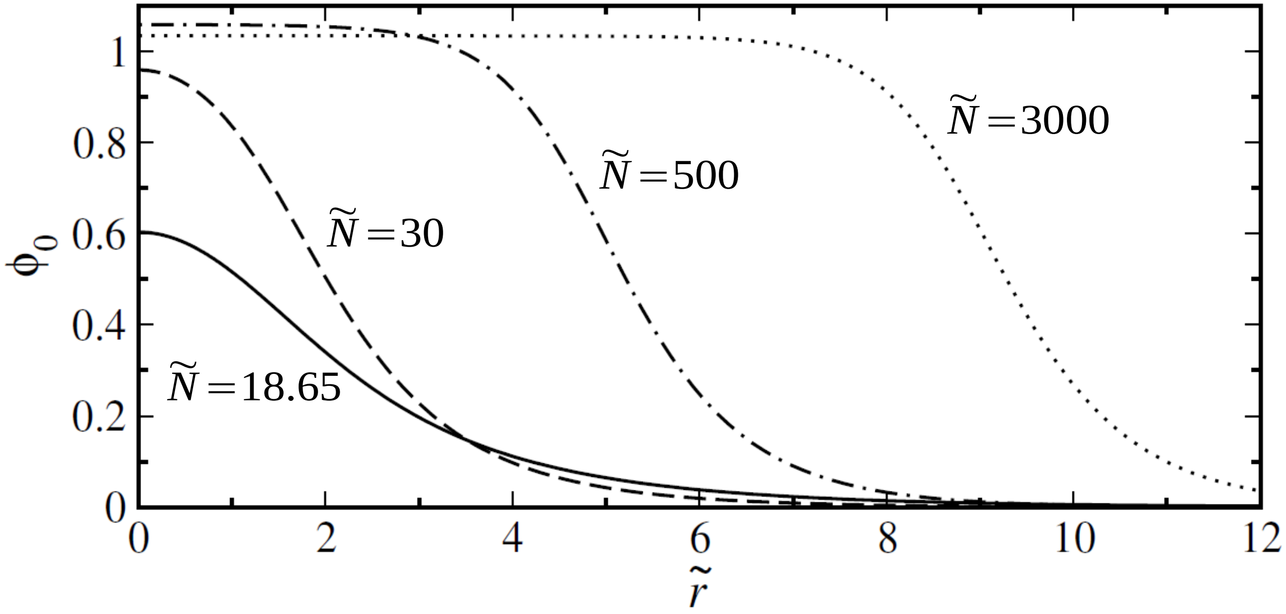
Gross-Pitaevskii eq., droplet shape

Rescaling $\vec{r}=\xi\vec{\tilde{r}},\ \ t=\tau\tilde{t},\ \ N=n\xi^{3}\tilde{N},$ where $\xi\propto 1/\sqrt{m|\delta g|n},\ \ \tau\propto 1/|\delta g|n$

$$i\partial_{\tilde{t}}\phi=\left(-\nabla_{\vec{\tilde{r}}}^{2}/2-3|\phi|^{2}+5|\phi|^{3}/2-\tilde{\mu}\right)\phi$$

$$\tilde{N}=\int|\phi|^{2}d^{3}\tilde{r}$$

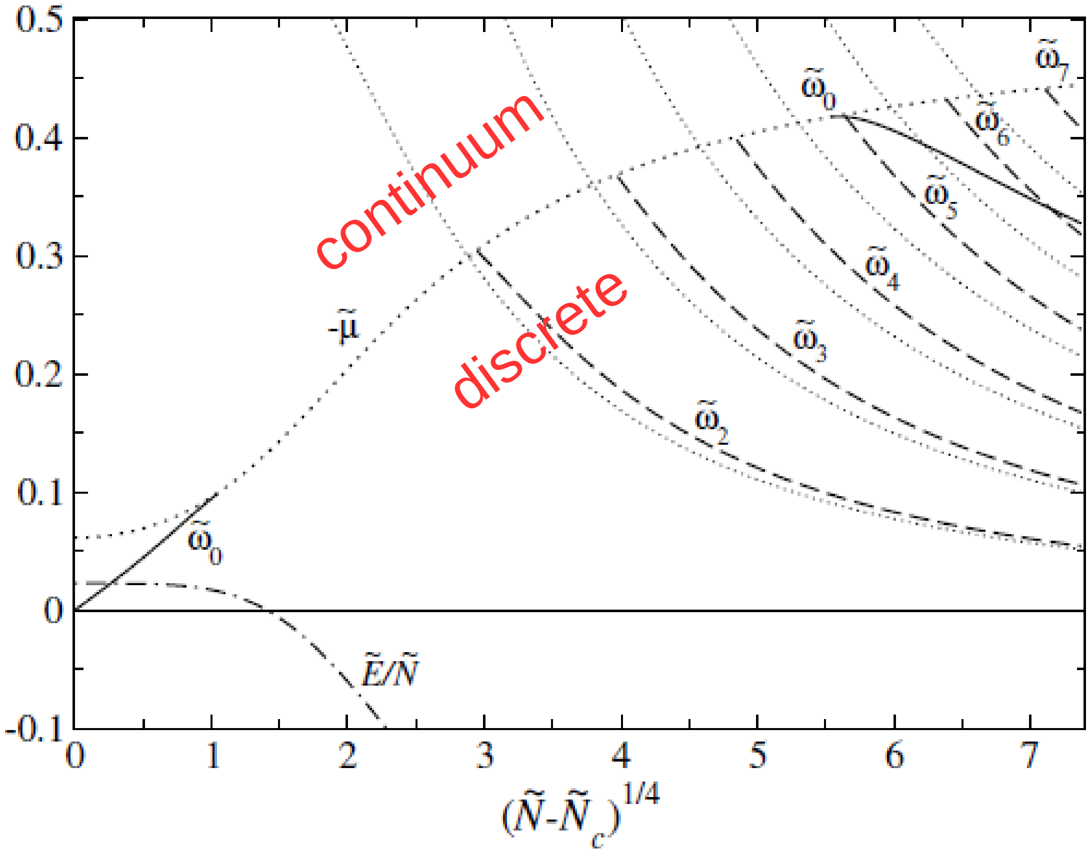
Modified Gross-Pitaevskii equation cubic-quartic nonlinearities



Bogoliubov-de Gennes eqs., excitations

$$\phi(\tilde{t},\vec{\tilde{r}})=\phi_0(\tilde{r})+\delta\phi(\tilde{t},\vec{\tilde{r}})$$

linearize $i\partial_{\tilde{t}}\phi=(-\nabla_{\vec{\tilde{r}}}{}^2/2-3|\phi|^2+5|\phi|^3/2-\tilde{\mu})\phi$ with respect to small $\delta\phi(\tilde{t},\vec{\tilde{r}})$



Surface modes

Bogoliubov method

Bogoliubov theory

Hamiltonian of the mixture:

$$\hat{H} = \sum_{\sigma,\mathbf{k}} \frac{k^2}{2m_\sigma} \hat{a}^\dagger_{\sigma,\mathbf{k}} \hat{a}_{\sigma,\mathbf{k}} + \frac{1}{2} \sum_{\sigma,\sigma',\mathbf{k}_1,\mathbf{k}_2,\mathbf{q}} \hat{a}^\dagger_{\sigma,\mathbf{k}_1+\mathbf{q}} \hat{a}^\dagger_{\sigma',\mathbf{k}_2-\mathbf{q}} U_{\sigma\sigma'}(\mathbf{q}) \hat{a}_{\sigma,\mathbf{k}_1} \hat{a}_{\sigma',\mathbf{k}_2}$$

$$\Big\downarrow \quad \hat{a}_{\sigma,0}, \hat{a}^\dagger_{\sigma,0} \to a_{\sigma,0} < \sqrt{n_\sigma}$$

$$\hat{H} = H_0 + \hat{H}_2 + \hat{H}_3 + \hat{H}_4$$

“Mean-field term”

$$H_0 = \frac{1}{2} \sum_{\sigma\sigma'} U_{\sigma\sigma'}(0) a^2_{\sigma,0} a^2_{\sigma',0}$$

“Quadratic Hamiltonian”

$$\hat{H}_2 = \sum_{\sigma,\mathbf{k}}{}' \left[\frac{k^2}{2m_\sigma} + \sum_{\sigma'} U_{\sigma\sigma'}(0) a^2_{\sigma',0} \right] \hat{a}^\dagger_{\sigma,\mathbf{k}} \hat{a}_{\sigma,\mathbf{k}} + \frac{1}{2} \sum_{\sigma,\sigma',\mathbf{k}}{}' U_{\sigma\sigma'}(\mathbf{k}) a_{\sigma,0} a_{\sigma',0} (\hat{a}^\dagger_{\sigma,\boldsymbol{k}} \hat{a}^\dagger_{\sigma',-\boldsymbol{k}} + \hat{a}_{\sigma,\boldsymbol{k}} \hat{a}_{\sigma',-\boldsymbol{k}} + 2\hat{a}^\dagger_{\sigma,\boldsymbol{k}} \hat{a}_{\sigma',\boldsymbol{k}})$$

$$\hat{H}_3 = \sum_{\sigma,\sigma',\boldsymbol{k}_1,\boldsymbol{k}_2}{}' U_{\sigma\sigma'}(\mathbf{k}_1) a_{\sigma,0} \hat{a}^\dagger_{\sigma',\boldsymbol{k}_1+\boldsymbol{k}_2} (\hat{a}_{\sigma,\boldsymbol{k}_1} + \hat{a}^\dagger_{\sigma,-\boldsymbol{k}_1}) \hat{a}_{\sigma',\boldsymbol{k}_2}$$

$$\hat{H}_4 = \frac{1}{2} \sum_{\sigma,\sigma',\mathbf{k}_1,\mathbf{k}_2,\mathbf{q}}{}' \hat{a}^\dagger_{\sigma,\mathbf{k}_1+\mathbf{q}} \hat{a}^\dagger_{\sigma',\mathbf{k}_2-\mathbf{q}} U_{\sigma\sigma'}(\mathbf{q}) \hat{a}_{\sigma,\mathbf{k}_1} \hat{a}_{\sigma',\mathbf{k}_2}$$

Bogoliubov theory

Quadratic part:

$$\hat{H}_2 = {\sum_{\sigma,\mathbf{k}}}' \left[\frac{k^2}{2m_\sigma} + \sum_{\sigma'} U_{\sigma\sigma'}(0) a_{\sigma',0}^2 \right] \hat{a}_{\sigma,\mathbf{k}}^\dagger \hat{a}_{\sigma,\mathbf{k}} + \frac{1}{2} {\sum_{\sigma,\sigma',\mathbf{k}}}' U_{\sigma\sigma'}(\mathbf{k}) a_{\sigma,0} a_{\sigma',0} (\hat{a}_{\sigma,\boldsymbol{k}}^\dagger \hat{a}_{\sigma',-\boldsymbol{k}}^\dagger + \hat{a}_{\sigma,\boldsymbol{k}} \hat{a}_{\sigma',-\boldsymbol{k}} + 2\hat{a}_{\sigma,\boldsymbol{k}}^\dagger \hat{a}_{\sigma',\boldsymbol{k}})$$

“Quadratic Hamiltonian” = “easy to diagonalize”

2 problems:

1) not so easy

2) the spectrum is gapped

Bogoliubov theory

Quadratic part:

$$\hat{H}_2 = {\sum_{\sigma,\mathbf{k}}}' \left[\frac{k^2}{2m_\sigma} + \sum_{\sigma'} U_{\sigma\sigma'}(0) a^2_{\sigma',0} \right] \hat{a}^\dagger_{\sigma,\mathbf{k}} \hat{a}_{\sigma,\mathbf{k}} + \frac{1}{2} {\sum_{\sigma,\sigma',\mathbf{k}}}' U_{\sigma\sigma'}(\mathbf{k}) a_{\sigma,0} a_{\sigma',0} (\hat{a}^\dagger_{\sigma,\boldsymbol{k}} \hat{a}^\dagger_{\sigma',-\boldsymbol{k}} + \hat{a}_{\sigma,\boldsymbol{k}} \hat{a}_{\sigma',-\boldsymbol{k}} + 2\hat{a}^\dagger_{\sigma,\boldsymbol{k}} \hat{a}_{\sigma',\boldsymbol{k}})$$

gap comes from this term. What is the problem?

The problem is the canonical description where n_σ are fixed. When we create an excited atom, we deplete the condensate, i.e.,

$$a^2_{\sigma,0} = n_\sigma - {\sum_{\mathbf{k}}}' \hat{a}^\dagger_{\sigma,\mathbf{k}} \hat{a}_{\sigma,\mathbf{k}}$$

$$H_0 = \frac{1}{2} \sum_{\sigma\sigma'} U_{\sigma\sigma'}(0) a^2_{\sigma,0} a^2_{\sigma',0} \approx \frac{1}{2} \sum_{\sigma\sigma'} U_{\sigma\sigma'}(0) n_\sigma n_{\sigma'} - {\sum_{\sigma,\sigma',\mathbf{k}}}' U_{\sigma\sigma'}(0) n_{\sigma'} \hat{a}^\dagger_{\sigma,\mathbf{k}} \hat{a}_{\sigma,\mathbf{k}}$$

$H_0 + \hat{H}_2$ becomes the Bogoliubov quadratic Hamiltonian:

$$\boxed{\hat{H}_{\text{Bog}} = \frac{1}{2} \sum_{\sigma\sigma'} U_{\sigma\sigma'}(0) n_\sigma n_{\sigma'} + {\sum_{\sigma,\mathbf{k}}}' \frac{k^2}{2m_\sigma} \hat{a}^\dagger_{\sigma,\mathbf{k}} \hat{a}_{\sigma,\mathbf{k}} + \frac{1}{2} {\sum_{\sigma,\sigma',\mathbf{k}}}' U_{\sigma\sigma'}(\mathbf{k}) \sqrt{n_\sigma n_{\sigma'}} (\hat{a}^\dagger_{\sigma,\boldsymbol{k}} \hat{a}^\dagger_{\sigma',-\boldsymbol{k}} + \hat{a}_{\sigma,\boldsymbol{k}} \hat{a}_{\sigma',-\boldsymbol{k}} + 2\hat{a}^\dagger_{\sigma,\boldsymbol{k}} \hat{a}_{\sigma',\boldsymbol{k}})}$$

*the “gap” problem does not show up in the grand canonical description

Bogoliubov theory

Diagonalization of the quadratic Hamiltonian (industrial)

$$\hat{H}_{\mathrm{Bog}} = \frac{1}{2}\sum_{\sigma\sigma'} U_{\sigma\sigma'}(0) n_\sigma n_{\sigma'} + {\sum_{\sigma,\mathbf{k}}}' \frac{k^2}{2m_\sigma}\hat{a}^\dagger_{\sigma,\mathbf{k}}\hat{a}_{\sigma,\mathbf{k}} + \frac{1}{2}{\sum_{\sigma,\sigma',\mathbf{k}}}' U_{\sigma\sigma'}(\mathbf{k})\sqrt{n_\sigma n_{\sigma'}}(\hat{a}^\dagger_{\sigma,\boldsymbol{k}}\hat{a}^\dagger_{\sigma',-\boldsymbol{k}} + \hat{a}_{\sigma,\boldsymbol{k}}\hat{a}_{\sigma',-\boldsymbol{k}} + 2\hat{a}^\dagger_{\sigma,\boldsymbol{k}}\hat{a}_{\sigma',\boldsymbol{k}})$$

symmetrize by using

$$\hat{a}^\dagger_{\sigma,\boldsymbol{k}}\hat{a}_{\sigma',\boldsymbol{k}} = (1/2)(\hat{a}^\dagger_{\sigma,\boldsymbol{k}}\hat{a}_{\sigma',\boldsymbol{k}} + \hat{a}_{\sigma,\boldsymbol{k}}\hat{a}^\dagger_{\sigma',\boldsymbol{k}}) - \delta_{\sigma\sigma'}/2$$

$$\begin{aligned}
\hat{H}_{\mathrm{Bog}} &= \frac{1}{2}\sum_{\sigma\sigma'} U_{\sigma\sigma'}(0) n_\sigma n_{\sigma'} - \frac{1}{2}{\sum_{\sigma,\mathbf{k}}}'\left[\frac{k^2}{2m_\sigma} + U_{\sigma\sigma}(\mathbf{k})n_\sigma\right] \\
&+\frac{1}{2}{\sum_{\mathbf{k}}}'(\hat{a}^\dagger_{\uparrow,\mathbf{k}}\hat{a}^\dagger_{\downarrow,\mathbf{k}}\hat{a}_{\uparrow,-\mathbf{k}}\hat{a}_{\downarrow,-\mathbf{k}})
\left(\begin{array}{cc|cc}
\frac{k^2}{2m_\uparrow} + U_{\uparrow\uparrow}(\mathbf{k})n_\uparrow & U_{\uparrow\downarrow}(\mathbf{k})\sqrt{n_\uparrow n_\downarrow} & U_{\uparrow\uparrow}(\mathbf{k})n_\uparrow & U_{\uparrow\downarrow}(\mathbf{k})\sqrt{n_\uparrow n_\downarrow} \\
U_{\uparrow\downarrow}(\mathbf{k})\sqrt{n_\uparrow n_\downarrow} & \frac{k^2}{2m_\downarrow} + U_{\downarrow\downarrow}(\mathbf{k})n_\downarrow & U_{\uparrow\downarrow}(\mathbf{k})\sqrt{n_\uparrow n_\downarrow} & U_{\downarrow\downarrow}(\mathbf{k})n_\downarrow \\
\hline
U_{\uparrow\uparrow}(\mathbf{k})n_\uparrow & U_{\uparrow\downarrow}(\mathbf{k})\sqrt{n_\uparrow n_\downarrow} & \frac{k^2}{2m_\uparrow} + U_{\uparrow\uparrow}(\mathbf{k})n_\uparrow & U_{\uparrow\downarrow}(\mathbf{k})\sqrt{n_\uparrow n_\downarrow} \\
U_{\uparrow\downarrow}(\mathbf{k})\sqrt{n_\uparrow n_\downarrow} & U_{\downarrow\downarrow}(\mathbf{k})n_\downarrow & U_{\uparrow\downarrow}(\mathbf{k})\sqrt{n_\uparrow n_\downarrow} & \frac{k^2}{2m_\downarrow} + U_{\downarrow\downarrow}(\mathbf{k})n_\downarrow
\end{array}\right)
\begin{pmatrix} \hat{a}_{\uparrow,\mathbf{k}} \\ \hat{a}_{\downarrow,\mathbf{k}} \\ \hat{a}^\dagger_{\uparrow,-\mathbf{k}} \\ \hat{a}^\dagger_{\downarrow,-\mathbf{k}} \end{pmatrix}
\end{aligned}$$

$$\begin{pmatrix} \hat{A} & \hat{B} \\ \hat{B} & \hat{A} \end{pmatrix} \quad \hat{A} \text{ and } \hat{B} \text{ are real symmetric matrices}$$

In fact, we have to diagonalize $\begin{pmatrix} \hat{A} & \hat{B} \\ -\hat{B} & -\hat{A} \end{pmatrix}$

Bogoliubov theory

Properties of $\begin{pmatrix} \hat{A} & \hat{B} \\ -\hat{B} & -\hat{A} \end{pmatrix}$:

1) spectrum symmetric wrt 0 $\begin{pmatrix} \hat{A} & \hat{B} \\ -\hat{B} & -\hat{A} \end{pmatrix} \begin{pmatrix} \vec{u} \\ \vec{v} \end{pmatrix} = \epsilon \begin{pmatrix} \vec{u} \\ \vec{v} \end{pmatrix} \Longrightarrow \begin{pmatrix} \hat{A} & \hat{B} \\ -\hat{B} & -\hat{A} \end{pmatrix} \begin{pmatrix} \vec{v} \\ \vec{u} \end{pmatrix} = -\epsilon \begin{pmatrix} \vec{v} \\ \vec{u} \end{pmatrix}$

2) left eigenvectors $\begin{pmatrix} \hat{A} & \hat{B} \\ -\hat{B} & -\hat{A} \end{pmatrix} \begin{pmatrix} \vec{u} \\ \vec{v} \end{pmatrix} = \epsilon \begin{pmatrix} \vec{u} \\ \vec{v} \end{pmatrix} \Longrightarrow \begin{pmatrix} \vec{u}^T & -\vec{v}^T \end{pmatrix} \begin{pmatrix} \hat{A} & \hat{B} \\ -\hat{B} & -\hat{A} \end{pmatrix} = \epsilon \begin{pmatrix} \vec{u}^T & -\vec{v}^T \end{pmatrix}$

3) normalization $\vec{u}_i^T \vec{u}_j - \vec{v}_i^T \vec{v}_j = \delta_{ij}$

assume $\hat{A}$ and $\hat{B}$ are $N \times N$ matrices

$$\hat{S} = \begin{pmatrix} \vec{u}_1 & \dots & \vec{u}_N & \vec{v}_1 & \dots & \vec{v}_N \\ \vec{v}_1 & \dots & \vec{v}_N & \vec{u}_1 & \dots & \vec{u}_N \end{pmatrix} \Longrightarrow \hat{S}^T \begin{pmatrix} \hat{A} & \hat{B} \\ \hat{B} & \hat{A} \end{pmatrix} \hat{S} = \begin{pmatrix} \epsilon_1 & & & & & 0 \\ & \ddots & & & & \\ & & \epsilon_N & & & \\ & & & \epsilon_1 & & \\ & & & & \ddots & \\ 0 & & & & & \epsilon_N \end{pmatrix}$$

eigenvectors of $\begin{pmatrix} \hat{A} & \hat{B} \\ -\hat{B} & -\hat{A} \end{pmatrix}$

Bogoliubov theory

Back to

$$\hat{H}_{\mathrm{Bog}} = \frac{1}{2}\sum_{\sigma\sigma'} U_{\sigma\sigma'}(0)\, n_\sigma n_{\sigma'} - \frac{1}{2}\sum_{\sigma,\mathbf{k}}{}' \left[\frac{k^2}{2m_\sigma} + U_{\sigma\sigma}(\mathbf{k})\, n_\sigma\right]$$

$$+\frac{1}{2}\sum_{\mathbf{k}}{}' \left(\hat{a}^\dagger_{\uparrow,\mathbf{k}} \hat{a}^\dagger_{\downarrow,\mathbf{k}} \hat{a}_{\uparrow,-\mathbf{k}} \hat{a}_{\downarrow,-\mathbf{k}}\right) \left(\begin{array}{cc|cc} \frac{k^2}{2m_\uparrow} + U_{\uparrow\uparrow}(\mathbf{k})n_\uparrow & U_{\uparrow\downarrow}(\mathbf{k})\sqrt{n_\uparrow n_\downarrow} & U_{\uparrow\uparrow}(\mathbf{k})n_\uparrow & U_{\uparrow\downarrow}(\mathbf{k})\sqrt{n_\uparrow n_\downarrow} \\ U_{\uparrow\downarrow}(\mathbf{k})\sqrt{n_\uparrow n_\downarrow} & \frac{k^2}{2m_\downarrow} + U_{\downarrow\downarrow}(\mathbf{k})n_\downarrow & U_{\uparrow\downarrow}(\mathbf{k})\sqrt{n_\uparrow n_\downarrow} & U_{\downarrow\downarrow}(\mathbf{k})n_\downarrow \\ \hline U_{\uparrow\uparrow}(\mathbf{k})n_\uparrow & U_{\uparrow\downarrow}(\mathbf{k})\sqrt{n_\uparrow n_\downarrow} & \frac{k^2}{2m_\uparrow} + U_{\uparrow\uparrow}(\mathbf{k})n_\uparrow & U_{\uparrow\downarrow}(\mathbf{k})\sqrt{n_\uparrow n_\downarrow} \\ U_{\uparrow\downarrow}(\mathbf{k})\sqrt{n_\uparrow n_\downarrow} & U_{\downarrow\downarrow}(\mathbf{k})n_\downarrow & U_{\uparrow\downarrow}(\mathbf{k})\sqrt{n_\uparrow n_\downarrow} & \frac{k^2}{2m_\downarrow} + U_{\downarrow\downarrow}(\mathbf{k})n_\downarrow \end{array}\right) \begin{pmatrix} \hat{a}_{\uparrow,\mathbf{k}} \\ \hat{a}_{\downarrow,\mathbf{k}} \\ \hat{a}^\dagger_{\uparrow,-\mathbf{k}} \\ \hat{a}^\dagger_{\downarrow,-\mathbf{k}} \end{pmatrix}$$

Diag. $\begin{pmatrix} \frac{k^2}{2m_\uparrow} + U_{\uparrow\uparrow}(\mathbf{k})n_\uparrow & U_{\uparrow\downarrow}(\mathbf{k})\sqrt{n_\uparrow n_\downarrow} & U_{\uparrow\uparrow}(\mathbf{k})n_\uparrow & U_{\uparrow\downarrow}(\mathbf{k})\sqrt{n_\uparrow n_\downarrow} \\ U_{\uparrow\downarrow}(\mathbf{k})\sqrt{n_\uparrow n_\downarrow} & \frac{k^2}{2m_\downarrow} + U_{\downarrow\downarrow}(\mathbf{k})n_\downarrow & U_{\uparrow\downarrow}(\mathbf{k})\sqrt{n_\uparrow n_\downarrow} & U_{\downarrow\downarrow}(\mathbf{k})n_\downarrow \\ -U_{\uparrow\uparrow}(\mathbf{k})n_\uparrow & -U_{\uparrow\downarrow}(\mathbf{k})\sqrt{n_\uparrow n_\downarrow} & -\frac{k^2}{2m_\uparrow} - U_{\uparrow\uparrow}(\mathbf{k})n_\uparrow & -U_{\uparrow\downarrow}(\mathbf{k})\sqrt{n_\uparrow n_\downarrow} \\ -U_{\uparrow\downarrow}(\mathbf{k})\sqrt{n_\uparrow n_\downarrow} & -U_{\downarrow\downarrow}(\mathbf{k})n_\downarrow & -U_{\uparrow\downarrow}(\mathbf{k})\sqrt{n_\uparrow n_\downarrow} & -\frac{k^2}{2m_\downarrow} - U_{\downarrow\downarrow}(\mathbf{k})n_\downarrow \end{pmatrix} \Longrightarrow E_{+,\mathbf{k}},\ E_{-,\mathbf{k}},\ -E_{+,\mathbf{k}},\ \text{and}\ -E_{-,\mathbf{k}}$

$\Downarrow$ no need to know eigenvectors!

$$\hat{H}_{\mathrm{Bog}} = \underbrace{\frac{1}{2}\sum_{\sigma\sigma'} U_{\sigma\sigma'}(0)\, n_\sigma n_{\sigma'}}_{\mathrm{MF} \propto g\, n^2} + \underbrace{\frac{1}{2}\sum_{\mathbf{k}}{}' \left[E_{+,\mathbf{k}} + E_{-,\mathbf{k}} - \frac{k^2}{2m_r} - U_{\uparrow\uparrow}(\mathbf{k})n_\uparrow - U_{\downarrow\downarrow}(\mathbf{k})n_\downarrow\right]}_{\mathrm{LHY} \propto (g\, n)^{5/2}} + \sum_{\pm,\mathbf{k}} E_{\pm,\mathbf{k}} \hat{b}^\dagger_{\pm,\mathbf{k}} \hat{b}_{\pm,\mathbf{k}}$$

Bogoliubov theory

$$\begin{pmatrix} \frac{k^2}{2m_\uparrow}+U_{\uparrow\uparrow}(\mathbf{k})n_\uparrow & U_{\uparrow\downarrow}(\mathbf{k})\sqrt{n_\uparrow n_\downarrow} & U_{\uparrow\uparrow}(\mathbf{k})n_\uparrow & U_{\uparrow\downarrow}(\mathbf{k})\sqrt{n_\uparrow n_\downarrow} \\ U_{\uparrow\downarrow}(\mathbf{k})\sqrt{n_\uparrow n_\downarrow} & \frac{k^2}{2m_\downarrow}+U_{\downarrow\downarrow}(\mathbf{k})n_\downarrow & U_{\uparrow\downarrow}(\mathbf{k})\sqrt{n_\uparrow n_\downarrow} & U_{\downarrow\downarrow}(\mathbf{k})n_\downarrow \\ -U_{\uparrow\uparrow}(\mathbf{k})n_\uparrow & -U_{\uparrow\downarrow}(\mathbf{k})\sqrt{n_\uparrow n_\downarrow} & -\frac{k^2}{2m_\uparrow}-U_{\uparrow\uparrow}(\mathbf{k})n_\uparrow & -U_{\uparrow\downarrow}(\mathbf{k})\sqrt{n_\uparrow n_\downarrow} \\ -U_{\uparrow\downarrow}(\mathbf{k})\sqrt{n_\uparrow n_\downarrow} & -U_{\downarrow\downarrow}(\mathbf{k})n_\downarrow & -U_{\uparrow\downarrow}(\mathbf{k})\sqrt{n_\uparrow n_\downarrow} & -\frac{k^2}{2m_\downarrow}-U_{\downarrow\downarrow}(\mathbf{k})n_\downarrow \end{pmatrix}$$

$$\Big\Downarrow \text{ Diag.}$$

$$E_{+,\mathbf{k}},\ E_{-,\mathbf{k}},\ -E_{+,\mathbf{k}},\ \text{and}\ -E_{-,\mathbf{k}}$$

$$E_{\pm,\mathbf{k}} = \sqrt{\frac{\omega_\uparrow^2(\mathbf{k})+\omega_\downarrow^2(\mathbf{k})}{2} \pm \sqrt{\frac{[\omega_\uparrow^2(\mathbf{k})-\omega_\downarrow^2(\mathbf{k})]^2}{4} + \frac{U_{\uparrow\downarrow}^2(\mathbf{k})n_\uparrow n_\downarrow k^4}{m_\uparrow m_\downarrow}}}$$

$$\omega_\sigma(\mathbf{k}) = \sqrt{U_{\sigma\sigma}(\mathbf{k})n_\sigma k^2/m_\sigma + (k^2/2m_\sigma)^2}$$

Bogoliubov spectra of individual components

Assume short-range potentials and the ultracold regime, i.e., typical *k* is much smaller than the range of $U_{\sigma\sigma'}(k)$ in momentum space. Can we replace $U_{\sigma\sigma'}(k) \to U_{\sigma\sigma'}(0)$?

Depends on the large-*k* behavior of this integral and on the space dimension!

$$\hat{H}_{\text{Bog}} = \frac{1}{2}\sum_{\sigma\sigma'} U_{\sigma\sigma'}(0)n_\sigma n_{\sigma'} + \frac{1}{2}\sum_{\mathbf{k}}{}' \left[E_{+,\mathbf{k}} + E_{-,\mathbf{k}} - \frac{k^2}{2m_r} - U_{\uparrow\uparrow}(\mathbf{k})n_\uparrow - U_{\downarrow\downarrow}(\mathbf{k})n_\downarrow\right] + \sum_{\pm,\mathbf{k}} E_{\pm,\mathbf{k}}\hat{b}_{\pm,\mathbf{k}}^\dagger \hat{b}_{\pm,\mathbf{k}}$$

Bogoliubov theory

$$\hat{H}_{\mathrm{Bog}} = \frac{1}{2}\sum_{\sigma\sigma'} U_{\sigma\sigma'}(0) n_\sigma n_{\sigma'} + \frac{1}{2}{\sum_{\mathbf{k}}}' \underbrace{\left[E_{+,\mathbf{k}} + E_{-,\mathbf{k}} - \frac{k^2}{2m_r} - U_{\uparrow\uparrow}(\mathbf{k}) n_\uparrow - U_{\downarrow\downarrow}(\mathbf{k}) n_\downarrow\right]}_{\text{large } k} + \sum_{\pm,\mathbf{k}} E_{\pm,\mathbf{k}} \hat{b}^\dagger_{\pm,\mathbf{k}} \hat{b}_{\pm,\mathbf{k}}$$

$$\int d^D k/(2\pi)^D$$

$$\approx -\frac{m_\uparrow U^2_{\uparrow\uparrow}(\mathbf{k}) n_\uparrow^2 + m_\downarrow U^2_{\downarrow\downarrow}(\mathbf{k}) n_\downarrow^2 + 4\mu_{\uparrow\downarrow} U^2_{\uparrow\downarrow}(\mathbf{k}) n_\uparrow n_\downarrow}{k^2}$$

$$\int \frac{d^D k}{2(2\pi)^D} \frac{m_\uparrow U^2_{\uparrow\uparrow}(0) n_\uparrow^2 + m_\downarrow U^2_{\downarrow\downarrow}(0) n_\downarrow^2 + 4\mu_{\uparrow\downarrow} U^2_{\uparrow\downarrow}(0) n_\uparrow n_\downarrow}{k^2}$$

1D: converges, straightforward replacement, delta potential is well behaving in 1D

2D: logarithmic divergence, handle by introducing a momentum cutoff

3D: diverges, “easy” to handle (as it converges at low *k*)

Renormalization in 3D

$$\hat{H}_{\mathrm{Bog}} = \frac{1}{2}\sum_{\sigma\sigma'} U_{\sigma\sigma'}(0) n_\sigma n_{\sigma'} + \frac{1}{2}{\sum_{\mathbf{k}}}' \left[E_{+,\mathbf{k}} + E_{-,\mathbf{k}} - \frac{k^2}{2m_r} - U_{\uparrow\uparrow}(\mathbf{k}) n_\uparrow - U_{\downarrow\downarrow}(\mathbf{k}) n_\downarrow \right] + \sum_{\pm,\mathbf{k}} E_{\pm,\mathbf{k}} \hat{b}^\dagger_{\pm,\mathbf{k}} \hat{b}_{\pm,\mathbf{k}}$$

subtract from here

add here

$$\frac{m_\uparrow U^2_{\uparrow\uparrow}(\mathbf{k}) n^2_\uparrow + m_\downarrow U^2_{\downarrow\downarrow}(\mathbf{k}) n^2_\downarrow + 4\mu_{\uparrow\downarrow} U^2_{\uparrow\downarrow}(\mathbf{k}) n_\uparrow n_\downarrow}{k^2}$$

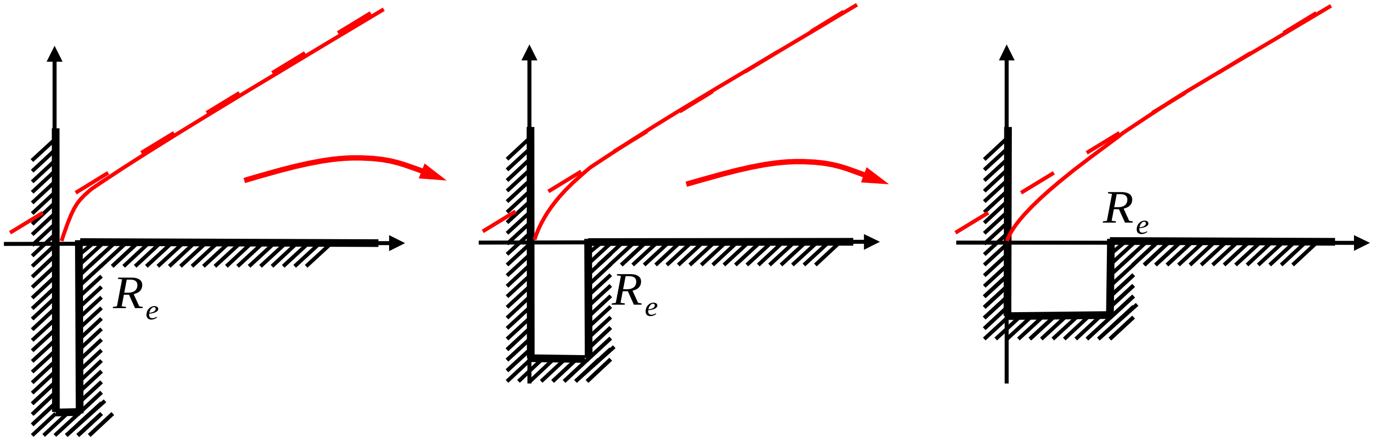
$$E_{\mathrm{MF}} = \frac{1}{2}\sum_{\sigma\sigma'} n_\sigma n_{\sigma'} \underbrace{\left[U_{\sigma\sigma'}(0) - {\sum_{\mathbf{k}}}' \frac{2\mu_{\sigma\sigma'} U^2_{\sigma\sigma'}(\mathbf{k})}{k^2} \right]}_{\text{Interaction shift per pair} = \; g_{\sigma\sigma'} = 2\pi a_{\sigma\sigma'}/\mu_{\sigma\sigma'}}$$

This perturbation theory cannot handle zero-range interactions

$U_{\sigma\sigma'}(\mathbf{k})$ is *effective potential*

Characterized by the desired low-energy scattering properties but suitable for perturbative expansion

$$|U_{\sigma\sigma'}| \ll 1/(\mu_{\sigma\sigma'}\kappa)$$



Interaction range in momentum space ~1/R_e

Renormalization in 3D

$$\hat{H}_{\mathrm{Bog}} = \frac{1}{2}\sum_{\sigma\sigma'} U_{\sigma\sigma'}(0) n_\sigma n_{\sigma'} + \frac{1}{2}\sum_{\mathbf{k}}{}' \left[E_{+,\mathbf{k}} + E_{-,\mathbf{k}} - \frac{k^2}{2m_r} - U_{\uparrow\uparrow}(\mathbf{k}) n_\uparrow - U_{\downarrow\downarrow}(\mathbf{k}) n_\downarrow\right] + \sum_{\pm,\mathbf{k}} E_{\pm,\mathbf{k}} \hat{b}^\dagger_{\pm,\mathbf{k}} \hat{b}_{\pm,\mathbf{k}}$$

subtract from here

add here

$$\frac{m_\uparrow U^2_{\uparrow\uparrow}(\mathbf{k}) n^2_\uparrow + m_\downarrow U^2_{\downarrow\downarrow}(\mathbf{k}) n^2_\downarrow + 4\mu_{\uparrow\downarrow} U^2_{\uparrow\downarrow}(\mathbf{k}) n_\uparrow n_\downarrow}{k^2}$$

⇓

$$E_{\mathrm{LHY}} = \frac{1}{2}\sum_{\mathbf{k}}{}' \left[E_{+,\mathbf{k}} + E_{-,\mathbf{k}} - \frac{k^2}{2\mu_{\uparrow\downarrow}} - U_{\uparrow\uparrow}(\mathbf{k}) n_\uparrow - U_{\downarrow\downarrow}(\mathbf{k}) n_\downarrow + \frac{m_\uparrow U^2_{\uparrow\uparrow}(\mathbf{k}) n^2_\uparrow + m_\downarrow U^2_{\downarrow\downarrow}(\mathbf{k}) n^2_\downarrow + 4\mu_{\uparrow\downarrow} U^2_{\uparrow\downarrow}(\mathbf{k}) n_\uparrow n_\downarrow}{k^2}\right]$$

Integral converges at ~healing momentum << κ ⇒ $U_{\sigma\sigma'}(k) \to U_{\sigma\sigma'}(0) \approx g_{\sigma\sigma'}$

$$E^{(3D)}_{\mathrm{LHY}} = \frac{8}{15\pi^2} m_\uparrow^{3/2} (g_{\uparrow\uparrow} n_\uparrow)^{5/2} f^{(3D)}\left(\frac{m_\downarrow}{m_\uparrow}, \frac{g^2_{\uparrow\downarrow}}{g_{\uparrow\uparrow} g_{\downarrow\downarrow}}, \frac{g_{\downarrow\downarrow} n_\downarrow}{g_{\uparrow\uparrow} n_\uparrow}\right)$$ where

$$f^{(3D)}(z,u,x) = \frac{15}{32}\int_0^\infty \left[\frac{1}{\sqrt{2}} \sum_{\pm} \sqrt{k^2 + \frac{xk^2}{z} + \frac{k^4}{4} + \frac{k^4}{4z^2} \pm \sqrt{\left(k^2 - \frac{xk^2}{z} + \frac{k^4}{4} - \frac{k^4}{4z^2}\right)^2 + \frac{4xuk^4}{z}}} - \frac{1+z}{2z} k^2 - 1 - x + \left(1 + x^2 z + \frac{4xzu}{1+z}\right)\frac{1}{k^2}\right] k^2 dk$$

Renormalization in 3D

$$E_{\mathrm{LHY}}^{(3D)} = \frac{8}{15\pi^2} m_\uparrow^{3/2} (g_{\uparrow\uparrow} n_\uparrow)^{5/2} f^{(3D)}\left(\frac{m_\downarrow}{m_\uparrow}, \frac{g_{\uparrow\downarrow}^2}{g_{\uparrow\uparrow}g_{\downarrow\downarrow}}, \frac{g_{\downarrow\downarrow}n_\downarrow}{g_{\uparrow\uparrow}n_\uparrow}\right)$$

where

$$f^{(3D)}(z,u,x) = \frac{15}{32}\int_0^\infty \left[\frac{1}{\sqrt{2}}\sum_{\pm}\sqrt{k^2+\frac{xk^2}{z}+\frac{k^4}{4}+\frac{k^4}{4z^2}\pm\sqrt{\left(k^2-\frac{xk^2}{z}+\frac{k^4}{4}-\frac{k^4}{4z^2}\right)^2+\frac{4xuk^4}{z}}}\right.$$
$$\left. - \frac{1+z}{2z}k^2 - 1 - x + \left(1 + x^2 z + \frac{4xzu}{1+z}\right)\frac{1}{k^2}\right] k^2 dk$$

⬇

Change of variable *k->t* $k^2 = \frac{4\sqrt{xuz^3}}{z^2-1}\left[t - \frac{1}{t} + \frac{x-z}{\sqrt{xuz}}\right] = \frac{4\sqrt{xuz^3}}{z^2-1}\frac{(t-b_1)(t-b_2)}{t}$ removes internal square root

⬇

$f^{(3D)}(z,u,x)$ is a combination of elementary and elliptic functions

For $g_{\uparrow\downarrow} = \pm\sqrt{g_{\uparrow\uparrow}g_{\downarrow\downarrow}}$ we obtain

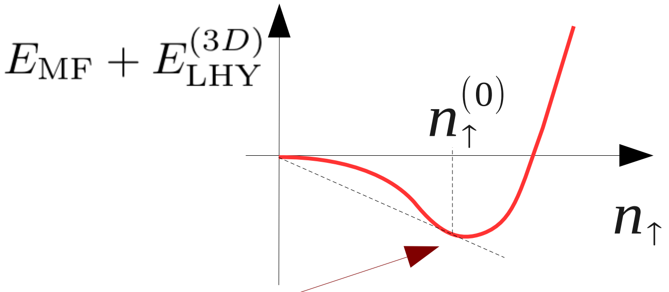
$$\begin{aligned} f^{(3D)}(z,1,x) = & \ (-2-7xz+2z^2+x^2z^2)\frac{\sqrt{x+z}}{2\sqrt{z}(z^2-1)} \\ & +(-2-7xz+3z^2+3x^2z^2-7xz^3-2x^2z^4)\frac{E[\arcsin(1/z)|-xz]-E(-xz)}{2(z^2-1)^{3/2}} \\ & +(2+8xz-3z^2+6x^2z^2-2xz^3+x^2z^4)\frac{F[\arcsin(1/z)|-xz]-K(-xz)]}{2(z^2-1)^{3/2}}. \end{aligned}$$

3D liquid properties

$$E_{\mathrm{MF}} + E^{(3D)}_{\mathrm{LHY}} = \frac{1}{2}\sum_{\sigma\sigma'} g_{\sigma\sigma'} n_\sigma n_{\sigma'} + \frac{8}{15\pi^2} m_\uparrow^{3/2} (g_{\uparrow\uparrow} n_\uparrow)^{5/2} f^{(3D)}\left(\frac{m_\downarrow}{m_\uparrow}, \frac{g^2_{\uparrow\downarrow}}{g_{\uparrow\uparrow} g_{\downarrow\downarrow}}, \frac{g_{\downarrow\downarrow} n_\downarrow}{g_{\uparrow\uparrow} n_\uparrow}\right)$$

Assuming $\delta g = g_{\uparrow\downarrow} + \sqrt{g_{\uparrow\uparrow} g_{\downarrow\downarrow}} < 0$ and $|\delta g| \ll \sqrt{g_{\uparrow\uparrow} g_{\downarrow\downarrow}}$ and $n_\uparrow / n_\downarrow = \sqrt{g_{\downarrow\downarrow}/g_{\uparrow\uparrow}}$

$$E_{\mathrm{MF}} + E^{(3D)}_{\mathrm{LHY}} \approx \delta g \sqrt{\frac{g_{\uparrow\uparrow}}{g_{\downarrow\downarrow}}} n_\uparrow^2 + \frac{8}{15\pi^2} m_\uparrow^{3/2} (g_{\uparrow\uparrow} n_\uparrow)^{5/2} f^{(3D)}\left(\frac{m_\downarrow}{m_\uparrow}, 1, \sqrt{\frac{g_{\downarrow\downarrow}}{g_{\uparrow\uparrow}}}\right)$$



$$n_\uparrow^{(0)} = \frac{25\pi}{1024} \frac{1}{f^2(m_\downarrow/m_\uparrow, 1, \sqrt{g_{\downarrow\downarrow}/g_{\uparrow\uparrow}})} \frac{1}{a^3_{\uparrow\uparrow}} \frac{\delta g^2}{g_{\uparrow\uparrow} g_{\downarrow\downarrow}}$$

Dilute liquid! $\sqrt{n a^3} \sim |\delta g| / g \ll 1$

Summary

Instability in one degree of freedom (density) can be prevented by quantum fluctuations in other degree(s) of freedom (spin)

Quantum droplets: BMF physics is essential in spite of the weakly interacting regime, dilute liquid phase with controllable parameters

LHY (nonanalytic) scaling of the stabilizing energy $\sim n^{5/2}$

Next lecture: other density scalings of the BMF term!

Next lecture: power of the Bogoliubov method

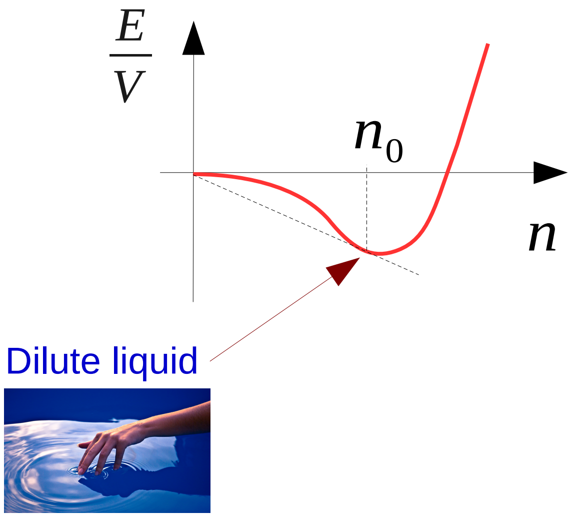
Lecture 2

Nonanalytic vs analytic beyond mean field

&

Three-body force

Dilute liquid phase



$$E = E_{\mathrm{MF}} + E_{\mathrm{BMF}}$$

$$E_{\mathrm{MF}}/V = g_2 n^2/2 < 0 \qquad E_{\mathrm{BMF}}/V \propto n^{\alpha \neq 2} > 0$$

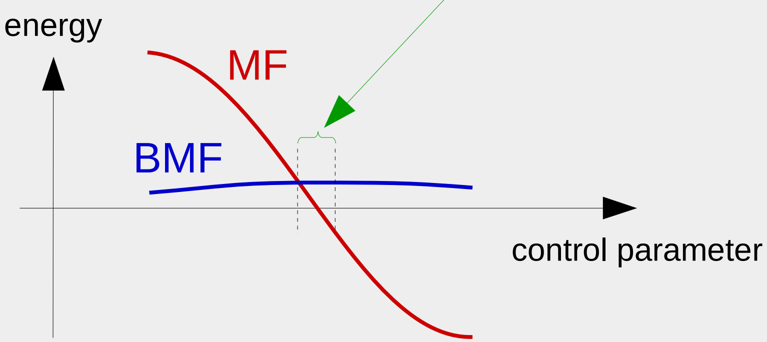
Need attractive mean-field (MF) and repulsive beyond-mean-field (BMF) terms

$\alpha = 3$ – “3-body” mechanism (Bulgac'02)

$\alpha = 5/2$ – “LHY” mechanism (DP'15)

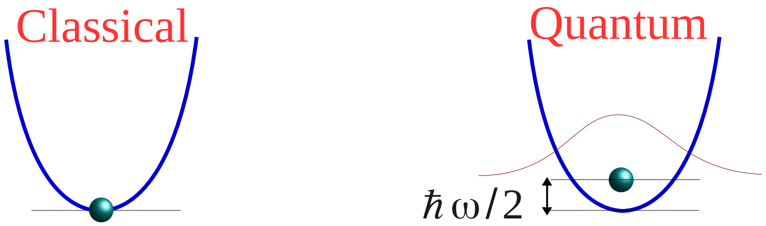
In the *dilute* regime

MF and BMF compete if MF is fine-tuned!



Interest in higher-order interactions near 2-body zero crossing

Lee-Huang-Yang (LHY) correction

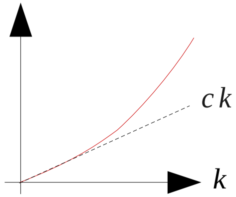


MF

LHY/BMF

$$\frac{E}{\text{Volume}}=\frac{g_2 n^2}{2}+\frac{1}{2}\sum_k\left[E(k)-k^2/2-c^2\right]=\frac{g_2 n^2}{2}+\frac{8}{15\pi^2}\overset{\propto n^{5/2}}{c^5}=\frac{g_2 n^2}{2}\left(1+\frac{128}{15}\sqrt{\frac{na^3}{\pi}}\right)$$

$$E(k)=\sqrt{c^2k^2+k^4/4}$$
$$c^2=g_2 n$$



$$g_2=4\pi a$$

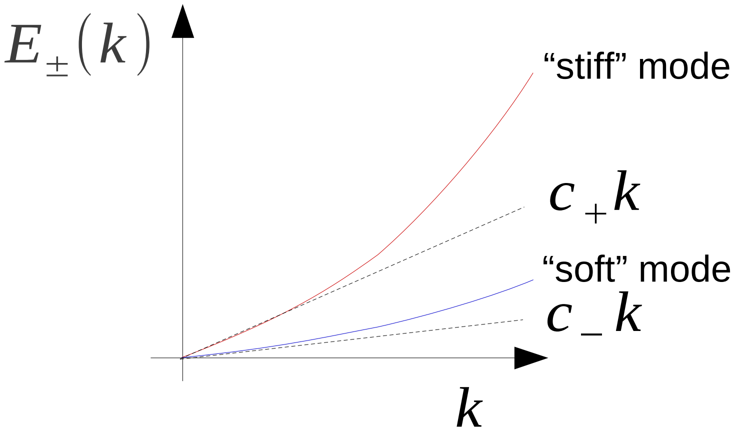
(Lee,Huang,&Yang'57)

LHY correction is QUANTUM (~ zero-point energy of Bogoliubov vacuum) and UNIVERSAL (depends only on the scattering length)

… but, unfortunately, VANISHES at the 2-body zero crossing!

Problem solved for systems with more control parameters

Two-component mixture $(a \to a_{11}, a_{22}, a_{12})$

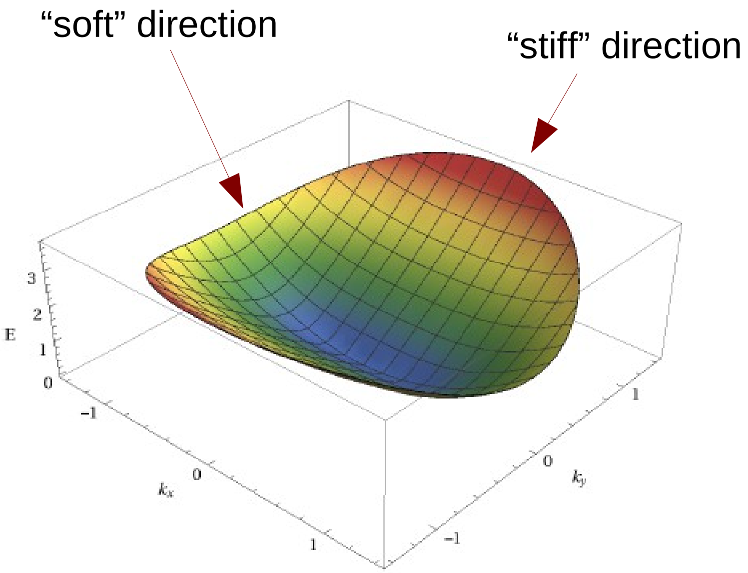


$$\frac{E}{\text{Volume}} = \frac{g_{11} n_1^2 + g_{22} n_2^2 + 2 g_{12} n_1 n_2}{2} + \frac{8}{15\pi^2} \overset{\propto n^{5/2}}{\left(c_+^5 + c_-^5\right)}$$

(Larsen'63)

Dipolar length

Dipolar gas ($a \to a, r_*$)



$$\frac{E_{\text{LHY}}}{\text{Volume}} = \frac{8}{15\pi^2} \frac{m^4}{\hbar^3} \overset{\propto n^{5/2}}{\langle c^5(\hat{k}) \rangle_{\hat{k}}}$$

(Lima&Pelster'11)

Independent control over MF and BMF terms

Low-dimensional examples

(DP, Astrakharchik'16)

3D:

$$\frac{E_{3D}}{\text{Volume}}=\frac{1}{2}\sum_{\sigma\sigma'} g_{\sigma\sigma'} n_\sigma n_{\sigma'}+\frac{8}{15\pi^2}\sum_{\pm} c_\pm^5 \sim \delta g n^2+(gn)^{5/2}$$

$$\sqrt{n g^3}\ll 1$$

2D:

$$\frac{E_{2D}}{\text{Surface}}=\frac{1}{2}\sum_{\sigma\sigma'} g_{\sigma\sigma'} n_\sigma n_{\sigma'}+\frac{1}{8\pi}\sum_{\pm} c_\pm^4 \ln\frac{c_\pm^2\sqrt{e}}{\kappa^2} \sim g^2 n^2 \ln\frac{n}{n_0}$$

$$g_{\sigma\sigma'}=2\pi/\ln\left(2e^{-\gamma}/a_{\sigma\sigma'}\kappa\right)\ll 1$$

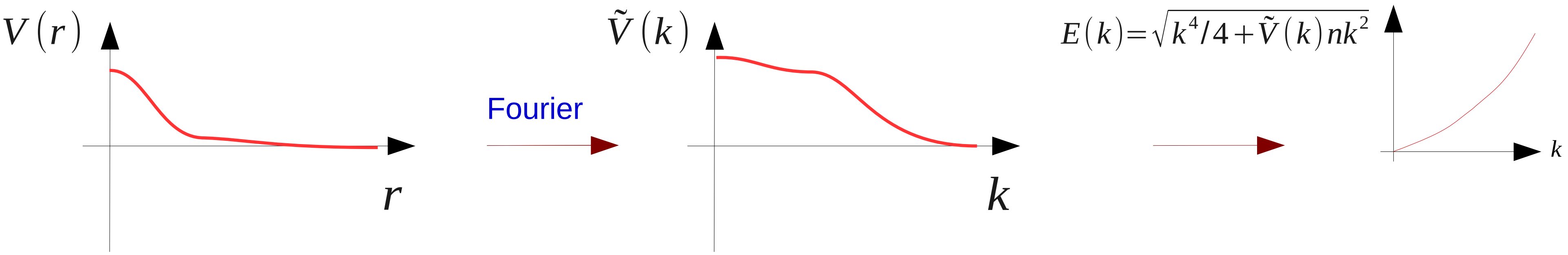
1D:

$$\frac{E_{1D}}{\text{Length}}=\frac{1}{2}\sum_{\sigma\sigma'} g_{\sigma\sigma'} n_\sigma n_{\sigma'}-\frac{2}{3\pi}\sum_{\pm} c_\pm^3 \sim \delta g n^2-(gn)^{3/2}$$

!

$$\sqrt{g/n}\ll 1$$

Where does nonanalyticity come from?



$$\frac{E}{\text{Volume}}=\frac{\tilde{V}(0)n^2}{2}+\frac{1}{2}\sum_k\left[\sqrt{k^4/4+\tilde{V}(k)nk^2}-k^2/2-\tilde{V}(k)n\right]$$

$$=\frac{\tilde{V}(0)n^2}{2}+\sum_k\left(-\frac{n^2\tilde{V}^2(k)}{2\mathrm{k}^2}+\frac{n^3\tilde{V}^3(k)}{k^4}+\ldots\right)$$

Renormalization of two-body interaction

Effective three-body force?

Analytic? $\sim n^3$

Not really. Integral is infrared divergent. Bogoliubov theory cuts it off at $k\sim\sqrt{\tilde{V}(0)n}$ making it nonanalytic

Analytic BMF (3-body):

1) if $\tilde{V}(0)=0$

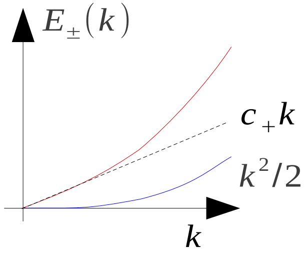
2) if the spectrum is gapped (different Hamiltonian)

Driven mixture (Cappellaro et al'17, Lavoine et al'21)

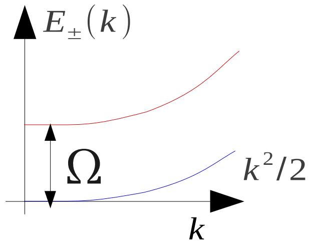
$$\hat{H}=\sum_{\sigma,\mathbf{k}}\frac{k^2}{2}\hat{a}^\dagger_{\sigma,\mathbf{k}}\hat{a}_{\sigma,\mathbf{k}}+\frac{1}{2}\sum_{\sigma,\sigma',\mathbf{k}_1,\mathbf{k}_2,\mathbf{q}}\hat{a}^\dagger_{\sigma,\mathbf{k}_1+\mathbf{q}}\hat{a}^\dagger_{\sigma',\mathbf{k}_2-\mathbf{q}}U_{\sigma\sigma'}(\mathbf{q})\hat{a}_{\sigma,\mathbf{k}_1}\hat{a}_{\sigma',\mathbf{k}_2}-\frac{\Omega}{2}(\hat{a}^\dagger_{\uparrow,\mathbf{k}}\hat{a}_{\downarrow,\mathbf{k}}+\hat{a}^\dagger_{\downarrow,\mathbf{k}}\hat{a}_{\uparrow,\mathbf{k}})$$

RF Rabi frequency,
or tunneling
amplitude (bilayer,
bitube geometry)

$\Omega=0$



finite Ω



$\sqrt{\Omega}$ is a new momentum scale which can cut off the infrared divergence instead of the healing momentum

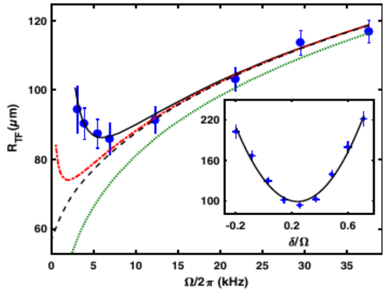
$$E_{\mathrm{LHY}}=\frac{8}{15\pi^2}c_+^5\propto n^{5/2}$$

$$E_{\mathrm{LHY}}\overset{gn\ll\Omega}{\approx}\frac{\sqrt{\Omega}g^2}{2\sqrt{2}\pi}\frac{n^2}{2}+\frac{3g^3}{4\sqrt{2}\pi\sqrt{\Omega}}\frac{n^3}{3!}+\dots$$

2-body renorm.

Effective three-body force
third order in the interaction

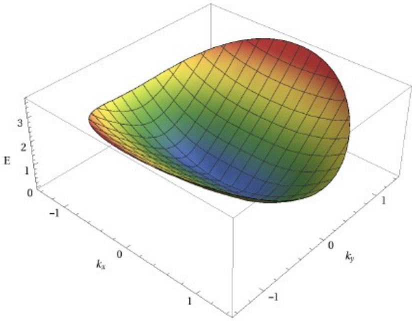
Agreement with 3-body analysis (DP'14)

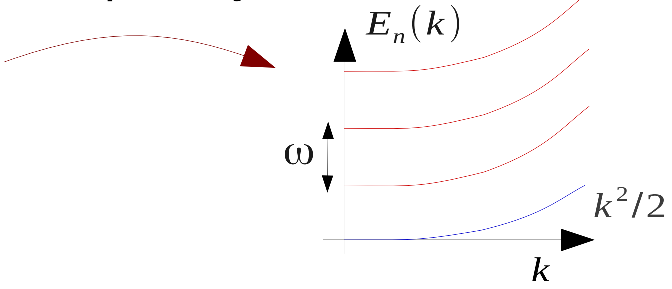


(Lavoine et al'21)

Dimensional crossover Edler et al'17, Zin et al'18&'19, Ilg et al'18

quasi-low-D
confinement with
frequency ω





Appearance of effective three-body terms
also in the third order

Effective 3-body force (2nd order)

Higher-order interactions in the Bose-Hubbard model
(Li et al.'06, Tiesinga et al.'09,11, Hazzard&Mueller'10)

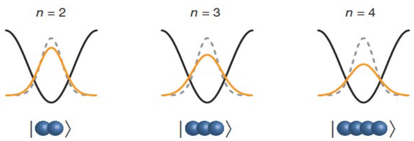
→ 3-body term – second order in interaction and attractive

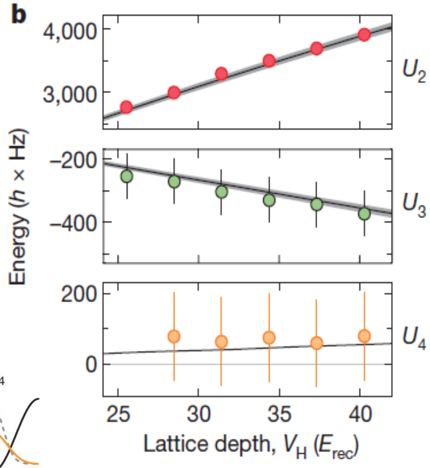
Time-resolved observation of coherent multi-body interactions in quantum phase revivals

Sebastian Will[1,2], Thorsten Best[1], Ulrich Schneider[1,2], Lucia Hackermüller[1], Dirk-Sören Lühmann[3] & Immanuel Bloch[1,2,4]

$$E(N)=U_2\frac{N(N-1)}{2!}+U_3\frac{N(N-1)(N-2)}{3!}+U_4\frac{N(N-1)(N-2)(N-3)}{4!}+\ldots$$

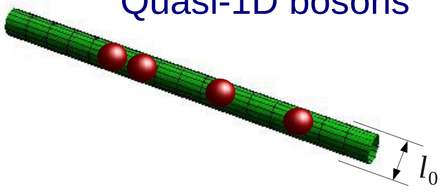
$$g\sim a\,\omega^{3/2}\gg g^2/\omega\gg g^3/\omega^2$$





Another example:

Quasi-1D bosons →



$$H_{\text{eff 1D}}=-\frac{1}{2}\sum_{i=1}^{N}\frac{\partial^2}{\partial x_i^2}+g_2\sum_{i<j}\delta(x_i-x_j)+g_3\sum_{i<j<k}\delta(x_i-x_j)\delta(x_j-x_k)$$

$$g_2=\frac{2a}{l_0}\left(1+\frac{Ca}{\sqrt{2}\,l_0}+\ldots\right)$$

(Olshanii'98)

$$g_3=-12\log\left(\frac{4}{3}\right)\frac{a^2}{l_0^2}$$

(Muryshev et al'02)

Three-body force:

Why interesting?

Is it second or third order?

Can Bogoliubov theory correctly handle it?

Why interesting?

Bosons + $g_2<0$ → Collapse

Bosons + $g_2<0$ + $g_3>0$ → Free space → self-trapped droplet state Bulgac'02:

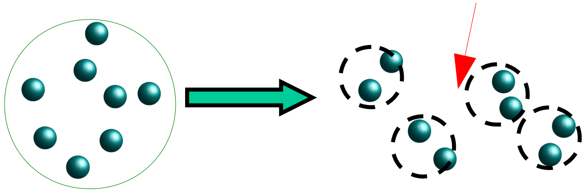
Neglecting surface tension, flat density profile $n=3|g_2|/2g_3$

Including surface tension → surface modes

Increasing $g_2<0$ → bosonic pairing Nozieres&Saint James'82

Topological transition, not crossover!

Radzihovsky et al., Romans et al., Lee&Lee'04



pairs repel because $g_3>0$

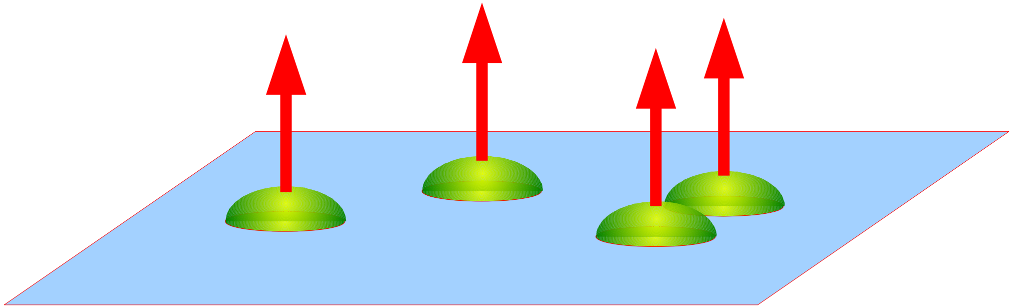
Pairing on a lattice with three-body constraint:

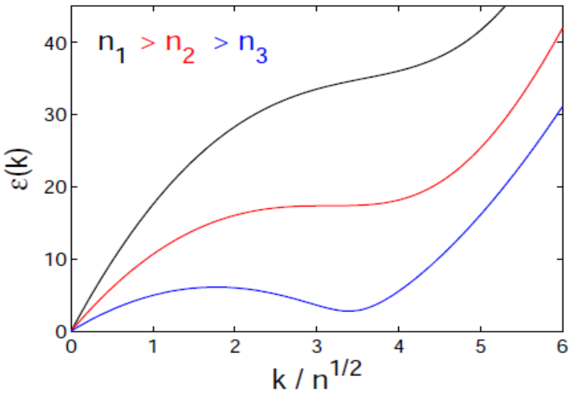
Daley et al.'09-, Ng&Yang'11, Bonnes&Wessel'12,...

g_3 is necessary! = Pauli pressure in the BCS-BEC crossover!

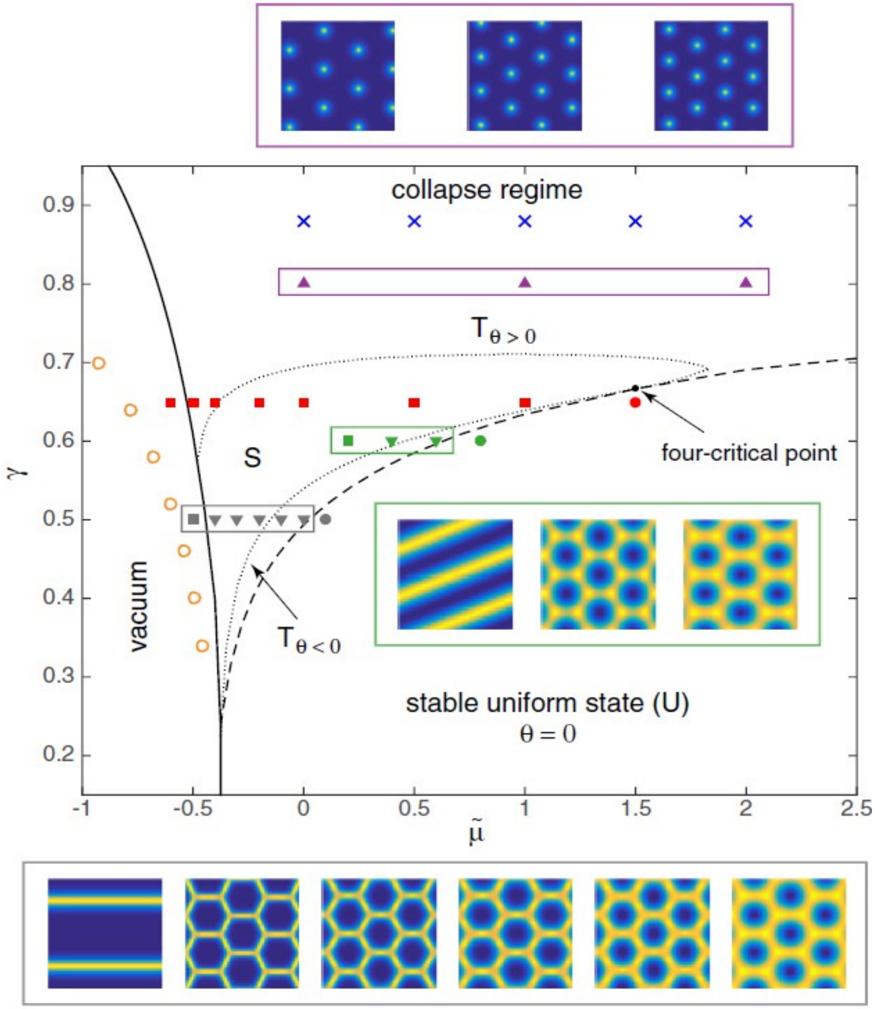
Why interesting?

2D dipoles





Phase diagram



Rotonized superfluid & supersolid

Mechanical stability for $g_3 > 0$

Lu et al'15

Dimensions of X-dimensional coupling constants

X-dimensional 2-body scattering → $g_2 \propto \left(\hbar^2/m\right)\times length^{X-2}$

⇓

X-dimensional 3-body scattering = (2X)-dimensional 2-body scattering

⇓

X-dimensional 3-body scattering → $g_3 \propto \left(\hbar^2/m\right)\times length^{2X-2}$

In particular,

3D: $g_3 \propto \left(\hbar^2/m\right)\times length^{4}$

2D: $g_3 \propto \left(\hbar^2/m\right)\times length^{2}$

1D: $g_3 \propto \left(\hbar^2/m\right)/\ln\left(k\times length\right)=$ small parameter

Dimensions of X-dimensional coupling constants

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⇓

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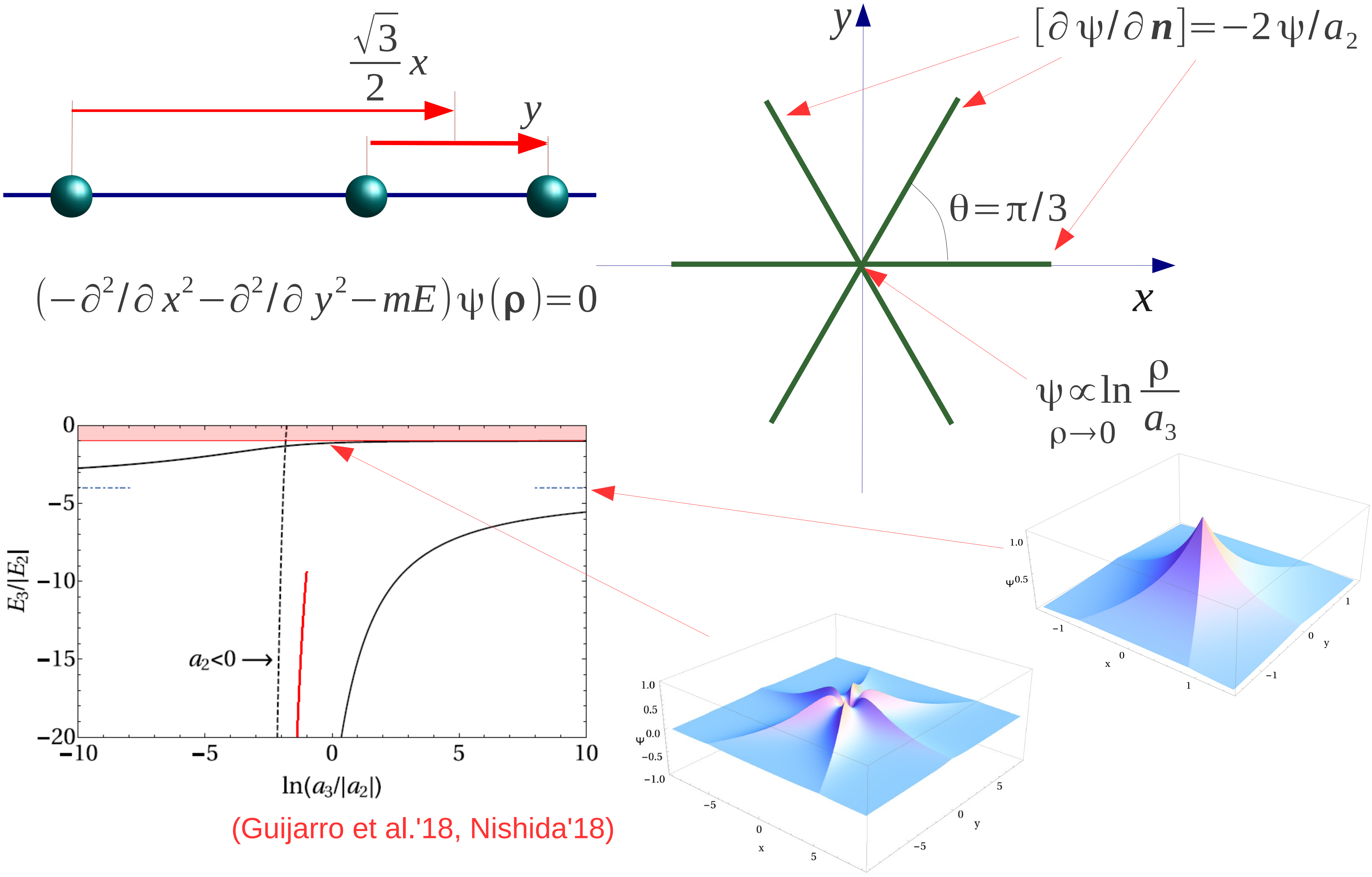
In particular,

3D: $g_3 \propto \left(\hbar^2/m\right) \times length^4$

2D: $g_3 \propto \left(\hbar^2/m\right) \times length^2$

1D: $g_3 \propto \left(\hbar^2/m\right) / \ln\left(k \times length\right) =$ small parameter

Three-body problem with a_2 and a_3



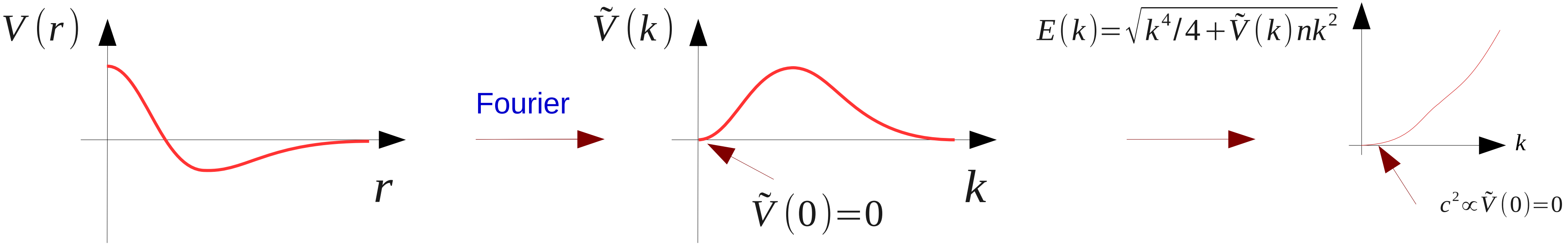
$$(-\partial^2/\partial x^2-\partial^2/\partial y^2-mE)\psi(\boldsymbol{\rho})=0$$

$$[\partial\psi/\partial\boldsymbol{n}]=-2\psi/a_2$$

$$\psi\underset{\rho\to0}{\propto}\ln\frac{\rho}{a_3}$$

(Guijarro et al.'18, Nishida'18)

2-body interaction of zero mean



$$\frac{E}{\text{Volume}}=\frac{\tilde V(0)n^2}{2}+\frac{1}{2}\sum_k[\sqrt{k^4/4+\tilde V(k)nk^2}-k^2/2-\tilde V(k)n]=\frac{\tilde V(0)n^2}{2}+\sum_k\left(-\frac{n^2\tilde V^2(k)}{2\mathrm{k}^2}+\frac{n^3\tilde V^3(k)}{k^4}+...\right)$$

Renormalization of two-body interaction

Effective three-body force
third order in the interaction

``3-body" 3^{rd} order scaling

Closed-form expression

$$g_3^{(3)}=6\sum_k\frac{\tilde V^3(k)}{k^4}$$

Converges since $\tilde V(0)=0$

Where could the 2^{nd} order 3-body term come from?

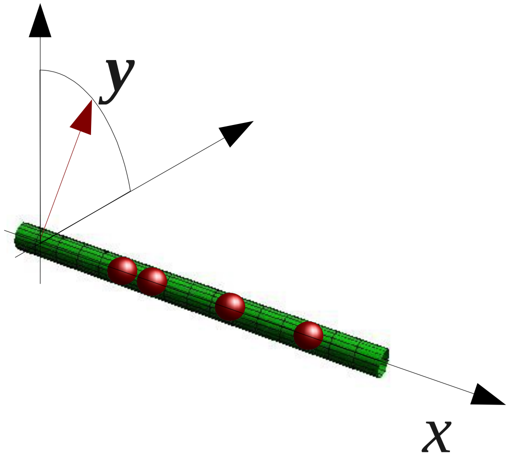
Not sure if Bogoliubov handles it correctly

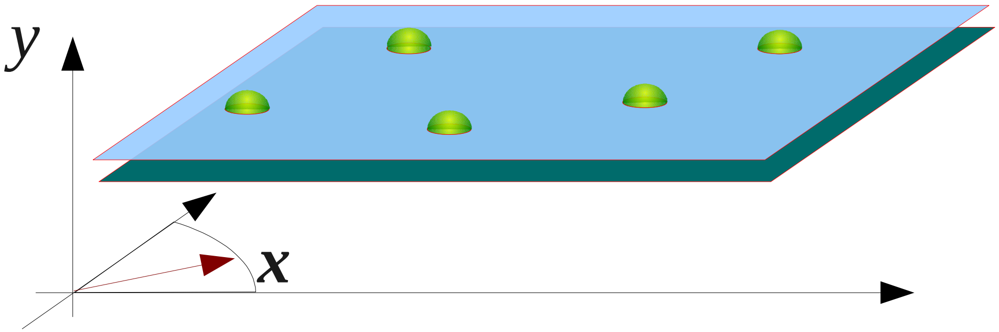
Model

Hamiltonian: $\hat{H} = \sum_{i=1}^{N} -\partial^2_{\mathbf{x}_i}/2 - \partial^2_{\mathbf{y}_i}/2 + U(\mathbf{y}_i) + \sum_{i>j} V(\mathbf{x}_i - \mathbf{x}_j, \mathbf{y}_i - \mathbf{y}_j)$

Quasi-low-D confinement

Interaction

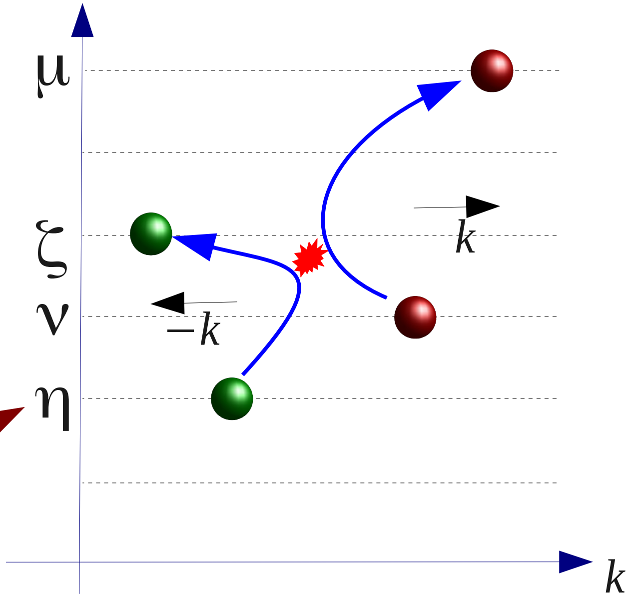




Interaction matrix elements [assume $V(\boldsymbol{r}) = V(-\boldsymbol{r})$]

$$V^{\zeta\eta}_{\boldsymbol{\mu\nu}}(\mathbf{k}) = [V^{\boldsymbol{\eta\zeta}}_{\boldsymbol{\nu\mu}}(-\boldsymbol{k})]^* = V^{\boldsymbol{\mu\nu}}_{\boldsymbol{\zeta\eta}}(-\boldsymbol{k})$$

$$= \int d\mathbf{y} d\mathbf{y}' d\mathbf{x} e^{i\mathbf{kx}} V(\mathbf{x}, \mathbf{y} - \mathbf{y}')\psi^*_{\boldsymbol{\zeta}}(\mathbf{y})\psi^*_{\boldsymbol{\mu}}(\mathbf{y}')\psi_{\boldsymbol{\eta}}(\mathbf{y})\psi_{\boldsymbol{\nu}}(\mathbf{y}')$$



single-particle states in *U*(*y*)

1st quantization vs Bogoliubov perturbation theory

$$\hat{H} = \sum_{i=1}^{N} -\partial^2_{\mathbf{x}_i}/2 - \partial^2_{\mathbf{y}_i}/2 + U(\mathbf{y}_i) + \sum_{i>j} V(\mathbf{x}_i - \mathbf{x}_j, \mathbf{y}_i - \mathbf{y}_j)$$

Landau & Lifshitz

$$E^{(1)}_{\bar{n}} = \bar{V}_{\bar{n}\bar{n}},$$

$$E^{(2)}_{\bar{n}} = -{\sum_{\bar{m}}}' |\bar{V}_{\bar{m}\bar{n}}|^2/\omega_{\bar{m}\bar{n}},$$

$$E^{(3)}_{\bar{n}} = {\sum_{\bar{k}}}'{\sum_{\bar{m}}}' \frac{\bar{V}_{\bar{n}\bar{m}}\bar{V}_{\bar{m}\bar{k}}\bar{V}_{\bar{k}\bar{n}}}{\omega_{\bar{m}\bar{n}}\omega_{\bar{k}\bar{n}}} - E^{(1)}_{\bar{n}}{\sum_{\bar{m}}}' \frac{|\bar{V}_{\bar{m}\bar{n}}|^2}{\omega^2_{\bar{m}\bar{n}}}$$

$$E[N] = E^{(1)}[N] + E^{(2)}[N] + E^{(3)}[N] + \ldots$$

$$g_2^{(1)}\binom{N}{2} = V^{\mathbf{00}}_{\mathbf{00}}(\mathbf{0})\binom{N}{2}$$

MF

$$g_2^{(2)}\binom{N}{2} + g_3^{(2)}\binom{N}{3}$$

$$g_2^{(2)} = -\sum_{\mathbf{k},\boldsymbol{\nu},\boldsymbol{\mu}} \frac{|V^{\mathbf{0}\boldsymbol{\nu}}_{\mathbf{0}\boldsymbol{\mu}}(\mathbf{k})|^2}{k^2 + \epsilon_{\boldsymbol{\nu}} + \epsilon_{\boldsymbol{\mu}}}$$

2-body renormalization

$$g_3^{(2)} = -6\sum_{\boldsymbol{\nu}} \frac{|V^{\mathbf{00}}_{\mathbf{0}\boldsymbol{\nu}}(\mathbf{0})|^2}{\epsilon_{\boldsymbol{\nu}}}$$

3-body 2nd order attraction

Vanishes in pure dimensions!

$$\hat{H} = \sum_{\boldsymbol{q},\boldsymbol{\nu}}(q^2/2 + \epsilon_{\boldsymbol{\nu}})\hat{a}^{\dagger}_{\boldsymbol{q},\boldsymbol{\nu}}\hat{a}_{\boldsymbol{q},\boldsymbol{\nu}} + \frac{1}{2}\sum_{\boldsymbol{q}_1,\boldsymbol{q}_2,\boldsymbol{k},\boldsymbol{\nu},\boldsymbol{\mu},\boldsymbol{\eta},\boldsymbol{\zeta}} V^{\boldsymbol{\zeta\eta}}_{\boldsymbol{\mu\nu}}(\mathbf{k})\hat{a}^{\dagger}_{\boldsymbol{q}_2+\boldsymbol{k},\boldsymbol{\mu}}\hat{a}^{\dagger}_{\boldsymbol{q}_1-\boldsymbol{k},\boldsymbol{\zeta}}\hat{a}_{\boldsymbol{q}_2,\boldsymbol{\nu}}\hat{a}_{\boldsymbol{q}_1,\boldsymbol{\eta}}$$

$$H_0 = V^{\mathbf{00}}_{\mathbf{00}}(\mathbf{0})n_0^2/2,$$

$$\hat{H}_{sp} = \sum_{\boldsymbol{q},\boldsymbol{\nu}}(q^2/2 + \epsilon_{\boldsymbol{\nu}})\hat{a}^{\dagger}_{\boldsymbol{q},\boldsymbol{\nu}}\hat{a}_{\boldsymbol{q},\boldsymbol{\nu}},$$

$$\hat{H}_1 = n_0^{3/2}{\sum_{\boldsymbol{\nu}}}' V^{\mathbf{00}}_{\boldsymbol{\nu}\mathbf{0}}(\mathbf{0})\hat{a}^{\dagger}_{\mathbf{0},\boldsymbol{\nu}} + V^{\mathbf{00}}_{\mathbf{0}\boldsymbol{\nu}}(\mathbf{0})\hat{a}_{\mathbf{0},\boldsymbol{\nu}},$$

$$\hat{H}_2 = \frac{n_0}{2}{\sum_{\boldsymbol{\nu},\boldsymbol{\mu},\boldsymbol{k}}}' V^{\boldsymbol{\nu}\mathbf{0}}_{\boldsymbol{\mu}\mathbf{0}}(\mathbf{k})\hat{a}^{\dagger}_{\boldsymbol{k},\boldsymbol{\mu}}\hat{a}^{\dagger}_{-\boldsymbol{k},\boldsymbol{\nu}} + V^{\mathbf{0}\boldsymbol{\nu}}_{\mathbf{0}\boldsymbol{\mu}}(\mathbf{k})\hat{a}_{-\boldsymbol{k},\boldsymbol{\mu}}\hat{a}_{\boldsymbol{k},\boldsymbol{\nu}} + 2[V^{\mathbf{00}}_{\boldsymbol{\mu\nu}}(\mathbf{0}) + V^{\mathbf{0}\boldsymbol{\nu}}_{\boldsymbol{\mu}\mathbf{0}}(\mathbf{k})]\hat{a}^{\dagger}_{\boldsymbol{k},\boldsymbol{\mu}}\hat{a}_{\boldsymbol{k},\boldsymbol{\nu}},$$

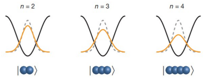
Bogoliubov

$$\hat{H}_3 = \sqrt{n_0}{\sum_{\boldsymbol{q},\boldsymbol{k},\boldsymbol{\nu},\boldsymbol{\mu},\boldsymbol{\eta}}}' V^{\boldsymbol{\mu\eta}}_{\boldsymbol{\nu}\mathbf{0}}(\mathbf{k})\hat{a}^{\dagger}_{\boldsymbol{k},\boldsymbol{\nu}}\hat{a}^{\dagger}_{\boldsymbol{q},\boldsymbol{\mu}}\hat{a}_{\boldsymbol{q}+\mathbf{k},\boldsymbol{\eta}} + V^{\mathbf{0}\boldsymbol{\nu}}_{\boldsymbol{\eta\mu}}(\mathbf{k})\hat{a}^{\dagger}_{\boldsymbol{q}+\mathbf{k},\boldsymbol{\eta}}\hat{a}_{\boldsymbol{k},\boldsymbol{\nu}}\hat{a}_{\boldsymbol{q},\boldsymbol{\mu}},$$

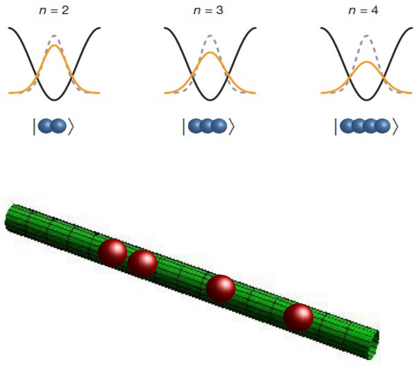
$$\hat{H}_4 = \frac{1}{2}{\sum_{\boldsymbol{q}_1,\boldsymbol{q}_2,\boldsymbol{k},\boldsymbol{\nu},\boldsymbol{\mu},\boldsymbol{\eta},\boldsymbol{\zeta}}}' V^{\boldsymbol{\zeta\eta}}_{\boldsymbol{\mu\nu}}(\mathbf{k})\hat{a}^{\dagger}_{\boldsymbol{q}_2+\boldsymbol{k},\boldsymbol{\mu}}\hat{a}^{\dagger}_{\boldsymbol{q}_1-\boldsymbol{k},\boldsymbol{\zeta}}\hat{a}_{\boldsymbol{q}_2,\boldsymbol{\nu}}\hat{a}_{\boldsymbol{q}_1,\boldsymbol{\eta}}$$

$$\langle \hat{H}_2|\hat{H}_{sp}^{-1}|\hat{H}_2\rangle \longrightarrow g_2^{(2)} \qquad \langle \hat{H}_1|\hat{H}_{sp}^{-1}|\hat{H}_1\rangle \longrightarrow g_3^{(2)}$$

$g_3^{(2)}$ also follows from GPE:



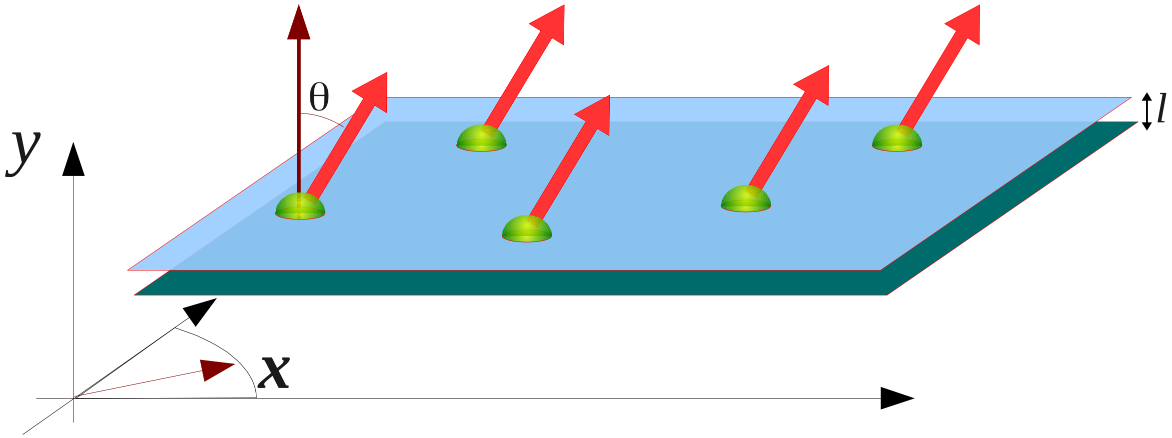
$$V^{00}_{00}(k=0)=0 \xrightarrow[\text{in “simple” cases}]{} V^{\eta\zeta}_{\nu\mu}(k=0)=0$$



In these “simple” cases

MF ~ $V^{00}_{00}(k=0)\propto a \longrightarrow g_3^{(2)}=-6\sum_{\nu}\frac{|V^{00}_{\nu 0}(0)|^2}{\epsilon_\nu}\propto -a^2$

Quasi-2D dipoles is also a simple case:



$$V(\mathbf{r}) = r_*\frac{r^2-3(x_1\sin\theta+y\cos\theta)^2}{r^5}+4\pi a\delta(\mathbf{r})$$

$$\Downarrow \quad \text{Fourier}$$

$$V(\mathbf{k},p) = 4\pi r_*\left[\frac{(k_1\sin\theta+p\cos\theta)^2}{k^2+p^2}-\frac{1}{3}+\frac{a}{r_*}\right]$$

$$\Downarrow$$

$$V^{\eta\zeta}_{\nu\mu}(k=0)=0 \longleftrightarrow a=a_*=(1/3-\cos^2\theta)r_*$$

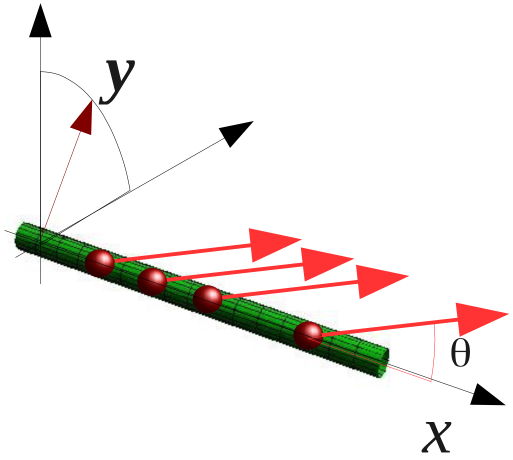
$$\Downarrow$$

$V^{00}_{00}(k=0)=0$ and $g_3^{(2)}=0$ simultaneously

$$V_{00}^{00}(k=0)=0 \;\cancel{\longrightarrow}\; V_{\nu\mu}^{\eta\zeta}(k=0)=0$$

not valid in general

Example is quasi-1D dipoles:



$$V(\mathbf{r}) = r_* \frac{r^2 - 3(x\cos\theta + y_1\sin\theta)^2}{r^5} + 4\pi a\delta(\mathbf{r})$$

$$\Downarrow$$

$$\int dx V(x,\mathbf{y}) = 4\pi(a-a_*)\delta(\mathbf{y}) + 2r_*\sin^2\theta\frac{y_2^2-y_1^2}{y^4}$$

$$a_* = \left(\frac{1}{3} - \frac{\sin^2\theta}{2}\right) r_*$$

$$\Downarrow$$

$$g_3^{(2)} = -12\ln\frac{4}{3}\left(\frac{a-a_*}{l}\right)^2 + \left(\frac{3}{2} - 6\ln\frac{4}{3}\right)\left(\frac{r_*}{l}\right)^2 \sin^4\theta$$

$V_{00}^{00}(k=0)=0$ when $a=a_*$, but all $V_{\nu\mu}^{\eta\zeta}(k=0)=0$

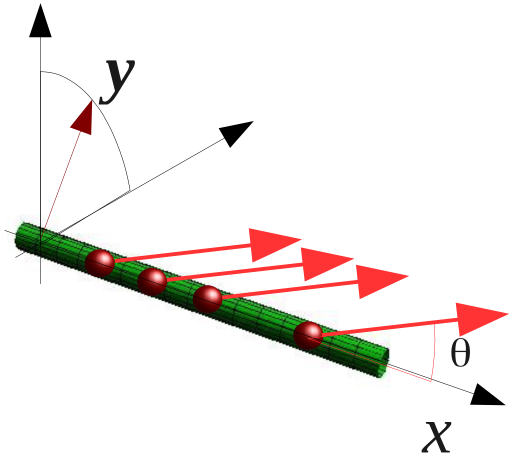
only when $a=a_*$ AND $\theta=0$

$$\Downarrow$$

Independent control of 2-body and 3-body interactions!

$$V_{00}^{00}(k=0)=0 \quad \underset{\text{not valid in general}}{\xcancel{\longrightarrow}} \quad V_{\nu\mu}^{\eta\zeta}(k=0)=0$$

Example is quasi-1D dipoles:



$$V(\mathbf{r}) = r_* \frac{r^2 - 3(x\cos\theta + y_1\sin\theta)^2}{r^5} + 4\pi a\delta(\mathbf{r})$$

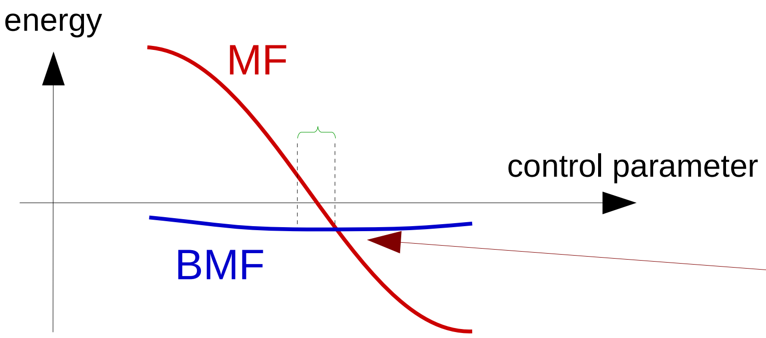
$$\Downarrow$$

$$\int dx V(x,\mathbf{y}) = 4\pi(a-a_*)\delta(\mathbf{y}) + 2r_*\sin^2\theta\frac{y_2^2-y_1^2}{y^4}$$

$$a_* = \left(\frac{1}{3} - \frac{\sin^2\theta}{2}\right) r_*$$

$$\Downarrow$$

$$g_3^{(2)} = -12\ln\frac{4}{3}\left(\frac{a-a_*}{l}\right)^2 + \left(\frac{3}{2} - 6\ln\frac{4}{3}\right)\left(\frac{r_*}{l}\right)^2\sin^4\theta$$



$V_{00}^{00}(k=0)=0$ when $a=a_*$, but all $V_{\nu\mu}^{\eta\zeta}(k=0)=0$

only when $a=a_*$ AND $\theta=0$

$$\Downarrow$$

Independent control of 2-body and 3-body interactions!

… although $g_3^{(2)} \leq 0$

Hereafter assume $\boxed{\int d\mathbf{x} V(\mathbf{x},\mathbf{y}) = 0 \longleftrightarrow V^{\eta\zeta}_{\nu\mu}(k=0)=0}$

(holds for quasi-2D dipoles when $a=a_*$ and for quasi-1D dipoles when $a=a_*$ and $\theta=0$)

$$E[N] = E^{(1)}[N] + E^{(2)}[N] + E^{(3)}[N] + \dots$$

$$g_3^{(3)}\binom{N}{3} + g_2^{(3)}\binom{N}{2}$$

where $$g_3^{(3)} = 6 \sum_{\mathbf{k},\boldsymbol{\nu},\boldsymbol{\mu},\boldsymbol{\eta}} \frac{V^{\mathbf{0}\boldsymbol{\eta}}_{\mathbf{0}\boldsymbol{\nu}}(\mathbf{k}) V^{\mathbf{0}\boldsymbol{\mu}}_{\boldsymbol{\eta}\mathbf{0}}(\mathbf{k}) V^{\boldsymbol{\nu}\mathbf{0}}_{\boldsymbol{\mu}\mathbf{0}}(\mathbf{k})}{(k^2+\epsilon_{\boldsymbol{\nu}}+\epsilon_{\boldsymbol{\eta}})(k^2+\epsilon_{\boldsymbol{\nu}}+\epsilon_{\boldsymbol{\mu}})}$$

and $$g_2^{(3)} = \sum_{\boldsymbol{\nu},\boldsymbol{\mu},\boldsymbol{\eta},\boldsymbol{\zeta},\mathbf{k},\mathbf{q}} \frac{V^{\mathbf{0}\boldsymbol{\zeta}}_{\mathbf{0}\boldsymbol{\eta}}(-\mathbf{q}) V^{\boldsymbol{\zeta}\boldsymbol{\mu}}_{\boldsymbol{\eta}\boldsymbol{\nu}}(\mathbf{q}-\mathbf{k}) V^{\boldsymbol{\mu}\mathbf{0}}_{\boldsymbol{\nu}\mathbf{0}}(\mathbf{k})}{(k^2+\epsilon_{\boldsymbol{\nu}}+\epsilon_{\boldsymbol{\mu}})(q^2+\epsilon_{\boldsymbol{\eta}}+\epsilon_{\boldsymbol{\zeta}})}$$

$$\hat{H}_{sp} = \sum_{\boldsymbol{q},\boldsymbol{\nu}} (q^2/2+\epsilon_{\boldsymbol{\nu}}) \hat{a}^\dagger_{\boldsymbol{q},\boldsymbol{\nu}} \hat{a}_{\boldsymbol{q},\boldsymbol{\nu}}$$

$$\hat{H}_2 = \frac{n_0}{2} {\sum_{\boldsymbol{\nu},\boldsymbol{\mu},\boldsymbol{k}}}' V^{\boldsymbol{\nu}\mathbf{0}}_{\boldsymbol{\mu}\mathbf{0}}(\mathbf{k}) \hat{a}^\dagger_{\boldsymbol{k},\boldsymbol{\mu}} \hat{a}^\dagger_{-\boldsymbol{k},\boldsymbol{\nu}} + V^{\mathbf{0}\boldsymbol{\nu}}_{\mathbf{0}\boldsymbol{\mu}}(\mathbf{k}) \hat{a}_{-\boldsymbol{k},\boldsymbol{\mu}} \hat{a}_{\boldsymbol{k},\boldsymbol{\nu}}$$
$$+ 2[V^{\mathbf{00}}_{\boldsymbol{\mu}\boldsymbol{\nu}}(\mathbf{0}) + V^{\mathbf{0}\boldsymbol{\nu}}_{\boldsymbol{\mu}\mathbf{0}}(\mathbf{k})] \hat{a}^\dagger_{\boldsymbol{k},\boldsymbol{\mu}} \hat{a}_{\boldsymbol{k},\boldsymbol{\nu}}$$

$$\langle \hat{H}_2 | \hat{H}_{sp}^{-1} | \hat{H}_2 | \hat{H}_{sp}^{-1} | \hat{H}_2 \rangle$$

$g_3^{(3)}$ is well described by Bogoliubov

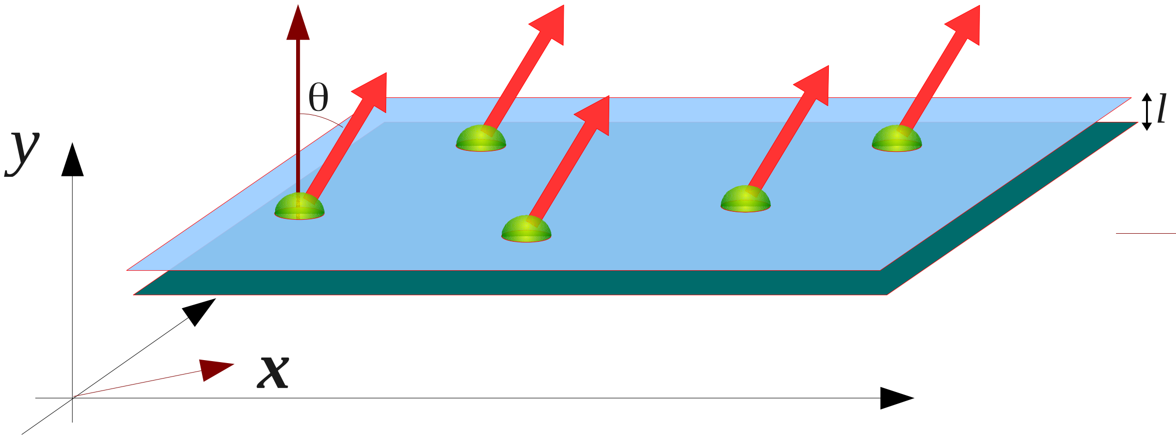
$$\langle \hat{H}_2 | \hat{H}_{sp}^{-1} | \hat{H}_4 | \hat{H}_{sp}^{-1} | \hat{H}_2 \rangle$$

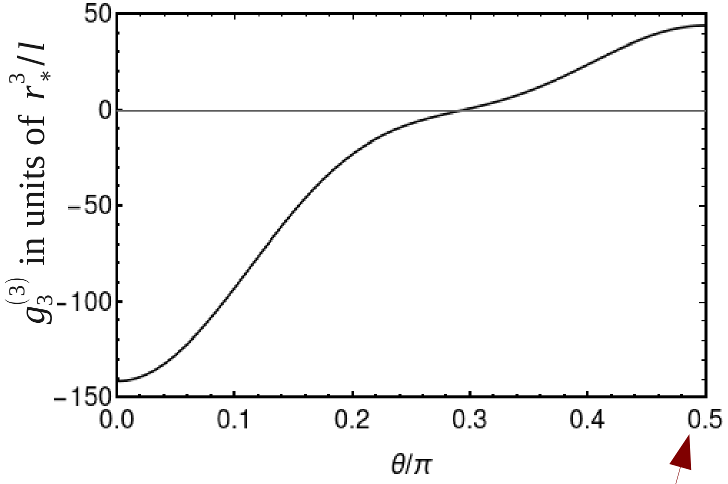
$g_2^{(3)}$ is formally beyond Bogoliubov, as we need

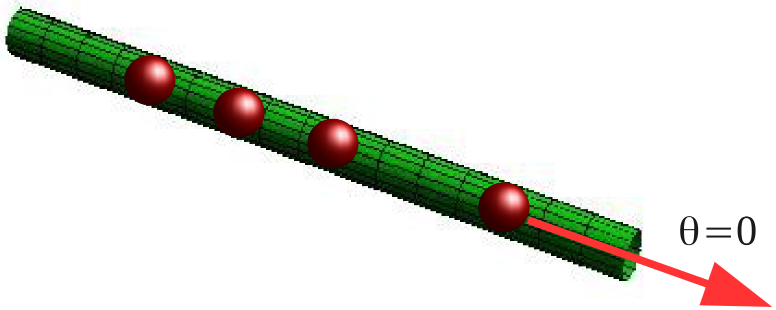
$$\hat{H}_4 = \frac{1}{2} {\sum_{\boldsymbol{q}_1,\boldsymbol{q}_2,\boldsymbol{k},\boldsymbol{\nu},\boldsymbol{\mu},\boldsymbol{\eta},\boldsymbol{\zeta}}}' V^{\boldsymbol{\zeta}\boldsymbol{\eta}}_{\boldsymbol{\mu}\boldsymbol{\nu}}(\mathbf{k}) \hat{a}^\dagger_{\boldsymbol{q}_2+\boldsymbol{k},\boldsymbol{\mu}} \hat{a}^\dagger_{\boldsymbol{q}_1-\boldsymbol{k},\boldsymbol{\zeta}} \hat{a}_{\boldsymbol{q}_2,\boldsymbol{\nu}} \hat{a}_{\boldsymbol{q}_1,\boldsymbol{\eta}}$$

Applications of

$$g_3^{(3)} = 6 \sum_{\mathbf{k},\boldsymbol{\nu},\boldsymbol{\mu},\boldsymbol{\eta}} \frac{V_{\mathbf{0}\boldsymbol{\nu}}^{\mathbf{0}\boldsymbol{\eta}}(\mathbf{k}) V_{\boldsymbol{\eta}\mathbf{0}}^{\mathbf{0}\boldsymbol{\mu}}(\mathbf{k}) V_{\boldsymbol{\mu}\mathbf{0}}^{\boldsymbol{\nu}\mathbf{0}}(\mathbf{k})}{(k^2+\epsilon_{\boldsymbol{\nu}}+\epsilon_{\boldsymbol{\eta}})(k^2+\epsilon_{\boldsymbol{\nu}}+\epsilon_{\boldsymbol{\mu}})}$$







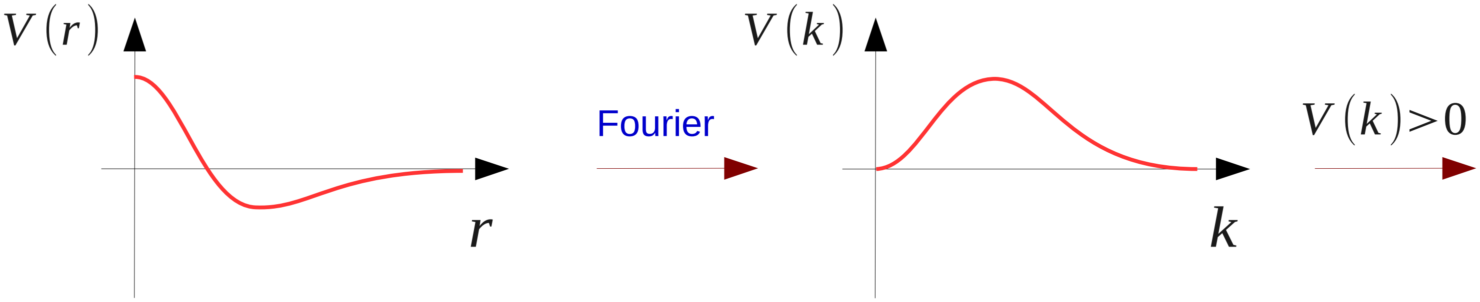
$$g_3^{(3)} = 4.65\,(r_*/l)^3$$

In both cases (if $r_* > 0$):

repulsive 3rd order 3-body interaction for dipoles oriented along unconfined direction(s) !

Happens when the 2-body potential is attracted at long range

2-body tail – 3-body sign correspondence



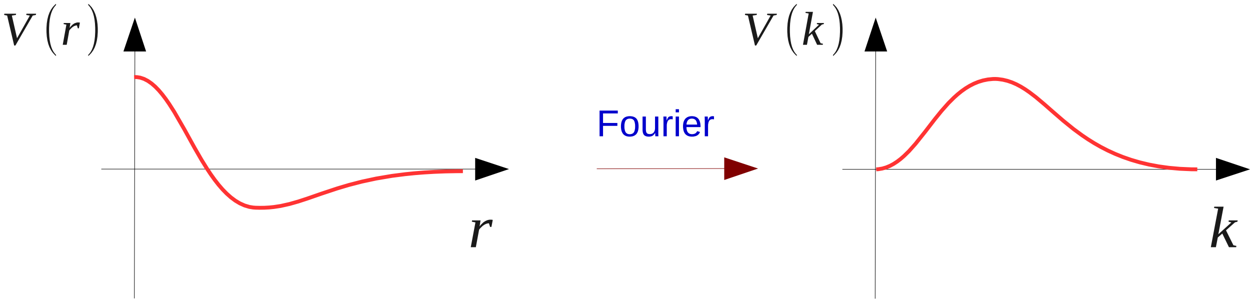
$V(k)>0$

$$g_3^{(3)}=6\sum_k \frac{V^3(k)}{k^4}>0$$

For ``not very exotic" 2-body potentials (double-Gaussian, Yukawa-plus-delta, quasi-low-D dipolar case) the rule is:

Attractive tail → repulsive $g_3^{(3)}>0$

2-body tail – 3-body sign correspondence

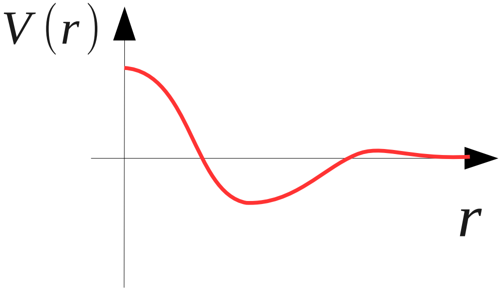


$$V(k)>0 \longrightarrow g_3^{(3)}=6\sum_k \frac{V^3(k)}{k^4}>0$$

For ``not very exotic" 2-body potentials (double-Gaussian, Yukawa-plus-delta, quasi-low-D dipolar case) the rule is:

Attractive tail $\xcancel{\longrightarrow}$ repulsive $g_3^{(3)}>0$

Counterexample:



Summary

- Bogoliubov spectrum ↔ type of the LHY term

    Phononic → nonanalytic LHY

    Gapped or V(k=0)=0 → regular expansion in powers of density

- Bogoliubov theory is a powerful three-body solver!

- Closed perturbative expressions:

$$g_3^{(2)} = -6 \sum_{\nu} \frac{|V_{\nu 0}^{00}(0)|^2}{\epsilon_\nu}$$

$$g_3^{(3)} = 6 \sum_{\mathbf{k},\boldsymbol{\nu},\boldsymbol{\mu},\boldsymbol{\eta}} \frac{V_{\mathbf{0}\boldsymbol{\nu}}^{\mathbf{0}\boldsymbol{\eta}}(\mathbf{k}) V_{\boldsymbol{\eta}\mathbf{0}}^{\mathbf{0}\boldsymbol{\mu}}(\mathbf{k}) V_{\boldsymbol{\mu}\mathbf{0}}^{\boldsymbol{\nu}\mathbf{0}}(\mathbf{k})}{(k^2 + \epsilon_{\boldsymbol{\nu}} + \epsilon_{\boldsymbol{\eta}})(k^2 + \epsilon_{\boldsymbol{\nu}} + \epsilon_{\boldsymbol{\mu}})}$$

- $g_2^{(3)}$ is not captured by Bogoliubov!

- Applications to quasi-low-D dipolar bosons

Thank you for your attention!