Varenna Summer School on
Quantum Mixtures of Ultracold Atomic Gases
18-23 July 2022

SPIN ORBIT COUPLED
BEC GASES

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CNR-INO

LECTURE 1

- The quantum phases of a spin-orbit coupled mixture of Bose-Einstein condensates

- Order parameter and nature of the phase transitions

- Sound and Dynamic properties of SOC BEC’s

LECTURE 2

- Superfluidity and rotation of SOC BEC’s

- Supersolidity and the novel Goldstone modes

Yesterday questions

Definition of effective mass

Eigenvalues of sp Hamiltonian $h_0 = \frac{1}{2m}[\left(p_x - \hbar k_0 \sigma_z\right)^2 + p_\perp^2] + \frac{1}{2}\hbar\Omega\sigma_x$

$$\varepsilon_\pm = \frac{p_x^2 + p_\perp^2}{2m} + E_R \pm \sqrt{\frac{\hbar^2 k_0^2 p_x^2}{m^2} + \frac{\hbar^2\Omega^2}{4}}$$

$$\frac{m}{m^*} = \frac{d^2\varepsilon}{dp_x^2}$$

At the two minima, where $p_x = \pm\hbar k_1 = \pm\hbar k_0 \sqrt{1 - \Omega^2 (m / 2\hbar k_0^2)^2}$,

One finds $\frac{m}{m^*} = 1 - \left(\frac{\Omega}{\Omega_C}\right)^2$ in Plane - Wave phase, $\Omega < \Omega_C = 4E_R/\hbar$

In Single Minimum phase,
where $p_x = 0$, one instead finds At $\frac{m}{m^*} = 1 - \frac{\Omega_C}{\Omega}$

Yesterday questions

What happens when one goes from lab to spin rotated frame

Unitary transformation $e^{i\Theta\sigma_z/2}$,
with $\Theta = 2k_0 x - \Delta\omega_L t$,
transforms the **time dependent**
sp Hamiltonian to

$$h_0^{lab} = \frac{\vec{p}^{\,2}}{2} + \frac{\Omega}{2}\sigma_x \cos(2k_0 x - \Delta\omega_L t) + \frac{\Omega}{2}\sigma_y \sin(2k_0 x - \Delta\omega_L t) - \frac{\omega_Z}{2}\sigma_z$$

time independent Hamiltonian
(no longer periodic potentials !)

$$h_0 = \frac{1}{2m}[\left(p_x - \hbar k_0 \sigma_z\right)^2 + p_\perp^2] + \frac{1}{2}\hbar\Omega\sigma_x$$

- System reaches equilibrium in spin rotated (SR) frame.
- Physical quantities $n(\vec{r},t),\ \mathrm{s_z}(\vec{r},t)$ are conserved when going back to the lab frame since they commute with $e^{i\Theta\sigma_z/2}$

$$< Lab\,|\,\hat{n}(\vec{r})\,|\,Lab >=< SR\,|\,\mathrm{e}^{i\theta\sigma_z}\,|\,\hat{n}(\vec{r})\,|\,e^{-i\theta\sigma_z}\,|\,\mathrm{SR} >=< SR\,|\,\hat{n}(\vec{r})\,|\,\mathrm{SR} >$$

$$< Lab\,|\,\hat{s}_z(\vec{r})\,|\,Lab >=< SR\,|\,\mathrm{e}^{i\theta\sigma_z}\,|\,\hat{s}_z(\vec{r})\,|\,e^{-i\theta\sigma_z}\,|\,\mathrm{SR} >=< SR\,|\,\hat{s}_z(\vec{r})\,|\,\mathrm{SR} >$$

Yesterday questions

Is the PW-SM transition a real quantum phase transition ?

Inclusion of interactions yields **divergent** behavior of magnetic **susceptibility** at

$$\boxed{\hbar\Omega_c = 4E_R - 2g_{ss}n \quad \text{with} \quad \mathrm{g}_{ss} = (g - g_{\uparrow\downarrow})/2}$$

- If $\boxed{\mathrm{g}_{ss} = 0}$ one recovers transition predicted by s.p. hamiltonian
- If $\boxed{E_R = 0}$ (no momentum transfer) one recovers transition predicted by Rabi configuration for negative values of g_{ss} (see Lamporesi and Oberthaler)

LECTURE 2

- Superfluidity and rotation of SOC BEC's

- Supersolidity and the novel Goldstone modes

Important issues concerning superfluidity

Hydrodynamic theory of superfluids, at T=0, is usually based on the 'classical' form of kinetic energy (liquid Helium, quantum gases)

$$E_{kin} = \int d\vec{r}\,\frac{1}{2}\rho_S \vec{v}_S^{\,2}$$

In Galilean invariant systems

$$\rho_S = \rho$$

Superfluid density coincides with total density at T=0

$$\vec{v}_S = \frac{\hbar}{m}\vec{\nabla}\varphi$$

Velocity field obeys **irrotationality** constraint (role of phase of the order parameter)

What happens in a spin-orbit coupled BEC ?

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What happens in a spin-orbit coupled BEC ?

- Violation of Galilean invariance **suppresses** value of $\boxed{\rho_S}$
- SOC causes violation of irrotationality and introduces **rigid body** effects in the rotation of the BEC condensate

Definition of **normal** (non superfluid) density

(G. Baym, The microscopic description of superfluidity, 1969):

$$\frac{\rho_n}{\rho} = \lim_{q\to 0} Q^{-1} \sum_{m,n} e^{-\beta E_m} \frac{|< m \,|\, J_x^T(q) \,|\, n >|^2}{E_n - E_m} + (q \to -q)$$

(Macroscopic response to transverse current)

$$J_x^T(q) = \sum_k (p_{k,x} - k_0 \sigma_{k,z}) e^{iqy_k}$$ transverse current operator

When $q \to 0$ current operator reduces to physical momentum

$$J_x^T(q \to 0) = P_x \equiv \sum (p_{k,x} - k_0 \sigma_{k,z})$$

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Non commutativity of P_x with H (violation of **Galilean** invariance)

$\boxed{\rho_n \neq 0 \ !!}$ **even at T=0**

In SOC systems non commutativity arises from spin terms and is compatible with translational invariance.

Effect is absent along y direction ⇨ tensor nature of ρ_n and ρ_s

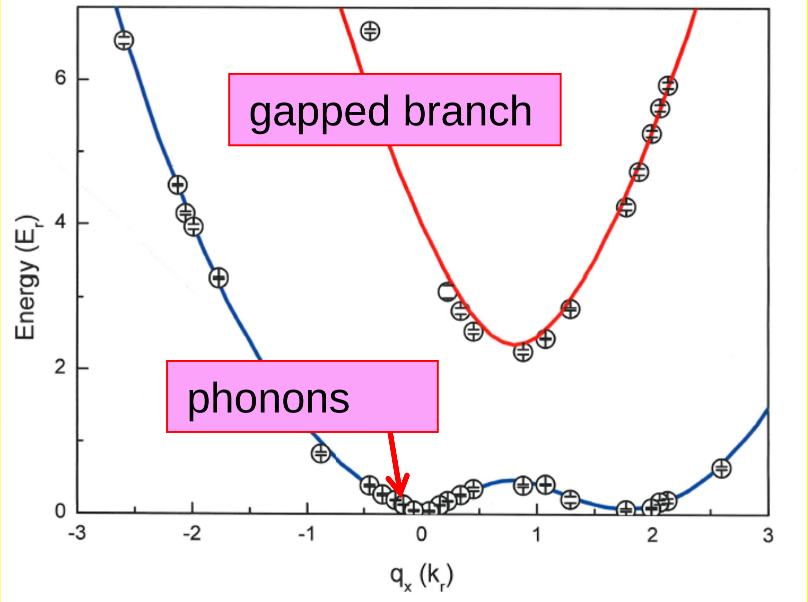
To calculate normal density at T=0

$$\frac{\rho_n}{\rho} = \lim_{q \to 0} \frac{1}{N} \sum_n \frac{|<0|J_x^T(q)|n>|^2}{E_n - E_0} + (q \to -q)$$

one needs knowledge of spectrum of elementary excitations

Two branches in the excitation spectrum of spinor BEC's

- Due to Raman coupling **only one branch is gapless** in PW and SM phases (one Goldstone mode)
- phonon behavior at small q



Exp: Si-Cong Ji et al., PRL 2015;
Khamehchi et al, PRA 2014
Theory: Martone et al., PRA 2012

Phonon branch has **longitudinal** nature and does not contribute to ρ_n. Contribution from gapped branch can be evaluated using **sum rule** arguments.

Results for superfluid density $\rho_s = \rho - \rho_n$

(Yi-Cai Zhang et al. PRA 2016)

Plane Wave

$$\Omega \le \Omega_c$$

$$\frac{\rho_s}{\rho} = \frac{\Omega_c(\Omega_c^2 - \Omega^2)}{\Omega_c^3 + 2g_{ss}n\Omega^2}$$

Single Minimum

$$\Omega \ge \Omega_c$$

$$\frac{\rho_s}{\rho} = \frac{\Omega - \Omega_c}{\Omega + 2g_{ss}n}$$

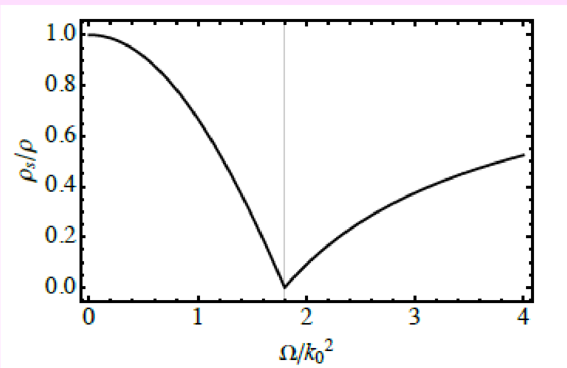
where $\hbar\Omega_{\mathrm{c}} = 4E_R - 2g_{ss}n$ is the transition value of Raman coupling and $\mathrm{g}_{\mathrm{ss}} = (g - g_{\uparrow\downarrow})/2$ is difference between intra and interspeicies coupling.

The superfluid density vanishes at the transition !!

(dramatic consequence of breaking of Galilean invariance in a uniform superfluid)

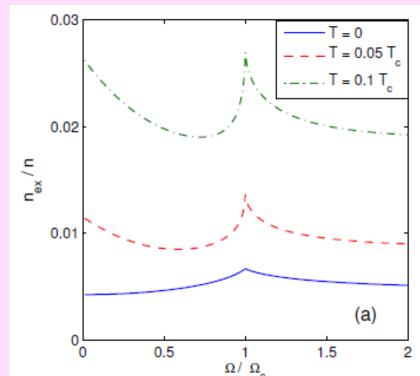
Superfluid density vs Bose-Einstein condensation

- **Superfluid density** strongly **suppressed** near the phase transition between the plane wave and zero-momentum phase
- **BEC fraction** is instead practically unperturbed (**quantum depletion** always remains very small, less than 1%)



Superfluid density

(Yi-Cai Zhang et al., PRA 2016)



Quantum depletion

(W. Zheng et al. JPhysB 2013)

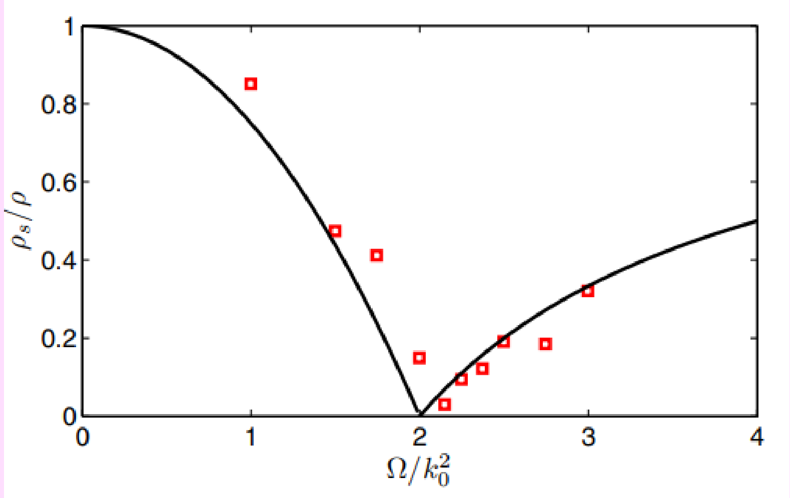
CAN ONE MEASURE ρ_s ?

At T=0 the superfluid density fixes the value of the sound velocity according to the hydrodynamic relationship

$$mc_+c_- = \frac{\rho_s^x}{\rho}\kappa^{-1} \quad \text{where} \quad \kappa^{-1} = \rho\partial_\rho\mu \quad \text{is inverse compressibility}$$

c_+ and c_- are values of sound velocity parallel and anti-parallel to x-direction. They differ because of violation of parity and time reversal symmetry (see Lecture 1)

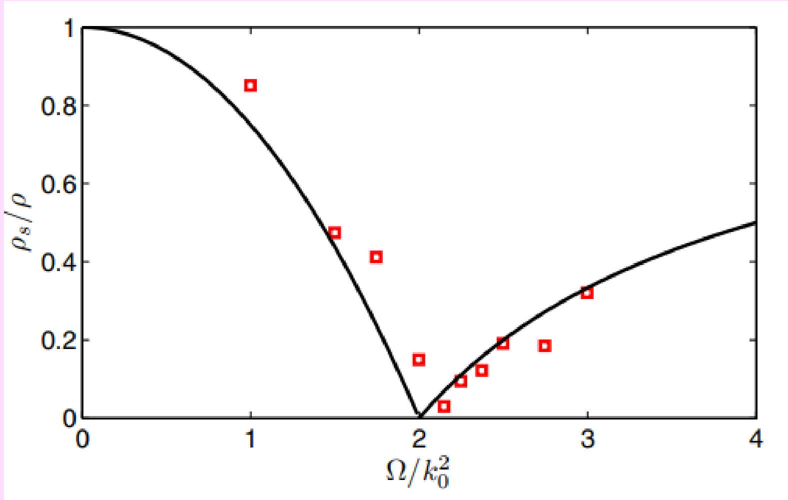
$$\rho_s^x = \rho m c_+ c_- \kappa$$



Using experimental data for sound velocities (Si-Cong Ji et al., PRL 2015) and evaluating theoretically the compressibility (weakly affected by SOC) one finds **good agreement** with predicted behavior of **superfluid density**.

[Yi-Cai Zhang, PRA 2016 (Trento-HongKong collaboration)]

$$\rho_s^x = \rho m c_+ c_- \kappa$$



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Fully model independent determination of could be provided by additional measurement of sound velocity along transverse y-direction where superfluid density is not affected by SOC (**superfluid density is actually a tensor**)

$$\rho_s^y = \rho m c_y^2 \kappa = \rho$$

$$\frac{\rho_s^x}{\rho} = \frac{c_+ c_-}{c_y^2}$$

Violation of **irrotationaly constraint** for velocity field

Can a **BEC rotate** like a **rigid body** ?

Calculation of **Moment of inertia** of a SOC BEC

Violation of **irrotationality**

SOC Hamiltonian yields result for the equation of continuity (see Lecture 1)

$$\partial_t n + \vec{\nabla} \cdot \vec{j} = 0$$

$$j_x \equiv n v_x = \frac{\hbar}{m} n \nabla_x \varphi - \frac{\hbar}{m} k_0 s_z$$

New behavior of the current density can yield **violation** of the **irrotationality** constraint for the velocity field, which is not uniquely fixed by the gradient of the phase (as in usual BECs)

$$v_x(\vec{r}) = j_x(\vec{r}) / n(\vec{r}) = \frac{\hbar}{m} \nabla_x \varphi - \frac{\hbar k_0}{m} \frac{s_z}{n}$$

Violation of **irrotationality**

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As a consequence of the spin term in the current and in the physical momentum $P_x = m v_x = (p_x - k_0 \sigma_z)$

also **angular momentum** acquires a novel spin dependent component:

$$L_z = x p_y - y(p_x - \hbar k_0 \sigma_z) = L_z^{can} + \hbar k_0 y \sigma_z$$

New spin term is crucial for determination of moment of inertia !!

The **moment of inertia** is defined as the **linear response**

$< L_z >_{\Omega \to 0} = \Omega_{rot}\theta$ to the **angular momentum** constraint

$$\mathrm{H} = \mathrm{H}_0 - \Omega_{rot}\mathrm{L}_z$$

$$\text{with } \mathrm{L}_z = \sum_k [x_k p_{k,y} - y(p_{k,x} - k_0 \sigma_{k,z})]$$

- In **usual BECs** moment of inertia takes **irrotational** form

$$\theta = \theta_{irr} \equiv \delta^2 \theta_{rig} \quad \text{with } \delta = \frac{< \mathrm{x}^2 - y^2 >}{< \mathrm{x}^2 + y^2 >})$$

$$\text{and } \theta_{\mathrm{rig}} = N(< \mathrm{x}^2 + y^2 >) \text{ the rigid value of moment of inertia}$$

Results for moment of inertia in SOC BECs in single minimum phase and $g_{\alpha\beta} = g$

Energy minimization in the presence of angular momentum constraint yields, for $\Omega \geq \Omega_c$ (isotropic trapping + LDA)

$$\boxed{\vec{v} = \frac{\Omega_c}{2\Omega - \Omega_c} \vec{\Omega}_{rot} \wedge \vec{r}}$$

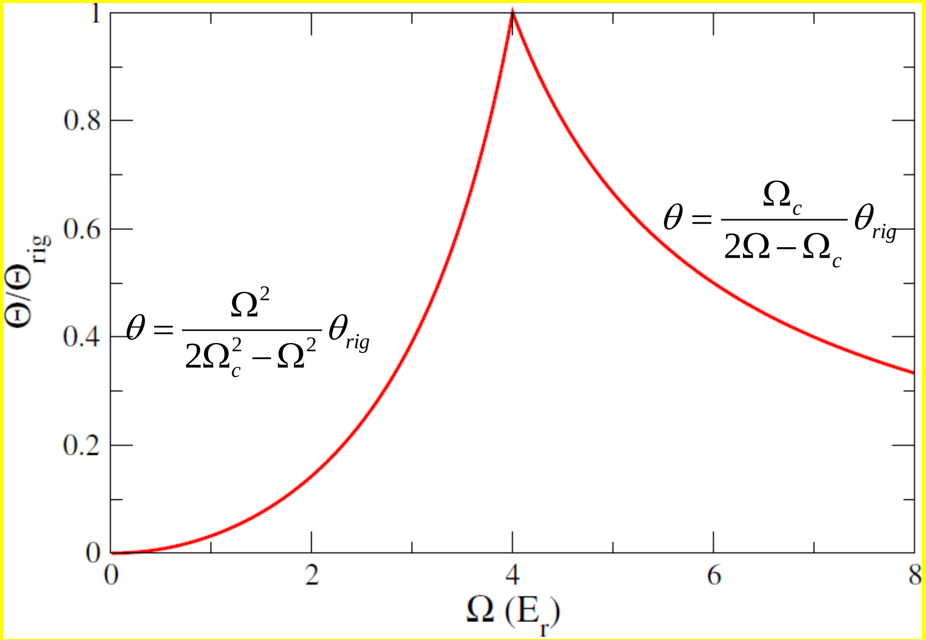
- **Diffused vorticity** despite BEC: $\vec{\nabla} \wedge \vec{v} = \frac{\Omega_c}{2\Omega - \Omega_c} 2\vec{\Omega}_{rot}$

- **Moment of inertia** takes **rigid value** $\boxed{\theta = \theta_{rig}}$ at the transition

- **BEC rotates like a rigid body !**

Behavior of moment of inertia

(S.S. PRL 18, 145302 2017)



Rigid value $\theta = \theta_{rig}$ at the transition between Plane wave and Single momentum phase.

Dramatic consequence of spin-orbit coupling

LECTURE 2

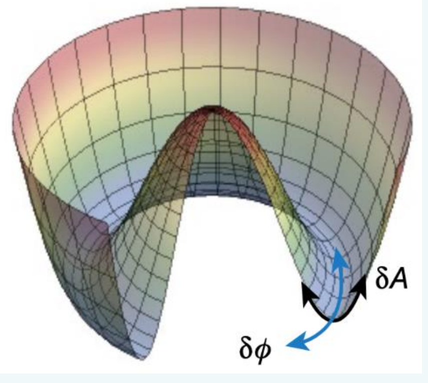
- **Superfluidity and rotation of SOC BEC's**

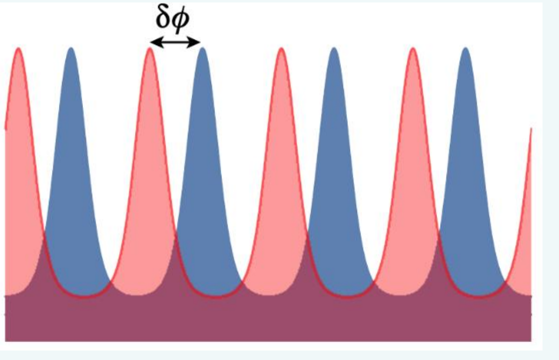
- **Supersolidity and the novel Goldstone modes**

Supersolids: many-body systems exhibiting spontaneous breaking of two continuous symmetries

Gauge symmetry (superfluidity)

Translational invariance (crystallization)





Spontaneous breaking of U(1) symmetry yields **Superfluidity**

Spontaneous breaking of translational symmetry yields **Crytsallization**

Supersolidity: long sought phase in solid helium: Atom vacancies can move, form a Bose-Eintein condensate, giving rise to superfluid effects

(Penrose&Onsager, Andreev&Lifshitz, Chester, Leggett …)

VOLUME 25, NUMBER 22 PHYSICAL REVIEW LETTERS 30 NOVEMBER 1970

Can a Solid Be "Superfluid"?

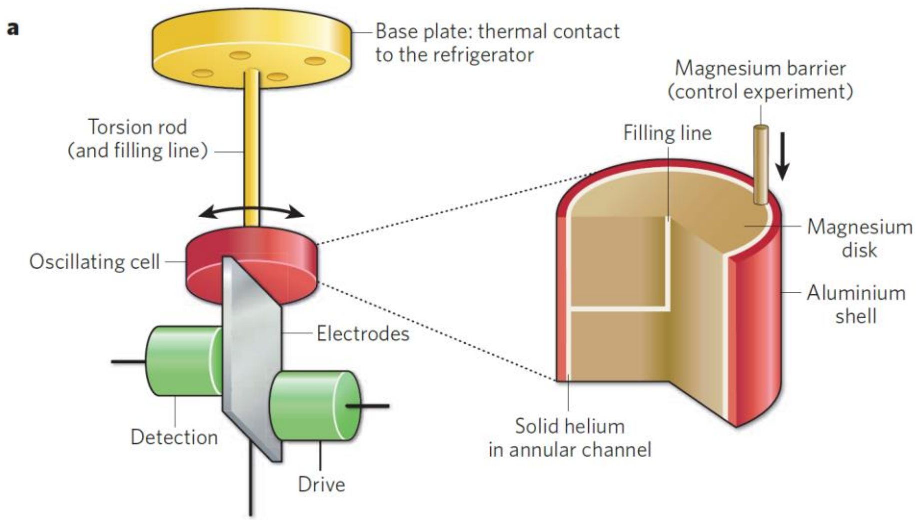
A. J. Leggett

School of Mathematical and Physical Sciences, University of Sussex, Falmer, Brighton, Sussex, England

(Received 15 September 1970)

It is suggested that the property of nonclassical rotational inertia possessed by superfluid liquid helium may be shared by some solids. In particular, nonclassical rotational inertia very probably occurs if the solid is Bose-condensed as recently proposed by Chester. Anomalous macroscopic effects are then predicted. However, the associated superfluid fraction is shown to be very small (probably $\lesssim 10^{-4}$) even at $T=0$, so that these effects could well have been missed. Direct tests are proposed.

Is solid helium a supersolid? Measure moment of inertia !



E. Kim and M. H. W. Chan, Probable observation of a supersolid helium phase, Nature 427, 225 (2004)

S. Balibar, **The enigma of supersolidity**
Nature 464, 176 (2010)

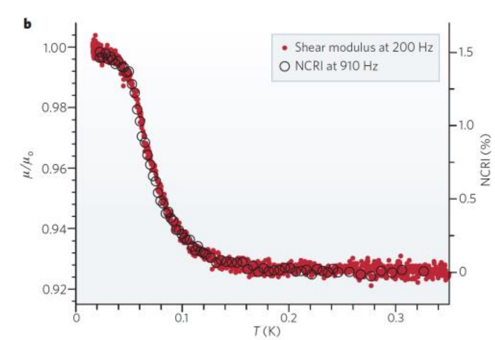
D. Y. Kim and M. H. W. Chan, Absence of supersolidity in solid helium in porous Vycor glass,
Phys. Rev. Lett. 109, 155301 (2012).

Torsion oscillator

$$\omega = \sqrt{K/\theta}$$

ω : oscillator frequency
θ : moment of inertia
K : elastic constant

Change of frequency can be however explained with change of elastic constant



Ultra-cold atomic gases have recently become successful platforms for supersolidity

- **Bec in optical resonators** (ETH 2017)
- **Spin-orbit coupled BEC's** (MIT 2017)
- **Dipolar gases** (Florence/Pisa, Stuggart, Innsbruck, 2019)

Key signatures associated with supersolidity:

- **Spontaneous density modulations**
- **Phase coherence**
- **Superfluid effects**
- **Novel Goldstone modes**

Quantum phase diagram predicted by SOC Hamiltonian at T=0

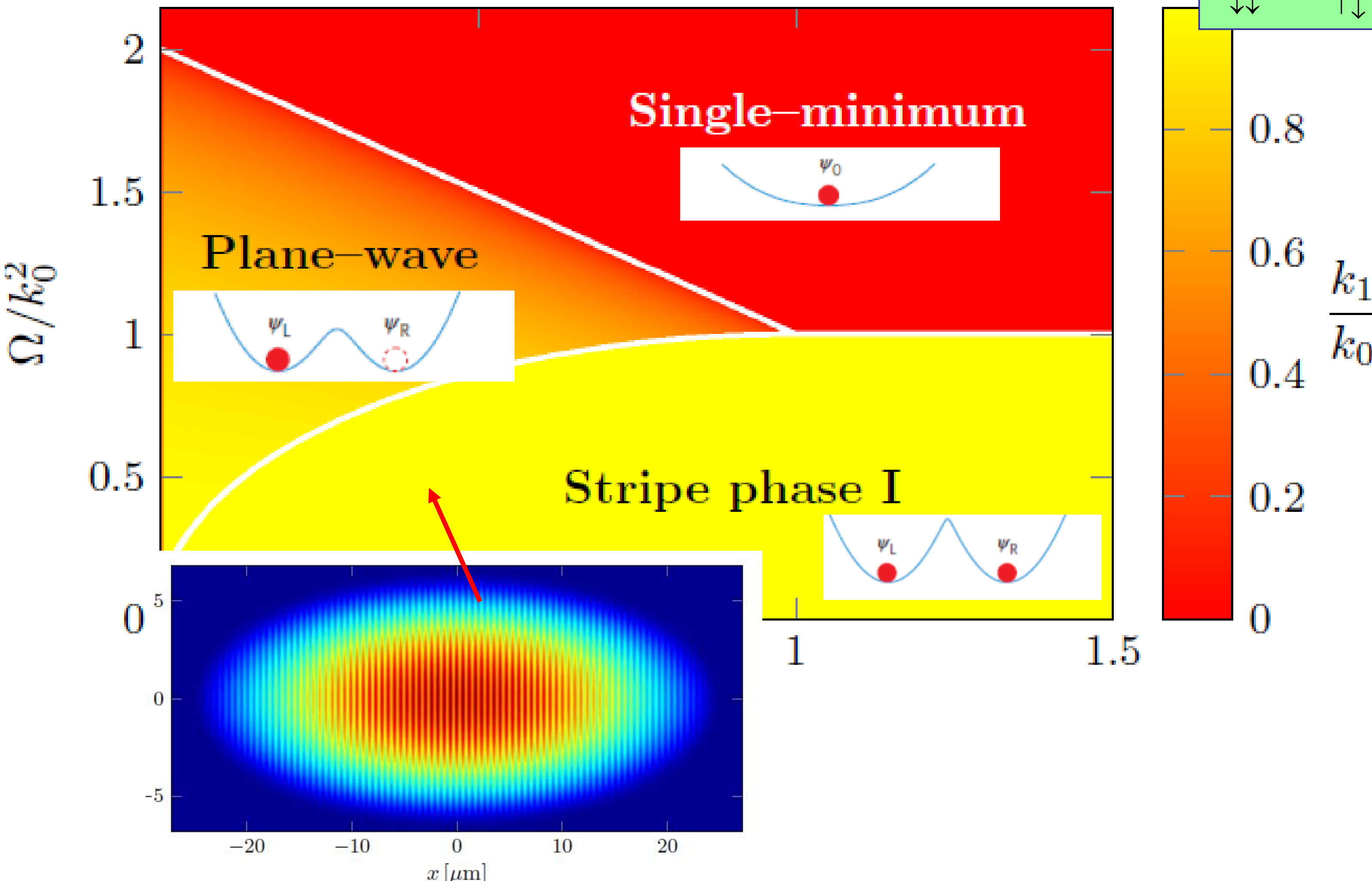
^{87}Rb

$|\uparrow>=|F=1, m_F=0>$

$|\uparrow>=|F=1, m_F=-1>$

$a_{\uparrow\uparrow}=101.41a_{\mathbf{B}}$

$a_{\downarrow\downarrow}=a_{\uparrow\updownarrow}=100.94a_B$



- The emergence of the stripe (supersolid) phase is a crucial consequence of **two-body interactions**:

The key interaction parameter

$$\gamma = \frac{g_{\uparrow\uparrow} + g_{\downarrow\downarrow} - 2g_{\uparrow\downarrow}}{g_{\uparrow\uparrow} + g_{\downarrow\downarrow} + 2g_{\uparrow\downarrow}}$$

fixes critical value of Raman coupling giving the transition between Supersolid and Plane Wave phase. The stripe phase is energetically favourable for

$$\Omega < \Omega_{cr} = 2k_0^2\sqrt{\frac{2\gamma}{1+2\gamma}}$$

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If $g_{\uparrow\uparrow} = g_{\downarrow\downarrow}$ one finds $\gamma = (g - g_{\uparrow\downarrow})/(g + g_{\uparrow\downarrow}) = g_{ss} / g_{dd}$, the relevant interaction parameters being related to the spin and density stiffness of the GP energy:

$$E_{\text{int}} = \frac{1}{2}\int d\vec{r}\,(g_{dd} n^2 + g_{ss} s_z^2)$$

Small values of g_{ss} (occurring if $g_{\uparrow\downarrow} \approx g$) favor spontaneous magnetization and realization of the Plane Wave phase.

If $g_{\uparrow\downarrow} \geq g$ the stripe phase disappears

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The relevant interaction parameters $g - g_{\uparrow\downarrow}$ and $g + g_{\uparrow\downarrow}$ are related to the spin and density stiffness of the GP energy:

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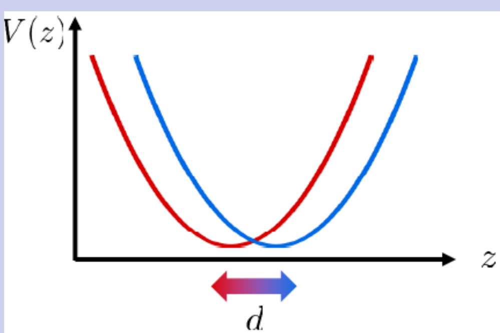
Small values of g_{ss} (occurring if $g_{\uparrow\downarrow} \approx g$) favor spontaneous magnetization and realization of the Plane Wave phase.

If $g_{\uparrow\downarrow} \geq g$ the Stripe Phase disappears

- In 87Rb

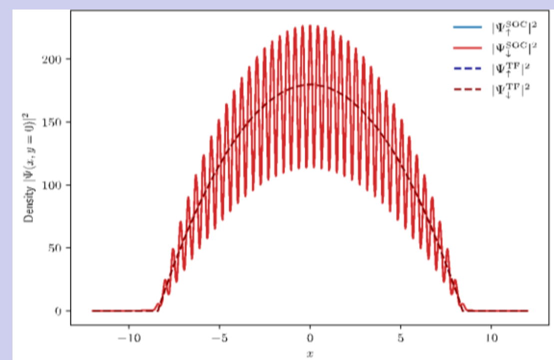
$$g_{ss} << g_{dd} \quad \text{and } \Omega_{cr} \quad \text{is very small} \left(g_{ss} / g_{dd} \approx 10^{-3}, \; \Omega_{cr} = 0.18 E_R\right)$$

- In order to increase the value of Ω_{cr} and have a wider region for the stripe phase (and hence a larger contrast of density fringes), a conveneient way to reduce the interspecies coupling $g_{\uparrow\downarrow}$, and hence increase $g_{ss} = (g - g_{\uparrow\downarrow})/2$, could be obtained by separating the wave functions of the two spin species withadditional spin selective external potential



$d = a_{ho}$

$\Omega_{cr} \approx 2.5 E_r$



$\Omega = 2E_r$

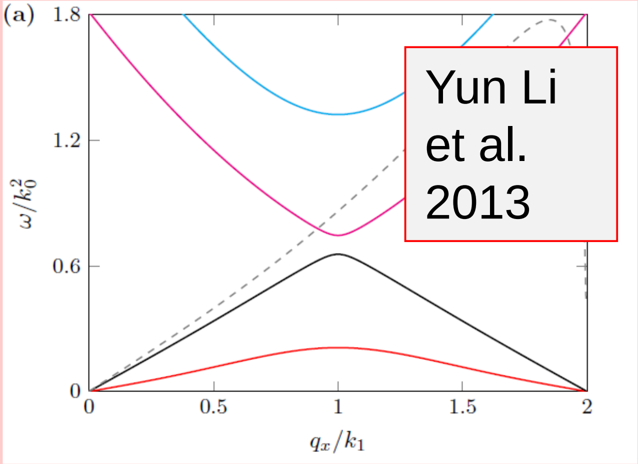
The excitation spectrum of a supersolid exhibits
two distinguished features

- **Band structure** typical of crystals

- **Two gapless excitations (Goldstone modes)** resulting from spontaneous breaking of two different symmetries

Gapless excitations in SOC BEC gas

- In the **stripe phase** (below Ω_{cr}) one finds **two** Goldstone modes:
  - **Superfluid** Goldstone mode (density mode)
  - **Crystal like** Goldstone mode (spin densitymode)



Yun Li et al. 2013

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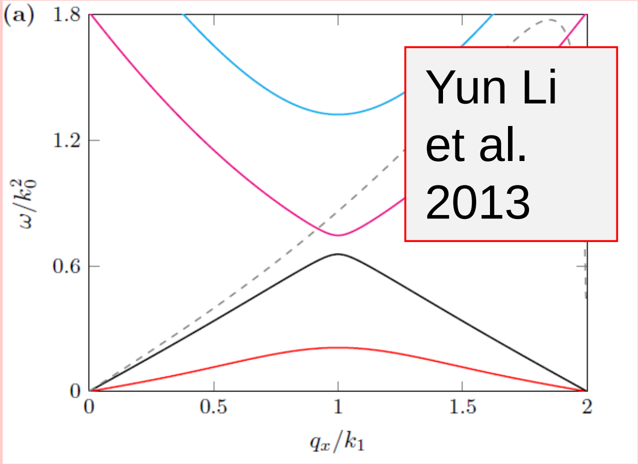
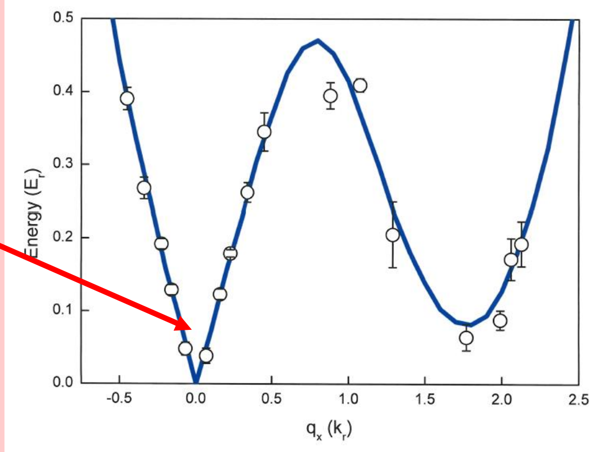
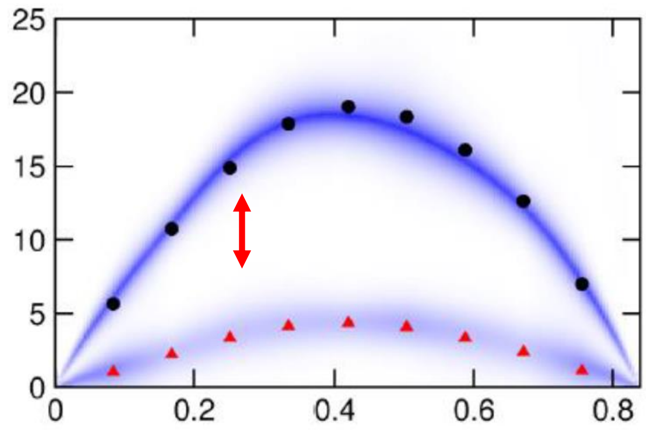


Yun Li et al. 2013

- **In the absence of stripes** (above Ω_{cr}) only one Goldstone mode of fully hybridized density and spin nature (see Lecture 1)



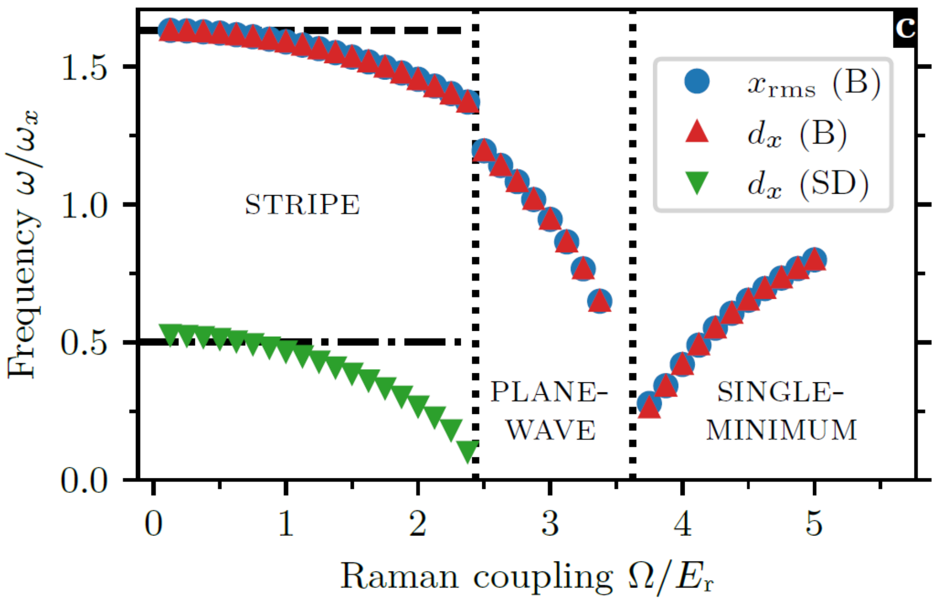
The two gapless branches of supersolid are coupled by density excitation operator



- For **spin-orbit** BECs the coupling is due to **hybridization between density** and **spin** degrees of freedom

- Exciting a compression mode (for example the **axial compression** mode) in a **harmonically trapped** gas is then expected to give rise to **beating effect** in the resulting oscillation

Dispersion of **axial breathing** (x^2) and **spin dipole** $x\sigma_z$) modes in harmonic trap, obtained applying a density perturbation proportional to x^2 and observing beating effect in the time dependent behavior of $< x\sigma_z >$. The same dispersion for the spin dipole mode is obtained by applying the perturbation $x\sigma_z$



K. Geier et al. PRL2021

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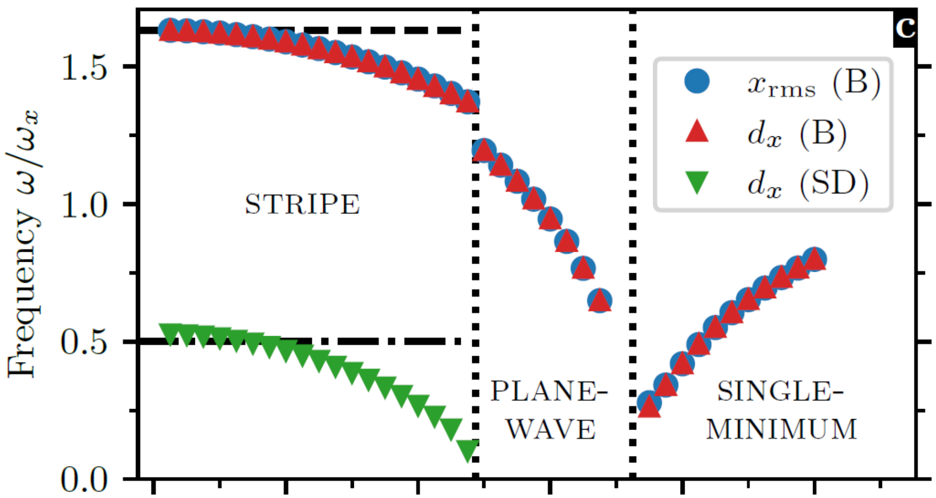


K. Geier et al.
PRL2021

- Spin dipole frequency decreases as a function of Raman coupling and vanishes at Ω_{cr} (consequence of **divergent** behavior of **spin polarizability** at the transition)
- Minimum at plane-wave/ single minimum transition is consequence of **large effective mass**.

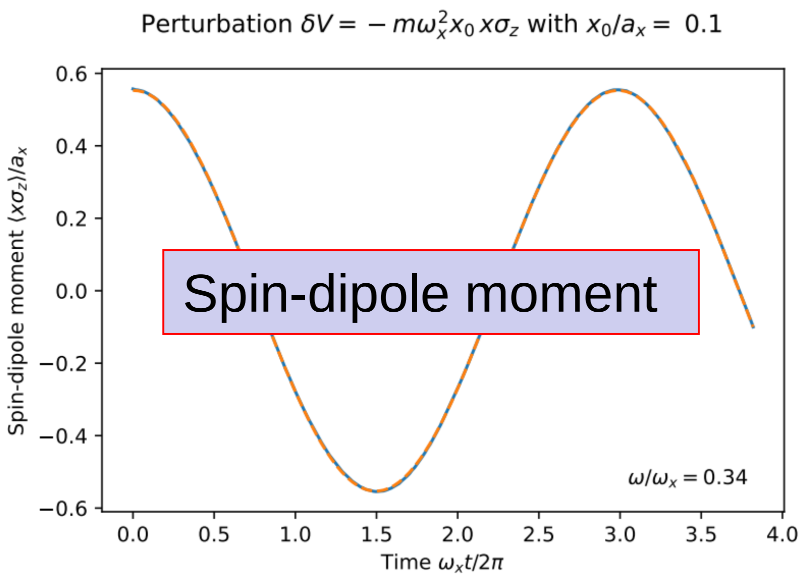
Key Question

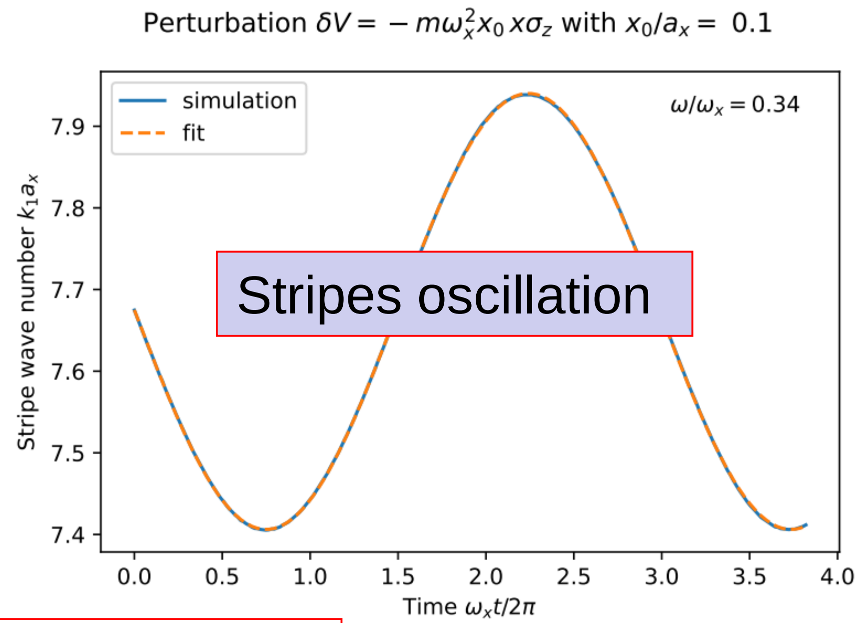
How is the novel **spin Goldstone mode** of a SOC connected with the **oscillation** (compression and dilation) of
of the **interstripe distance** characterizing the **crystal** nature of the Goldstone mode of a supersolid ?

[Question emerged during the very recent debate in the supersolid community and triggered by Wolfgang Ketterle, communication addressed to dipolar gas super solid community]

- We have recently checked that the **Goldstone spin dipole** mode is actually accompanied by the **oscillation** of the **interstripe distance** at the same frequency

Relevant features of the novel spin dipole Goldstone mode





K. Geier et al., In preparation

These results prove that the **stripe pattern** for the SO-BEC system **is not rigid** and actually characterizes the oscillating behavior of the **novel crystal-like Goldstone mode**, as a expected for a general supersolid.

Some conclusions

Spin orbit coupled BEC gase provide an excelllent laboratory for exploring fundamental questions of many body physics:

- Violation of **Galilean invariance**
- Violation of **irrotationality** **constraint** for superfluid motion
- Novel magnetization effects **in supersolidity**

Some perspectives

- Understanding the physics of quantum mixtures with **non abelian gauge fields**: new superfluid and topological features

- Waiting for efficient **experimental approaches** to the **supersolid phase**
  (observation of novel Goldstone modes, dynamics of stripes)

Thank you for the attention

Main collaborators (old and new) on **SOC-BECs**

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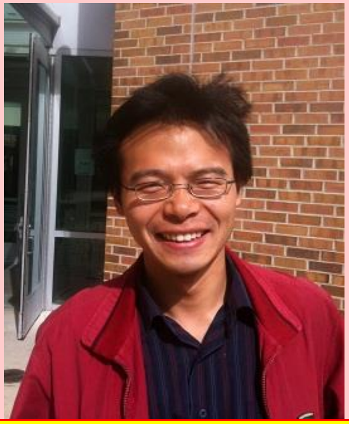

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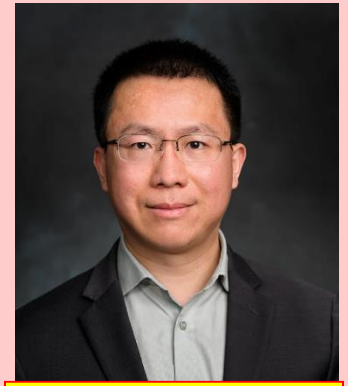

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