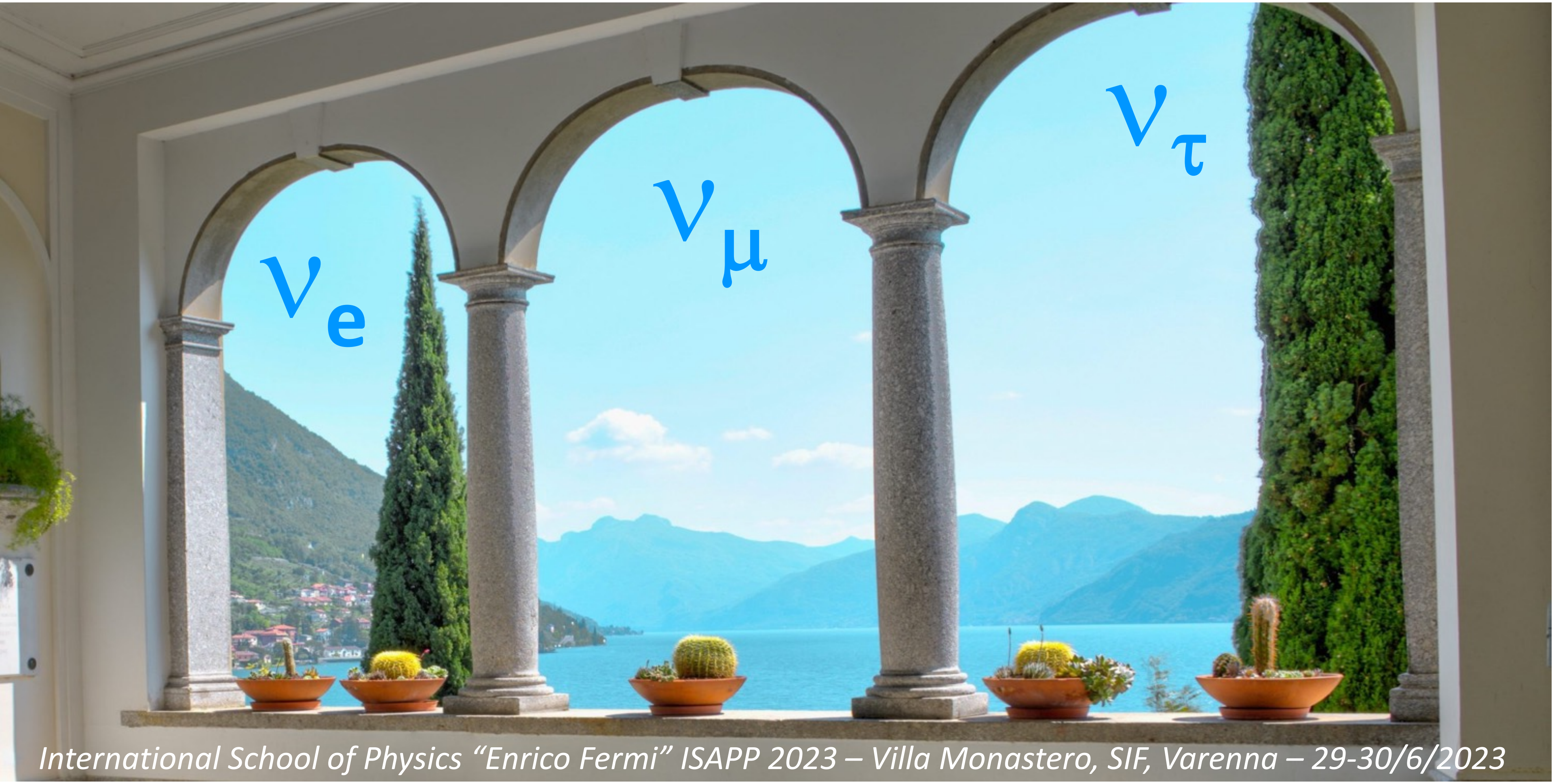
Neutrino Oscillations

Lecture II



Eligio Lisi
(INFN, Bari, Italy)

Outline of lectures:

Lecture I

Pedagogical introduction + warm-up exercise

Lecture II

3ν osc. in vacuum and matter: notation and basic math

Lecture III

2ν approximations of phenomenological interest

Lecture IV

Back to 3ν oscillations: Status and Perspectives

The "standard" 3ν oscillation framework

Physics facts and mass notation:

- There are three mass states ν_1, ν_2, ν_3 with masses m_1, m_2, m_3
- Neutrino oscillations probe the differences $\Delta E \propto \Delta m^2_{ij}$
- There are only two independent Δm^2_{ij} , say, δm^2 and Δm^2
- Experimentally, very different scales: $\delta m^2/\Delta m^2 \sim 1/30$
  Difficult to observe both! Current expts sensitive to a dominant one.

The "standard" 3ν oscillation framework

Physics facts and mass notation:

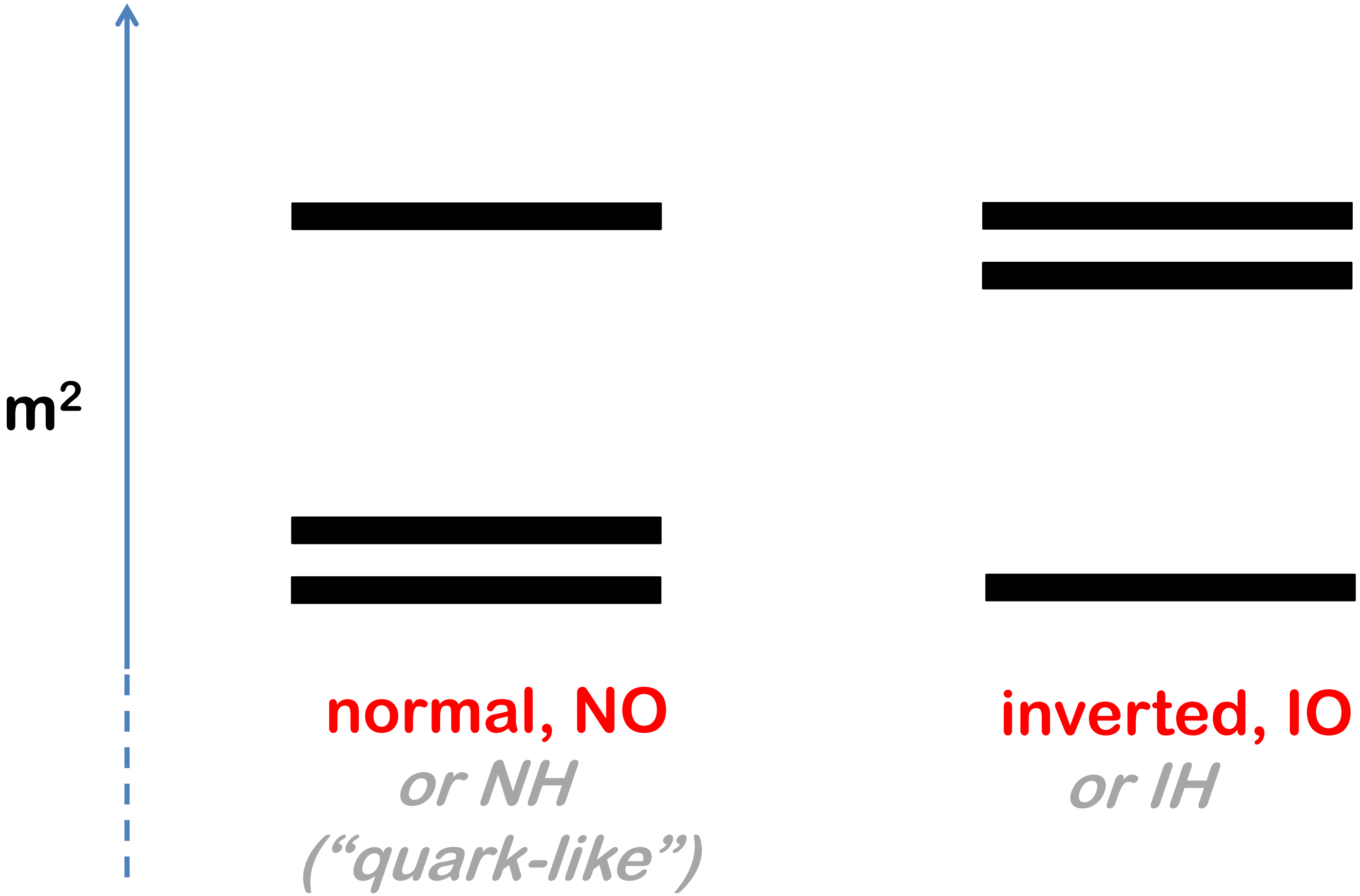
- There are three mass states ν_1, ν_2, ν_3 with masses m_1, m_2, m_3
- Neutrino oscillations probe the differences $\Delta E \propto \Delta m^2_{ij}$
- There are only two independent Δm^2_{ij} , say, δm^2 and Δm^2
- Experimentally, very different scales: $\delta m^2 / \Delta m^2 \sim 1/30$
  Difficult to observe both! Current expts sensitive to a dominant one.

$\delta m^2 \simeq 7.5 \times 10^{-5}\ \mathrm{eV}^2$ ← "small" *or "solar"* splitting

$\Delta m^2 \simeq 2.5 \times 10^{-3}\ \mathrm{eV}^2$ ← "large" *or "atmospheric"* splitting

(somewhat obsolete terms)

Two possible mass orderings (*or hierarchies*)



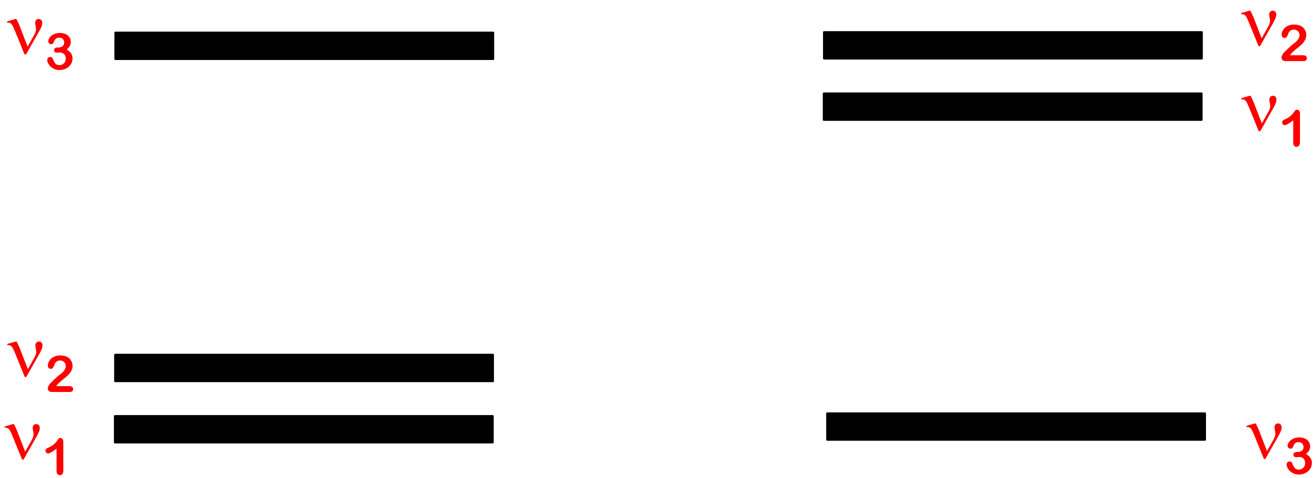
0 ?

Absolute mass scale still unknown, but upper limits exist: **m < O(0.1-1) eV**

PDG convention for 3ν masses:

(ν_1,ν_2) = “close” states, with $m_2 > m_1$ always

ν_3 = “lone” state, with $m_3 > m_{1,2}$ in NO ($m_3 < m_{1,2}$ in IO)



Our notation for splittings: Define as independent ones

$$\delta m^2 = m_2^2 - m_1^2 > 0$$

$$\Delta m^2 = \frac{1}{2}(\Delta m_{31}^2 + \Delta m_{32}^2) \begin{cases} > 0 & \text{NO} \\ < 0 & \text{IO} \end{cases}$$

PDG convention for 3ν mixing:

Three Euler rotations, one being complex

$$\nu_\alpha = U_{\alpha i}\nu_i \qquad \begin{matrix}\alpha = e,\ \mu,\ \tau \\ i = 1,\ 2,\ 3\end{matrix}$$

$$U = \begin{pmatrix} 1 & 0 & 0 \\ 0 & c_{23} & s_{23} \\ 0 & -s_{23} & c_{23} \end{pmatrix} \begin{pmatrix} c_{13} & 0 & s_{13}e^{-i\delta} \\ 0 & 1 & 0 \\ -s_{13}e^{i\delta} & 0 & c_{13} \end{pmatrix} \begin{pmatrix} c_{12} & s_{12} & 0 \\ -s_{12} & c_{12} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

This rotation ordering happens to be particularly useful for phenomenologically interesting limits

$$\begin{bmatrix} U_{e1} & U_{e2} & U_{e3} \\ U_{\mu 1} & U_{\mu 2} & U_{\mu 3} \\ U_{\tau 1} & U_{\tau 2} & U_{\tau 3} \end{bmatrix} = \begin{bmatrix} c_{12}c_{13} & s_{12}c_{13} & s_{13}e^{-i\delta_{CP}} \\ -s_{12}c_{23} - c_{12}s_{23}s_{13}e^{i\delta_{CP}} & c_{12}c_{23} - s_{12}s_{23}s_{13}e^{i\delta_{CP}} & s_{23}c_{13} \\ s_{12}s_{23} - c_{12}c_{23}s_{13}e^{i\delta_{CP}} & -c_{12}s_{23} - s_{12}c_{23}s_{13}e^{i\delta_{CP}} & c_{23}c_{13} \end{bmatrix}$$

$$UU^\dagger = 1 \qquad U \rightarrow U^* \text{ for } \overline{\nu} \qquad c_{ij} = \cos\theta_{ij} \quad s_{ij} = \sin\theta_{ij}$$

Phase: **$\delta = \delta_{CP}$ = "Dirac" phase**. Governs possible CP violation in oscillations

If ν are Majorana: two additional (relative) **"Majorana" phases ϕ_{21}, ϕ_{31}**

$$U \to U U_M \ , \ U_M = \mathrm{diag}[1, \ \phi_{21}, \phi_{31}]$$

The Majorana phases are probed in $0\nu\beta\beta$ decay, but not in oscillations.

Phase: **$\delta = \delta_{CP}$ = "Dirac" phase**. Governs possible CP violation in oscillations

If ν are Majorana: two additional (relative) **"Majorana" phases ϕ_{21}, ϕ_{31}**

$$U \to UU_M \ , \ U_M = \mathrm{diag}[1, \ \phi_{21}, \phi_{31}]$$

The Majorana phases are probed in $0\nu\beta\beta$ decay, but not in oscillations.

Proof: The flavor evolution operator (Lecture I) reads

$$S_f = US_mU^\dagger \qquad \text{with} \qquad S_m = \mathrm{diag}\left[e^{-im_i^2 x/2E}\right]$$

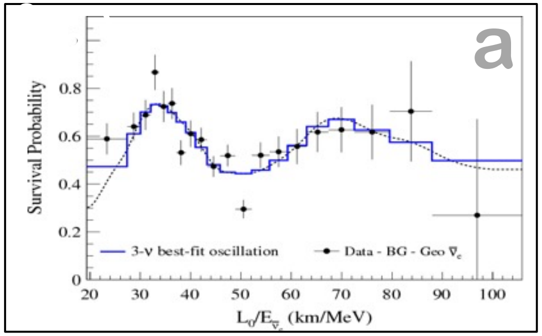
and is unaffected by $U \to UU_M$

$$S_f \to UU_M S_m (UU_M)^\dagger = U(U_M S_m U_M^\dagger)U^\dagger = US_mU^\dagger = S_f$$

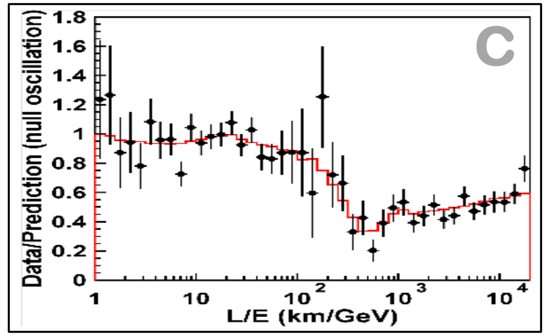
In general, standard 3ν oscillations do not distinguish Dirac/Majorana ν

3ν can explain α → β oscillations seen in vacuum and matter...

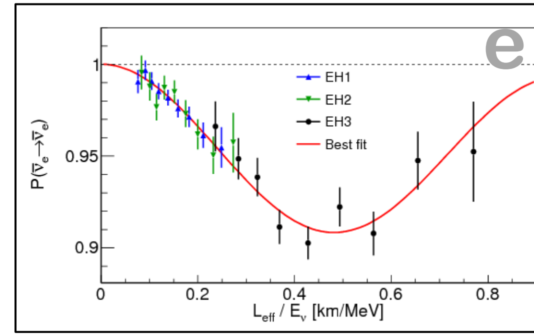
e→e (KamLAND, KL)



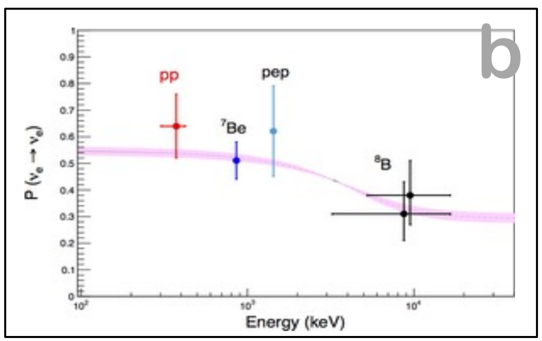
μ→μ (Atmospheric)



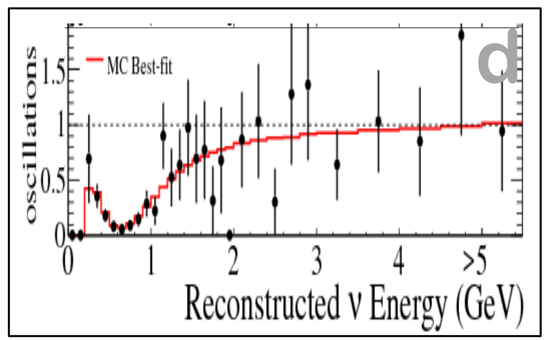
e→e (SBL Reac.)



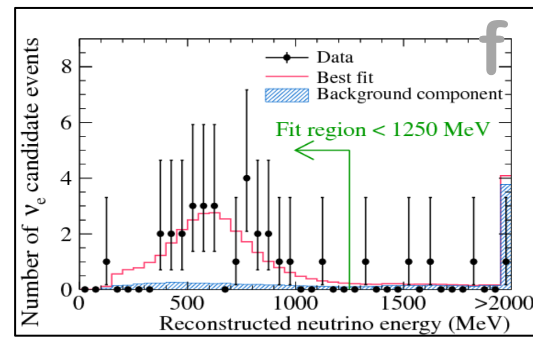
e→e (Solar)



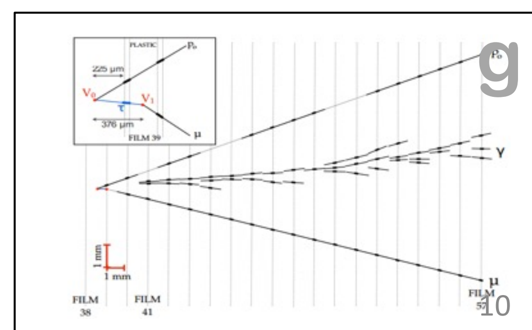
μ→μ (LBL Accel)



μ→e (LBL Accel)



μ→τ (OPERA, SK, DC)

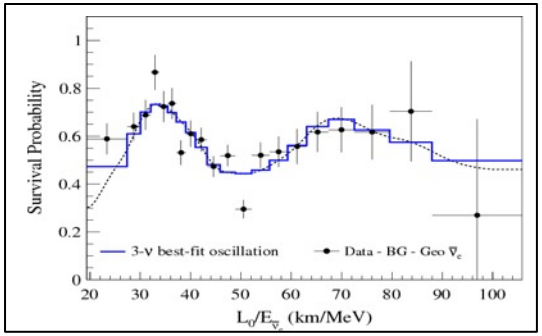


LBL = Long baseline (few x 100 km); **SBL = short baseline** (~1 km)

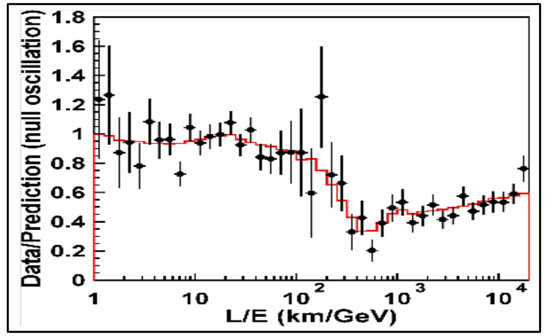
(a) KamLAND reactor [plot]; (b) Borexino [plot], Homestake, Super-K, SAGE, GALLEX/GNO, SNO; (c) Super-K atmosph. [plot], DeepCore, MACRO, MINOS etc.; (d) T2K (plot), NOvA, MINOS, K2K LBL accel.; (e) Daya Bay [plot], RENO, Double Chooz SBL reactor; (f) T2K [plot], MINOS, NOvA LBL accel.; (g) OPERA [plot] LBL accel., Super-K and IC-CD atmospheric.

...with dominant parameters (Lecture III):

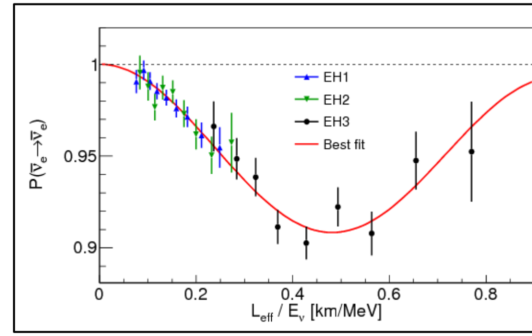
$e \rightarrow e$ (δm^2 , θ_{12})



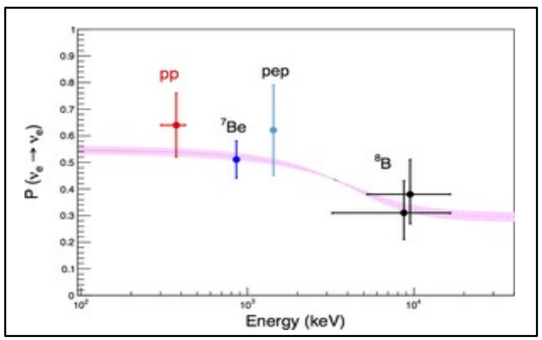
$\mu \rightarrow \mu$ (Δm^2 , θ_{23})



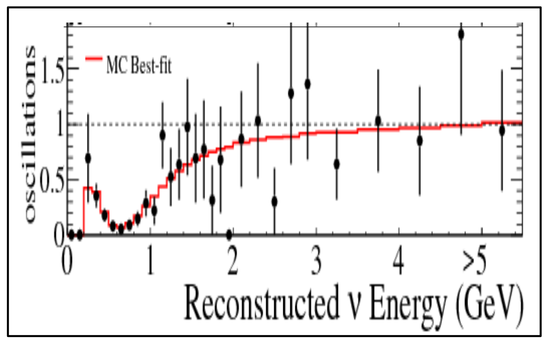
$e \rightarrow e$ (Δm^2 , θ_{13})



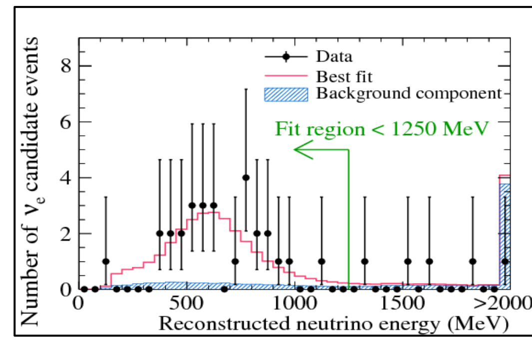
$e \rightarrow e$ (δm^2 , θ_{12})



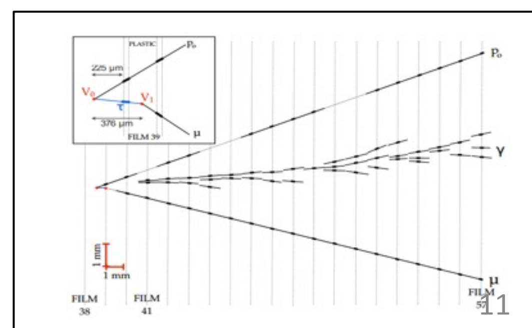
$\mu \rightarrow \mu$ (Δm^2 , θ_{23})



$\mu \rightarrow e$ (Δm^2 , θ_{13} , θ_{23})



$\mu \rightarrow \tau$ (Δm^2 , θ_{23})



Established so far:

δm^2 $|\Delta m^2|$ θ_{12} θ_{23} θ_{13}

δm^2	$\sim 7 \times 10^{-5}$ eV^2	← "small" splitting
$\lvert\Delta m^2\rvert$	$\sim 2 \times 10^{-3}$ eV^2	← "large" splitting
$\sin^2\theta_{12}$	~ 0.3	← "large" 12 mixing
$\sin^2\theta_{23}$	~ 0.5	← "nearly maximal" 23 mixing
$\sin^2\theta_{13}$	~ 0.02	← "small" 13 mixing

We shall now deal with

3ν flavor evolution

in vacuum and in matter

δm^2	$\sim 7 \times 10^{-5}$ eV^2
$\lvert\Delta m^2\rvert$	$\sim 2 \times 10^{-3}$ eV^2
$\sin^2\theta_{12}$	~ 0.3
$\sin^2\theta_{23}$	~ 0.5
$\sin^2\theta_{13}$	~ 0.02

The presence of two small dimensionless parameters,

$$\delta m^2 / \Delta m^2 \sim 3 \times 10^{-2}$$

$$\sin^2\theta_{13} \sim 2 \times 10^{-2}$$

will allow useful $3\nu \rightarrow 2\nu$ approximations and simplify the understanding of phenomenology (next Lecture)

We shall now deal with

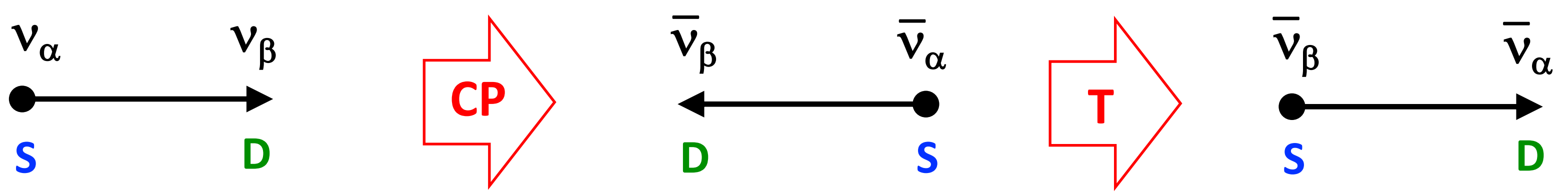
3ν flavor evolution

in vacuum and in matter

Three-neutrino flavor evolution: CP, T symmetries *(intuitive approach)*

C = charge conjugation (particle-antiparticle exchange)
P = parity (space reversal)
T = time reversal

Action of **CP** and **T** on $\nu_\alpha \rightarrow \nu_\beta$ oscillations from source **S** to detector **D**:

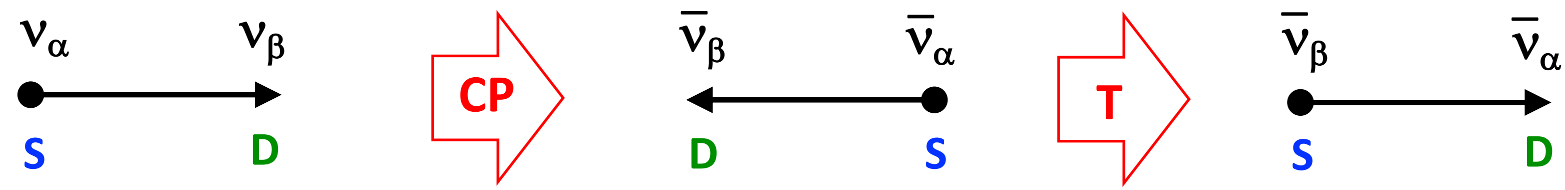


(CP violation: one of the Sakharov conditions to generate matter-antimatter asymmetry in the Universe)

Three-neutrino flavor evolution: CP, T symmetries *(intuitive approach)*

C = charge conjugation (particle-antiparticle exchange)
P = parity (space reversal)
T = time reversal

Action of **CP** and **T** on $\nu_\alpha \to \nu_\beta$ oscillations from source **S** to detector **D**:

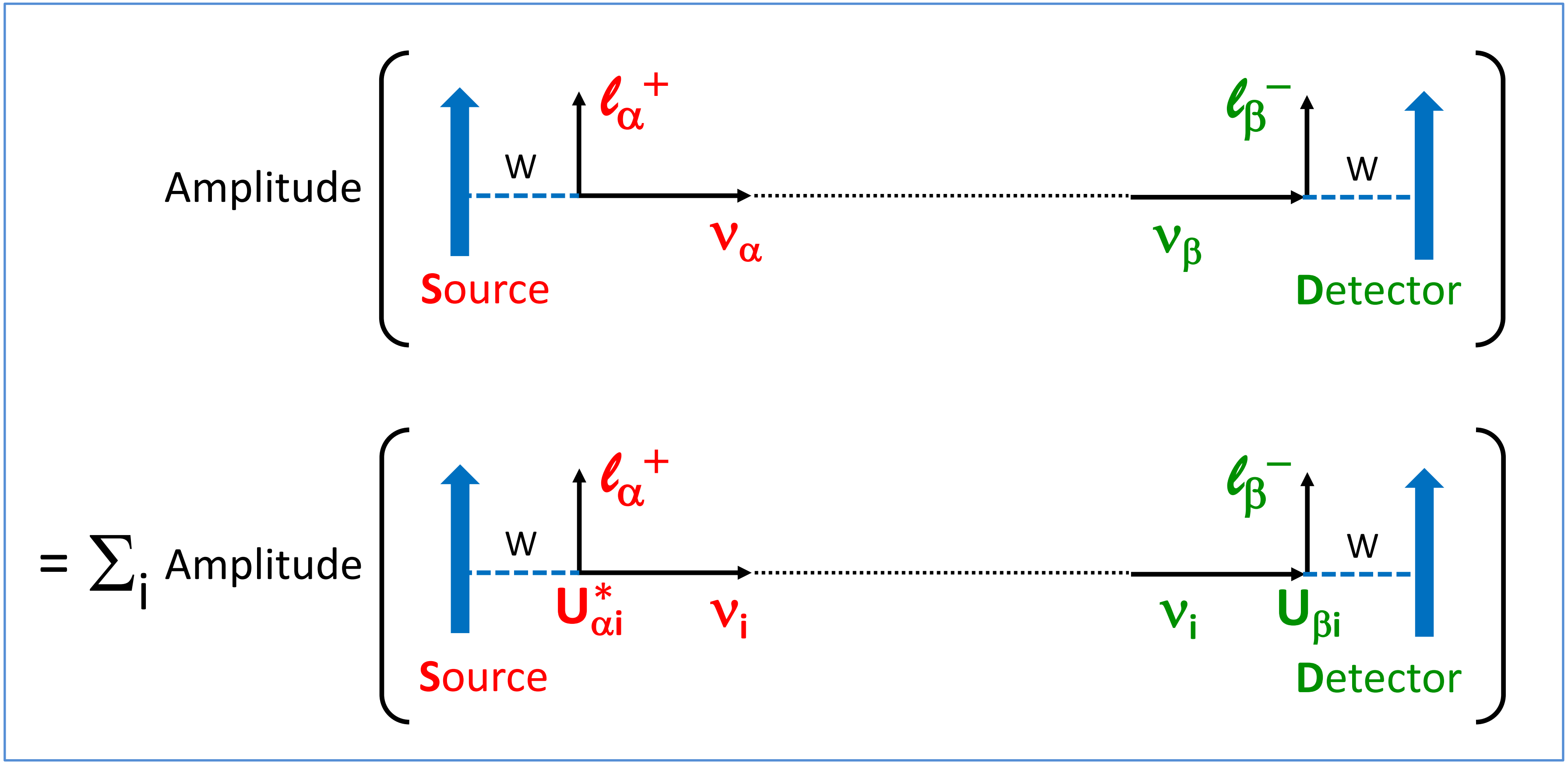


If **CP** invariance holds, then $P(\nu_\alpha \to \nu_\beta) = P(\bar{\nu}_\alpha \to \bar{\nu}_\beta) \quad \Leftarrow \quad (\nu \leftrightarrow \bar{\nu})$

If **T** invariance holds, then $\begin{cases} P(\nu_\alpha \to \nu_\beta) = P(\nu_\beta \to \nu_\alpha) \\ P(\bar{\nu}_\alpha \to \bar{\nu}_\beta) = P(\bar{\nu}_\beta \to \bar{\nu}_\alpha) \end{cases} \quad \Leftarrow \quad (\alpha \leftrightarrow \beta)$

If **CPT** invariance holds, then $P(\nu_\alpha \to \nu_\beta) = P(\bar{\nu}_\beta \to \bar{\nu}_\alpha) \quad \Leftarrow \quad (\nu \leftrightarrow \bar{\nu}) \oplus (\alpha \leftrightarrow \beta)$

From Lecture I:

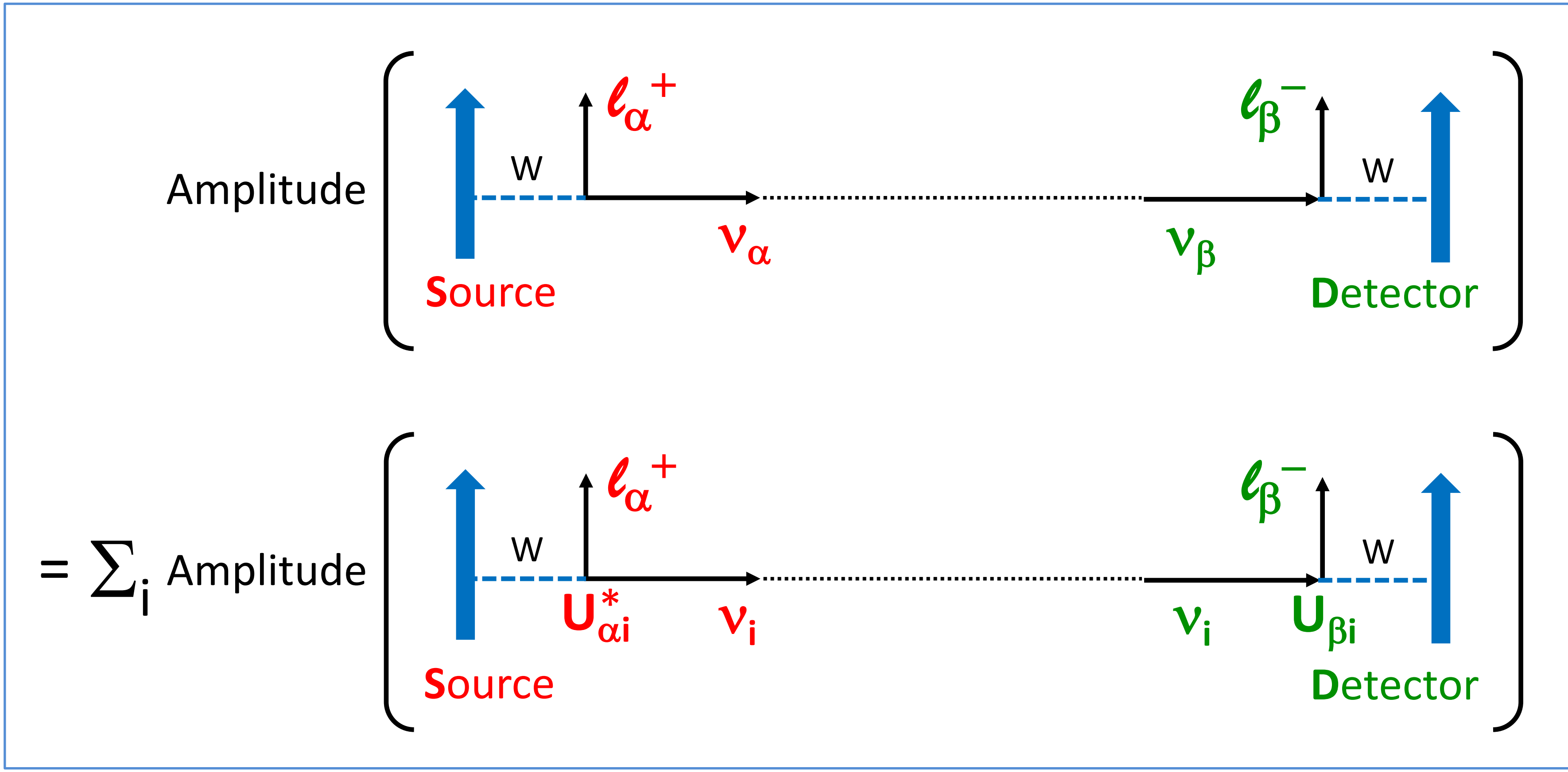


for $\nu \rightarrow \bar{\nu}$

$\ell_\alpha^\pm \rightarrow \ell_\alpha^\mp$

$U \rightarrow U^*$

From Lecture I:



Also, swapping initial/final flavors is equivalent to

$\mathbf{U \rightarrow U^*}$

for $\nu \rightarrow \bar{\nu}$

$\ell_\alpha^\pm \rightarrow \ell_\alpha^\mp$

$\mathbf{U \rightarrow U^*}$

$$(\nu \to \bar{\nu}) \Leftrightarrow U \to U^*$$
$$(\alpha \leftrightarrow \beta) \Leftrightarrow U \to U^*$$

If **CP** invariance holds, then $P(\nu_\alpha \to \nu_\beta) = P(\bar{\nu}_\alpha \to \bar{\nu}_\beta) \quad \Leftarrow \quad (\nu \leftrightarrow \bar{\nu})$

If **T** invariance holds, then $\begin{cases} P(\nu_\alpha \to \nu_\beta) = P(\nu_\beta \to \nu_\alpha) \\ P(\bar{\nu}_\alpha \to \bar{\nu}_\beta) = P(\bar{\nu}_\beta \to \bar{\nu}_\alpha) \end{cases} \quad \Leftarrow \quad (\alpha \leftrightarrow \beta)$

If **CPT** invariance holds, then $P(\nu_\alpha \to \nu_\beta) = P(\bar{\nu}_\beta \to \bar{\nu}_\alpha) \quad \Leftarrow \quad (\nu \leftrightarrow \bar{\nu}) \oplus (\alpha \leftrightarrow \beta)$

CP and **T** invariance hold iff **U real** (sin $\delta = 0$)
CPT invariance holds for **any U** (as it should)

Exercise: General 3ν oscillations in vacuum

The oscillation probability in vacuum can be written as:

$$P_{\alpha\beta} = \delta_{\alpha\beta} - 4\sum_{i<j} \mathrm{Re}\, J^{ij}_{\alpha\beta} \sin^2\left(\frac{\Delta m^2_{ij} x}{4E}\right) - 2\sum_{i<j} \mathrm{Im}\, J^{ij}_{\alpha\beta} \sin\left(\frac{\Delta m^2_{ij} x}{2E}\right)$$

where:

$$\Delta m^2_{ij} = m^2_i - m^2_j$$

$$J^{ij}_{\alpha\beta} = U_{\alpha i} U^*_{\beta i} U^*_{\alpha j} U_{\beta j}$$

This formula applyes to any flavor α and β:

$\alpha \neq \beta$ → *appearance channel, appearance probability*

$\alpha = \beta$ → *disappearance channel, survival probability*

Despite its simplicity, it contains a lot of physics…

CP properties:

$$P_{\alpha\beta} = \delta_{\alpha\beta} \underbrace{- 4\sum_{i<j} \mathrm{Re}\, J^{ij}_{\alpha\beta} \sin^2\left(\frac{\Delta m^2_{ij} x}{4E}\right)} \underbrace{- 2\sum_{i<j} \mathrm{Im}\, J^{ij}_{\alpha\beta} \sin\left(\frac{\Delta m^2_{ij} x}{2E}\right)}$$

CP-conserving part P^{CP}
not changing for U↔U*

CP-violating part P^{CPV}
changes sign for U↔U*

Re

Im

$$J^{ij}_{\alpha\beta} = U_{\alpha i} U^*_{\beta i} U^*_{\alpha j} U_{\beta j}$$

For $\alpha = \beta :\ \mathrm{Im}(J^{ij}_{\alpha\beta}) = 0$ → **CP violation** must be probed in appearance, $\alpha \neq \beta$

Exercise: J = Jarlskog invariant

Using the previous PDG convention for the mixing matrix, it's easy to find that:

$$J = \mathrm{Im}(J^{12}_{e\mu}) = \frac{1}{8} \sin 2\theta_{12} \sin 2\theta_{23} \sin 2\theta_{13} \cos \theta_{13} \sin \delta$$

Then, using the unitarity of U, for $\alpha \neq \beta$ it is:

$$\mathrm{Im}(J^{ij}_{\alpha\beta}) = \begin{cases} +J & \text{for } (\alpha,\, \beta) = (e,\, \mu),\ (\mu,\, \tau),\ (\tau,\, e) & \leftarrow \text{flavor cyclic} \\ +J & \text{for } (i,\, j) = (1,\, 2),\ (2,\, 3),\ (3,\, 1) & \leftarrow \text{generation cyclic} \\ -J & \text{otherwise} \end{cases}$$

Note that J changes sign from neutrinos to antineutrinos in the same channel

EPS Prize 2023 to Cecilia Jarlskog:
for the discovery of an invariant measure of CP violation in both quark and lepton sectors
EPS Prize 2023 also to … wait for a few more slides!

Exercise: P^{CPV} in product form

Using J, the term P^{CPV} can be written as

$$\boxed{P_{\alpha\beta}^{\mathrm{CPV}} = \pm 8J \sin\left(\frac{\Delta m_{12}^2 x}{4E}\right) \sin\left(\frac{\Delta m_{23}^2 x}{4E}\right) \sin\left(\frac{\Delta m_{31}^2 x}{4E}\right)}$$

Summary of conditions to have CP violation

$$J = \mathrm{Im}(J^{12}_{e\mu}) = \frac{1}{8}\sin 2\theta_{12}\sin 2\theta_{23}\sin 2\theta_{13}\cos\theta_{13}\sin\delta$$

$$P^{\mathrm{CPV}}_{\alpha\beta} = \pm 8J\sin\left(\frac{\Delta m^2_{12}x}{4E}\right)\sin\left(\frac{\Delta m^2_{23}x}{4E}\right)\sin\left(\frac{\Delta m^2_{31}x}{4E}\right)$$

$\delta \neq 0, \pi$ ← U must be complex, sinδ≠0

$\alpha \neq \beta$ ← Need appearance experiments

$\theta_{ij} \neq 0$ ← All mixing angles should be ≠0

$\Delta m^2_{ij} \neq 0$ ← Need some sensitivity to both δm^2 and Δm^2

CP-violating oscillations involve all the mass-mixing parameters

(genuine 3ν phenomenon, experimentally challenging!)

Summary of conditions to have CP violation

$$J = \mathrm{Im}(J_{e\mu}^{12}) = \frac{1}{8} \sin 2\theta_{12} \sin 2\theta_{23} \sin 2\theta_{13} \cos \theta_{13} \sin \delta$$

2012: θ_{13} is nonzero!

2023 EPS High Energy and Particle Physics Prize is awarded to

Cecilia Jarlskog for the discovery of an invariant measure of CP violation in both quark and lepton sectors; and to the

Daya Bay and RENO collaborations for the observation of short-baseline reactor electron-antineutrino disappearance, providing the first determination of the neutrino mixing angle θ_{13}, which paves the way *for* the detection of CP violation in the lepton sector.

At present: only hints of leptonic CPV… (Lecture IV)

In principle… All the oscill. parameters + CP phase + NO/IO might be determined by precise measurements of $P_{\alpha\beta}$ for selected channels and L/E:

$$P_{\alpha\beta} = \delta_{\alpha\beta} - 4\sum_{i<j} \mathrm{Re}\, J^{ij}_{\alpha\beta} \sin^2\left(\frac{\Delta m^2_{ij} x}{4E}\right) - 2\sum_{i<j} \mathrm{Im}\, J^{ij}_{\alpha\beta} \sin\left(\frac{\Delta m^2_{ij} x}{2E}\right)$$

In principle... All the oscill. parameters + CP phase + NO/IO might be determined by precise measurements of $P_{\alpha\beta}$ for selected channels and L/E:

$$P_{\alpha\beta} = \delta_{\alpha\beta} - 4\sum_{i<j} \mathrm{Re}\, J^{ij}_{\alpha\beta} \sin^2\left(\frac{\Delta m^2_{ij} x}{4E}\right) - 2\sum_{i<j} \mathrm{Im}\, J^{ij}_{\alpha\beta} \sin\left(\frac{\Delta m^2_{ij} x}{2E}\right)$$

In practice... We never measure $P_{\alpha\beta}$ but event rates R, and do it only for experimentally feasible oscillation channels and L/E:

$$\mathbf{R}_\beta \sim \int \mathbf{\Phi}_\alpha \otimes \mathbf{P}_{\alpha\beta} \otimes \boldsymbol{\sigma}_\beta \otimes \boldsymbol{\epsilon}_\beta$$

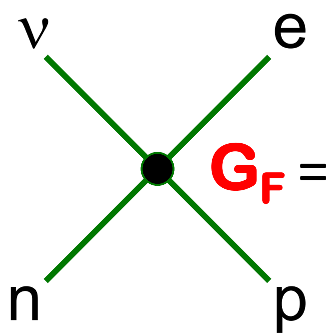
Observable event rate	Source flux (production)	Propagation (flavor change)	Interaction and detection

→ each ingredient of R is a research field of its own

→ need to account for a vast ν phenomenology

→ must consider realistic propagation, e.g., in matter

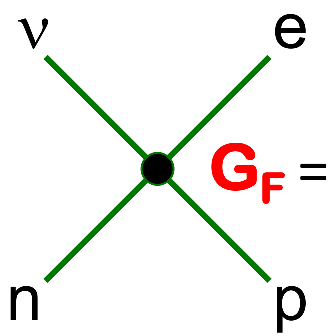
Three-neutrino flavor evolution in matter *(intuitive approach)*



$\mathbf{G_F}$ = 1.166 x 10^{-5} GeV^{-2} → Sets the energy scale $\sqrt{(1/G_F)}$~O(few 10^2) GeV of weak interactions

Amplitude squared → Neutrino cross section ∝ $\mathbf{G_F}^2$

Three-neutrino flavor evolution in matter *(intuitive approach)*



$G_F = 1.166 \times 10^{-5}\ \mathrm{GeV}^{-2}$ → Sets the energy scale $\sqrt{(1/G_F)} \sim O(\text{few } 10^2)$ GeV of weak interactions

Amplitude squared → Neutrino cross section $\propto G_F^2$

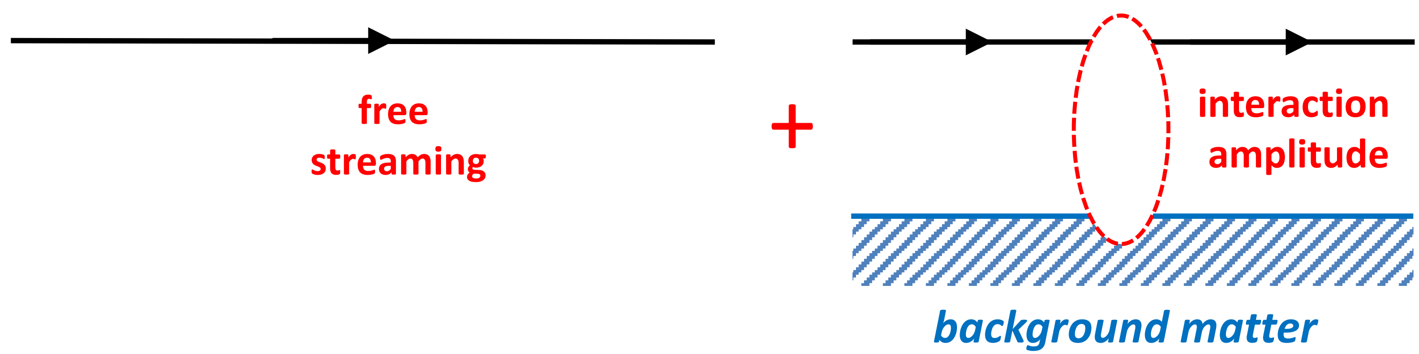
During propagation, **absorption processes** $\propto G_F^2$ are negligible, except at very high energy. E.g., Earth starts to be opaque to ν for E > O (10) TeV.

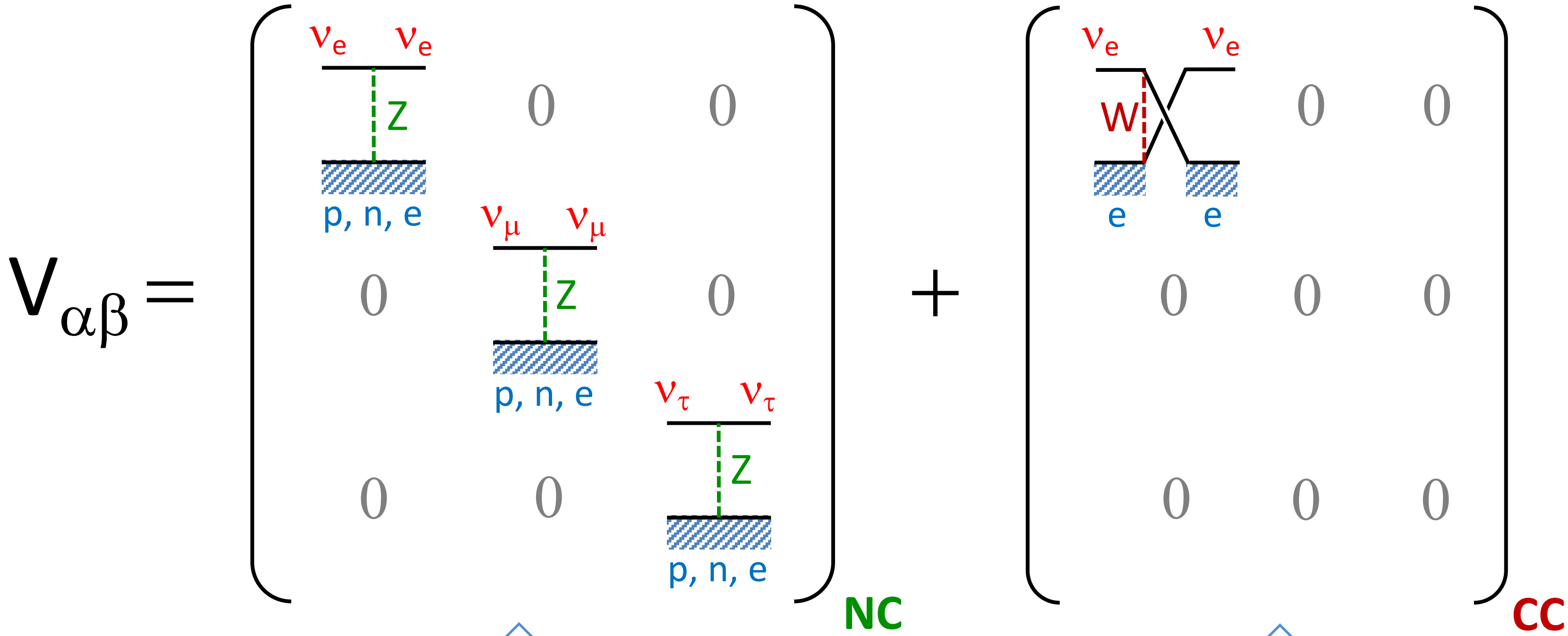
But: Are also **scattering amplitudes** $\propto G_F$ irrelevant for ν flavor evolution? Not necessarily, if the process occurs along the direction of propagation!

It was first realized by Wolfenstein, and later elaborated by Mykheev and Smirnov (**MSW**), that neutrinos travelling in a fermion background receive a contribution to coherent forward scattering (i.e., along the same direction of propagation) in the form of a tiny interaction energy $\mathbf{V_{\alpha\beta}}$

3ν Hamiltonian in matter:

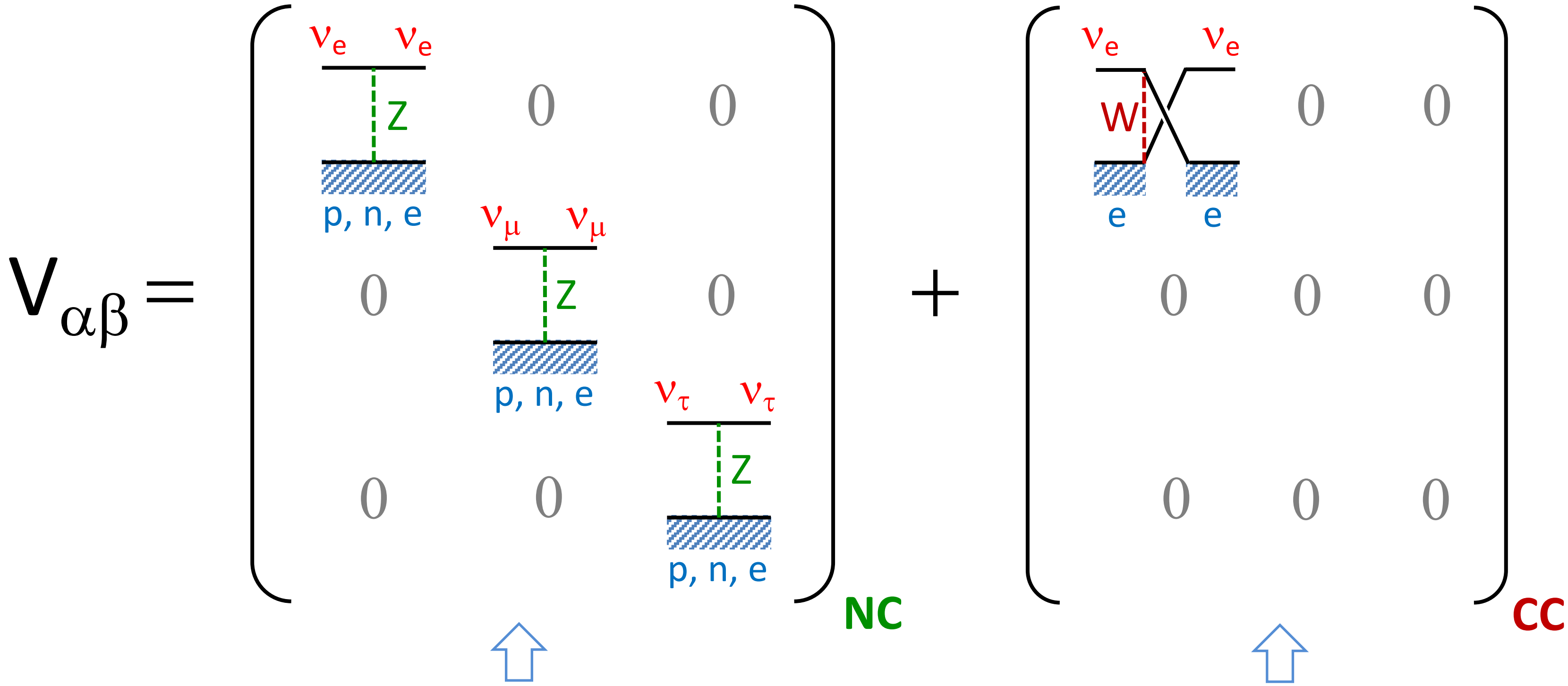
$$H_f = \overbrace{\frac{1}{2E} U \begin{pmatrix} m_1^2 & & \\ & m_2^2 & \\ & & m_3^2 \end{pmatrix} U^\dagger}^{\text{vacuum (kinematics)}} + \overbrace{\begin{pmatrix} V_{ee} & V_{e\mu} & V_{e\tau} \\ V_{\mu e} & V_{\mu\mu} & V_{\mu\tau} \\ V_{\tau e} & V_{\tau\mu} & V_{\tau\tau} \end{pmatrix}}^{\text{matter (dynamics)}}$$





proportional to unity
→ unobservable

observable in
ν_e-related oscill.



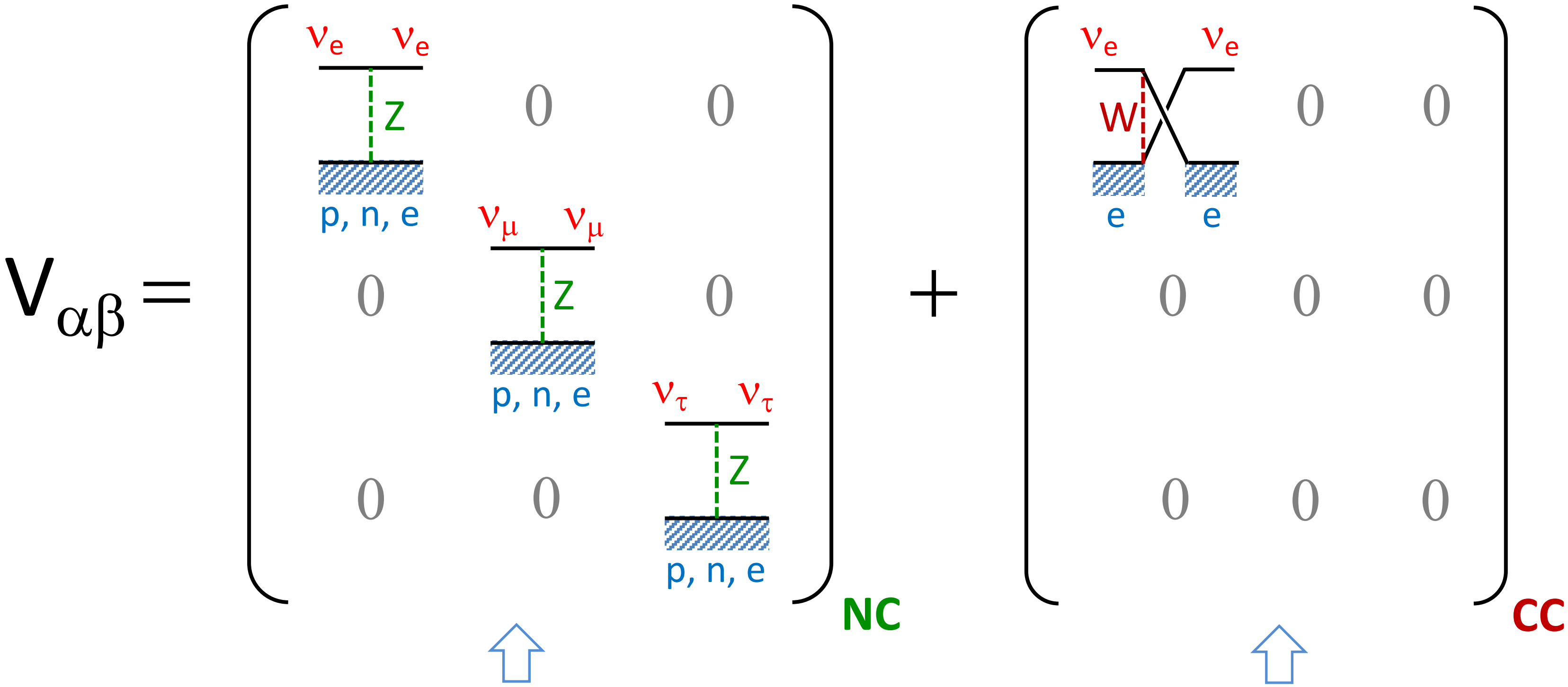
proportional to unity
→ unobservable

observable in
ν_e-related oscill.

Relevant term is the extra "*ee*" energy (ν potential) **$V_{CC} \propto G_F$**

[No μ,τ in ordinary matter → no "$\mu\mu$" or "$\tau\tau$" CC term]

Potential proportional to e^- number density **$V_{CC} \propto N_e$**



proportional to unity
→ unobservable

observable in
ν_e-related oscill.

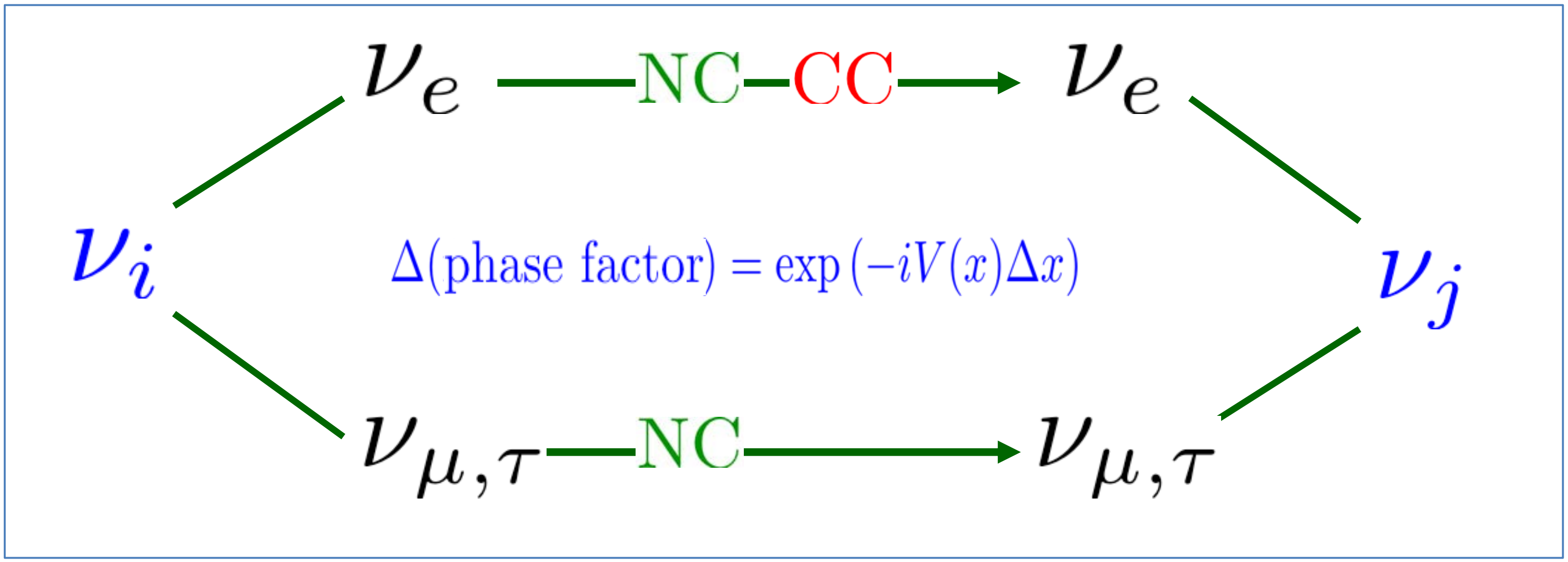
Relevant term is the extra "*ee*" energy (ν potential) $V_{CC} \propto G_F$

Potential proportional to e^- number density $V_{CC} \propto N_e$

Full calculation (omitted): $V\ (= V_{CC}) = \sqrt{2}\, G_F\, N_e$

As anticipated in Lecture I:

Analogy of matter effects with double-slit experiment:
one "arm" (e-flavor) feels a different "refraction index"
through coherent forward scattering (not absorption!)



Governed by a tiny ν "interaction energy" or "potential" V
In general, V=V(x) via $N_e=N_e(x)$ → x-dependent hamiltonian.
Not necessarily periodic effects: oscillations → transitions

3ν MSW hamiltonian in matter:

$$H_f(x) = \frac{1}{2E}\left[U\begin{pmatrix} m_1^2 & & \\ & m_2^2 & \\ & & m_3^2 \end{pmatrix}U^\dagger + \begin{pmatrix} A(x) & & \\ & 0 & \\ & & 0 \end{pmatrix}\right]$$

where A=2EV is introduced to make the vacuum and matter terms more similar:

$$A(x) = 2EV = 2\sqrt{2}G_F N_e(x)E$$ [$-A(x)$ for anti-ν]

Huge literature on solutions for various A(x). In general:

Expect **sizeable matter effects** when the two terms in **H** are comparable,

$$\frac{A}{\delta m^2} \sim O(1)$$ or $$\frac{A}{\Delta m^2} \sim O(1)$$

Exercise: Units for matter effects

$$\frac{A}{\Delta m^2_{ij}} = 1.526 \times 10^{-7} \left(\frac{N_e}{\mathrm{mol/cm^3}} \right) \left(\frac{E}{\mathrm{MeV}} \right) \left(\frac{\mathrm{eV}^2}{\Delta m^2_{ij}} \right)$$

Note: **N_e** is the number of electrons per unit volume. If the chemical composition is known (say, the **average Z/A**), one can connect N_e and the matter density ρ:

$$\frac{N_e}{\mathrm{mol/cm^3}} \simeq \left\langle \frac{Z}{A} \right\rangle \frac{\rho}{\mathrm{g/cm^3}}$$

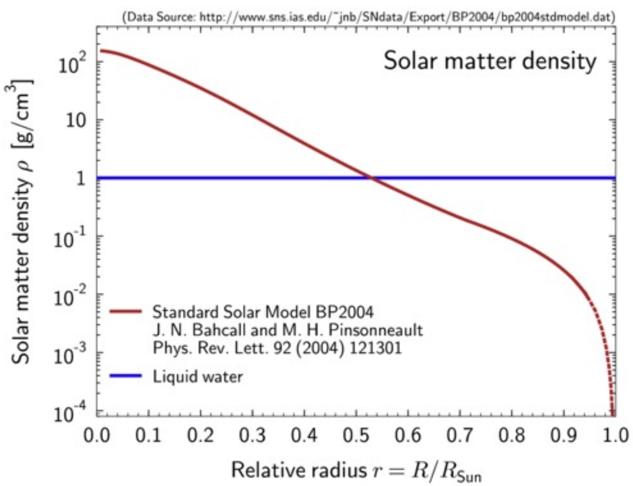
↑

electron fraction $Y_e \sim 1/2$

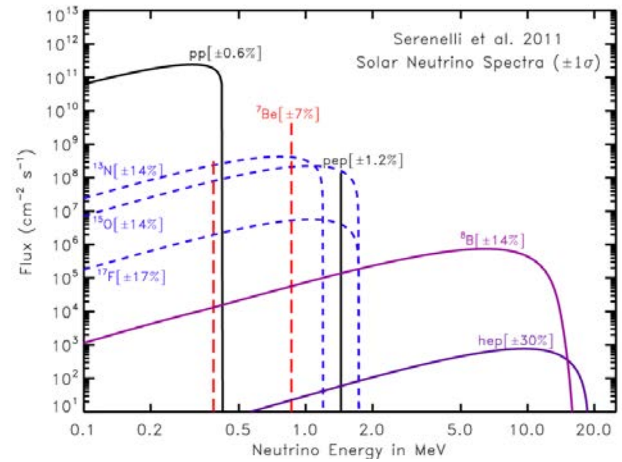
Order-of-magnitude expectations →

Solar neutrinos

Solar density profile



Solar neutrino spectrum

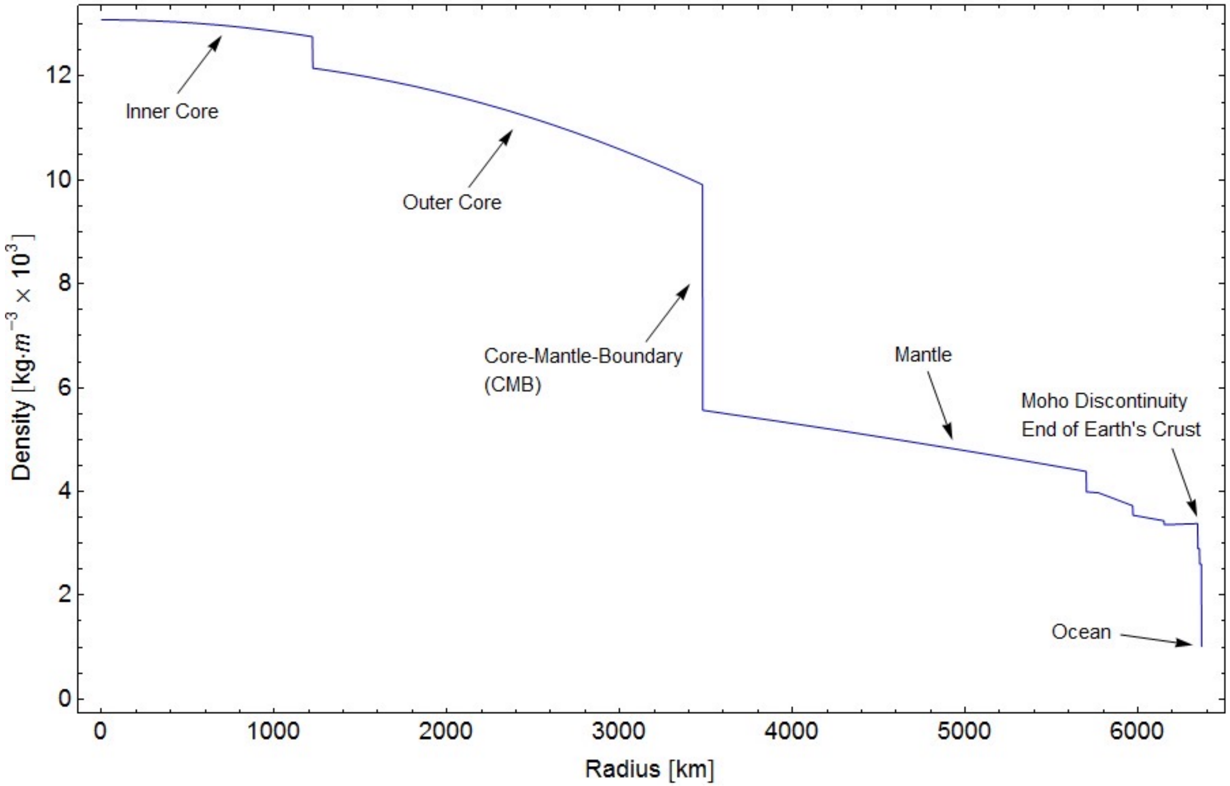


$$\frac{A_{\rm Sun}(0)}{\delta m^2} \sim O(1)$$ just in the middle of the E spectrum…

$$\frac{A_{\rm Earth}}{\delta m^2} \sim O(\text{few} \times 10^{-2})$$ at $E \sim O(10)$ MeV….

Large effects in solar matter; subleading day-night effects in Earth

Earth density profile

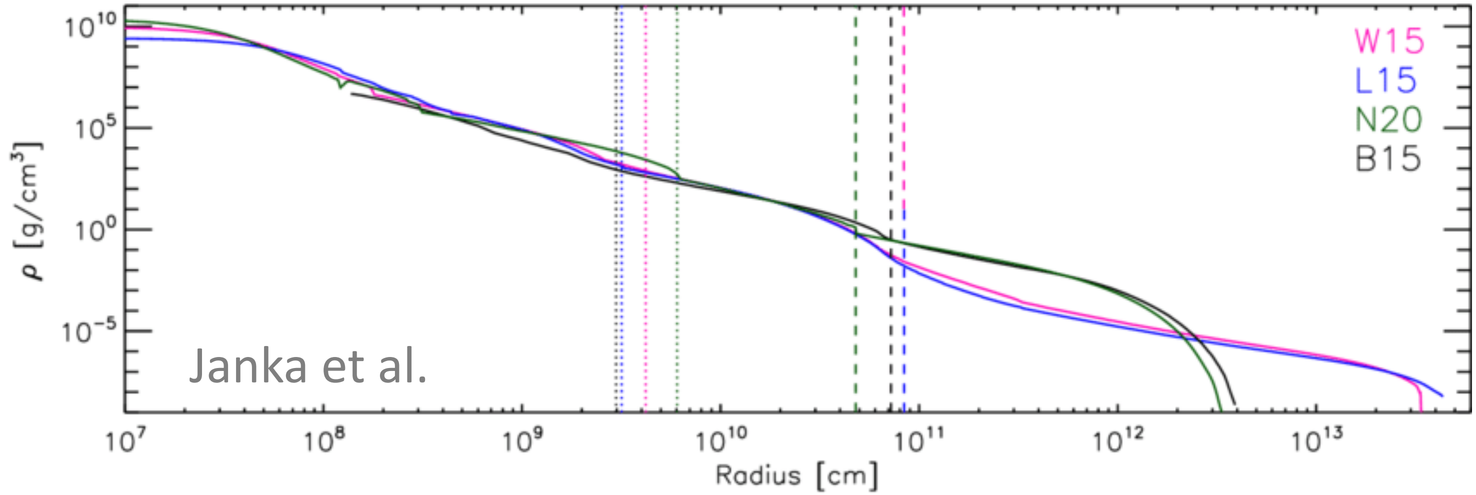


$$\frac{A_{\rm crust}}{\Delta m^2} \sim O(10^{-1})$$ at $E \sim O(1)$ GeV….

Subleading matter effects in long-baseline accelerator neutrinos

Depend on sign of numerator vs denominator: handle on NO/IO

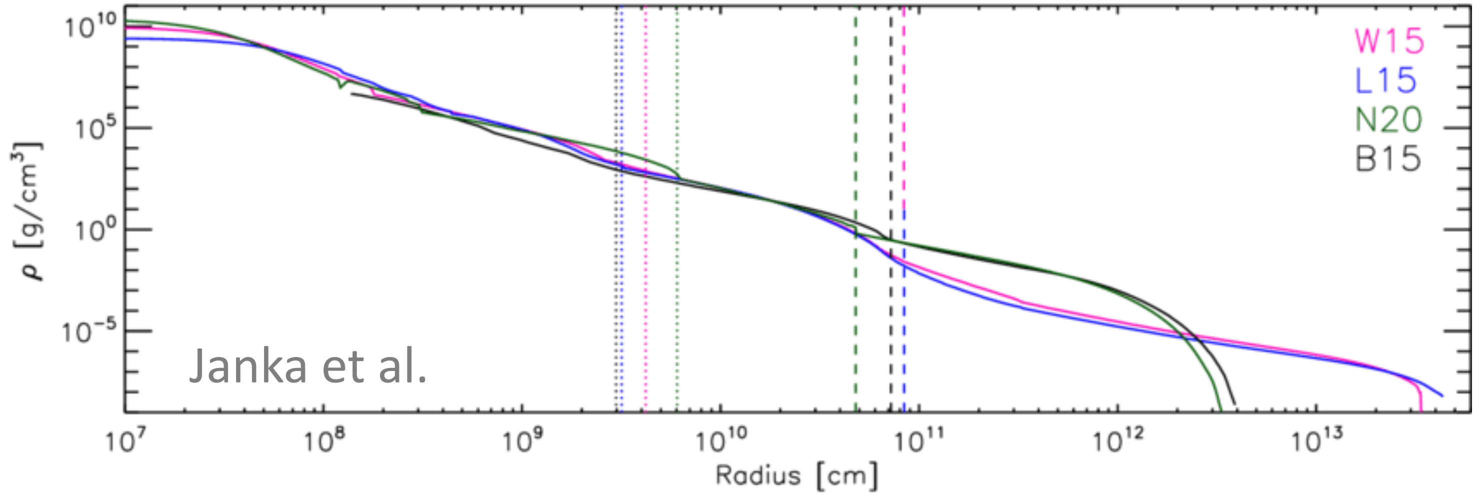
Supernova neutrino density profile(s)



The most dynamical (radius- and time-dependent) matter background:

$\mathbf{N_e=N_e(x,t)}$

Supernova neutrino density profile(s)



The most dynamical (radius- and time-dependent) matter background:

$\mathbf{N_e=N_e(x,t)}$

Moreover, for a few seconds, neutrinos are a background to themselves!

$\mathbf{N_\nu \sim O(N_e)}$

→ "Self-interaction" effects, "collective" highly-nonlinear flavor evolution

H=H(ν) → density matrix formalism

Physics & math to be (better) understood: a research topic in its own!

3ν MSW hamiltonian in matter:

$$H_f(x) = \frac{1}{2E}\left[U\begin{pmatrix} m_1^2 & & \\ & m_2^2 & \\ & & m_3^2 \end{pmatrix}U^\dagger + \begin{pmatrix} A(x) & & \\ & 0 & \\ & & 0 \end{pmatrix}\right]$$

Huge related literature on numerical and (semi)analytical solutions at given A(x)

Oscillating behavior makes numerical solutions prone to error accumulation: brute-force application of Runge-Kutta codes may fail [published examples…]
(Semi)Analytical solutions/approximations useful whenever A(x) is "simple"

We shall consider the two simplest cases of phenomenological interest:

A(x) = constant **[dA/dx = 0]** ← **constant density case**
A(x) = slowly varying **[dA/dx = "small"]** ← **adiabatic case**

[A(x) = rapidly varying [dA/dx = "large"] ← non-adiabatic case]

Effective parameters. In general, $H_f(x)$ can be diagonalized at each point x:

$$H_f(x) = \frac{1}{2E}\left[U \begin{pmatrix} m_1^2 & & \\ & m_2^2 & \\ & & m_3^2 \end{pmatrix} U^\dagger + \begin{pmatrix} A(x) & & \\ & 0 & \\ & & 0 \end{pmatrix} \right]$$

$$= \frac{1}{2E} \tilde{U}(x) \begin{pmatrix} \tilde{m}_1^2(x) & & \\ & \tilde{m}_2^2(x) & \\ & & \tilde{m}_3^2(x) \end{pmatrix} \tilde{U}^\dagger(x)$$

where

$$\begin{bmatrix} \nu_e \\ \nu_\mu \\ \nu_\tau \end{bmatrix} = \tilde{U}(x) \begin{bmatrix} \tilde{\nu}_1(x) \\ \tilde{\nu}_2(x) \\ \tilde{\nu}_3(x) \end{bmatrix}$$

"effective..." ...mixing matrix ...mass states ...squared masses... ...in matter

A = constant: diagonalize only once (at any energy E) and exponentiate

$$\tilde{S}_f = e^{-i\tilde{H}_f x}$$

Get the same probability functions as in vacuum, but with "effective" energy-dependent mass-mixing oscillation parameters in matter.
Relevant application: matter effects in LBL accelerator experiments

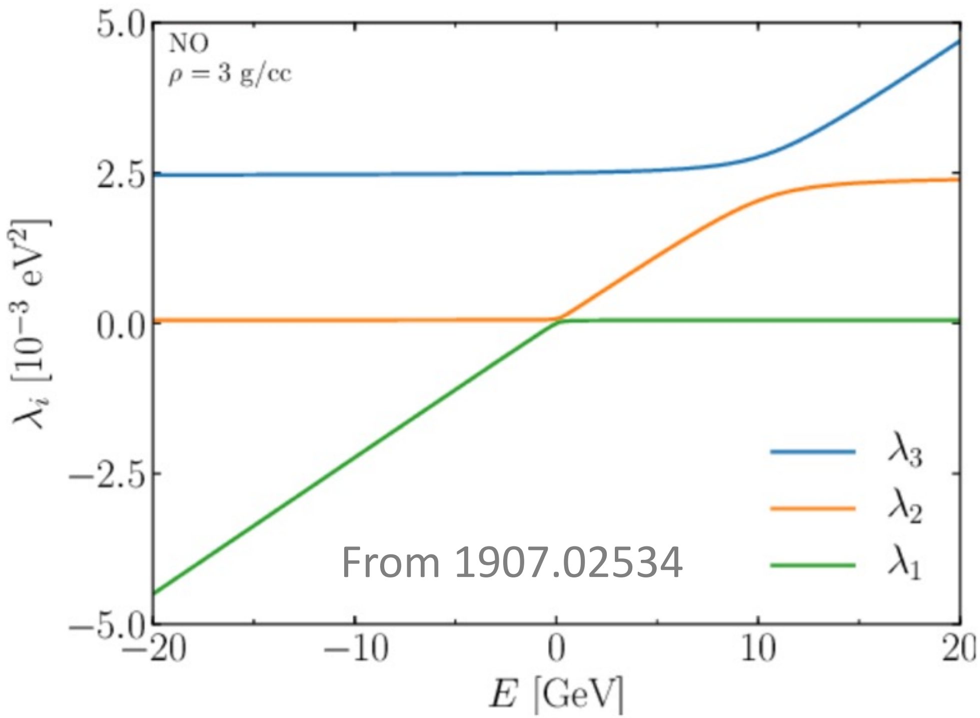
A = constant: diagonalize only once (at any energy E) and exponentiate

$$\tilde{S}_f = e^{-i\tilde{H}_f x}$$

Get the same probability functions as in vacuum, but with "effective" energy-dependent mass-mixing oscillation parameters in matter.
Relevant application: matter effects in LBL accelerator experiments

Examples of effective $\tilde{m}^2_i$ for Earth crust density as functions of energy E

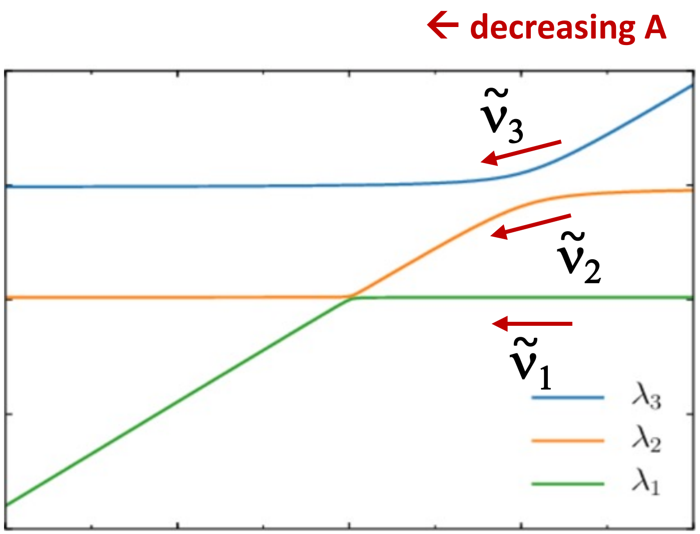
[Here, formally E<0 for anti-nu, since A → −A]



A(x) slowly varying: diagonalize step by step, and patch the solutions

Exercise: $$d\tilde{U}/dx \simeq 0 \;\Rightarrow\; \tilde{\nu}_i(0) \to \tilde{\nu}_i(x)$$

i.e., the effective mass states in matter evolve independently.
The adiabaticity condition can be formulated precisely (omitted).
Applicable, e.g., in stars with smoothly decreasing density…



…. for the oscillation parameters chosen by Nature!

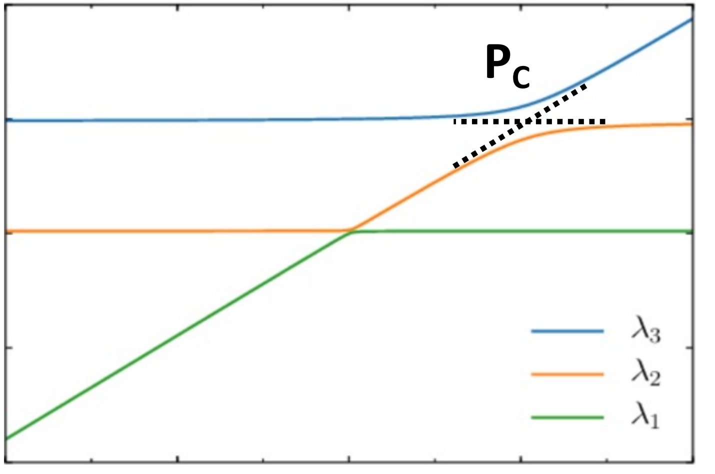
Relevant applications: (1) matter effects for solar neutrinos
(2) for SN neutrinos, up to shock-wave and collective effects

A(x) rapidly varying: "crossing" probability between effective states

[In QM: "tunnelling" between E eigenstates subject to rapid externals field variations]

$$P_c = P(\tilde{\nu}_i \to \tilde{\nu}_j)$$

For a 2-level QM system:
Solved independently by
Majorana, Landau, Zener,
Stueckelberg in 1932
(at leading order)



This would have happened to solar neutrinos at very small mixing!

[so-called Small Mixing Angle MSW solution, a prejudice for many years…]

Relevant applications: (1) steps in Earth density profile
(2) Shock-wave front in SN neutrinos (discussion omitted)

Similar effects (re)analyzed independently in many subfields of physics but with different "jargon" – e.g. recently in qbit manipulations across QM levels

Recap

3ν oscillations in vacuum (may be CP violating):

$$P_{\alpha\beta} = \delta_{\alpha\beta} - 4\sum_{i<j} \mathrm{Re}\, J_{\alpha\beta}^{ij} \sin^2\left(\frac{\Delta m_{ij}^2 x}{4E}\right) - 2\sum_{i<j} \mathrm{Im}\, J_{\alpha\beta}^{ij} \sin\left(\frac{\Delta m_{ij}^2 x}{2E}\right)$$

Vacuum oscillation effects expected to be significant when:

$$\frac{\delta m^2 L}{4E} \sim O(1) \quad \text{or} \quad \frac{\Delta m^2 L}{4E} \sim O(1)$$

Recap

3ν oscillations in vacuum (may be CP violating):

$$P_{\alpha\beta} = \delta_{\alpha\beta} - 4 \sum_{i<j} \mathrm{Re}\, J_{\alpha\beta}^{ij} \sin^2 \left(\frac{\Delta m_{ij}^2 x}{4E} \right) - 2 \sum_{i<j} \mathrm{Im}\, J_{\alpha\beta}^{ij} \sin \left(\frac{\Delta m_{ij}^2 x}{2E} \right)$$

Vacuum oscillation effects expected to be significant when:

$$\frac{\delta m^2 L}{4E} \sim O(1) \qquad \text{or} \qquad \frac{\Delta m^2 L}{4E} \sim O(1)$$

$P_{\alpha\beta}$ in matter (constant or slowly changing): vacuum param. → effective param.

Matter effects expected to be significant when:

$$\frac{A}{\delta m^2} \sim O(1) \qquad \text{or} \qquad \frac{A}{\Delta m^2} \sim O(1)$$

End of Lecture II

Solutions to exercises: extra slides →

Exercise: General 3ν oscillations in vacuum

- Flavor evolution operator: $S_f = U S_m U^+$

  where: $S_m = \mathrm{diag}\left(e^{-i\frac{m_1^2 x}{2E}},\ e^{-i\frac{m_2^2 x}{2E}},\ e^{-i\frac{m_3^2 x}{2E}}\right)$

- Inserting indices: $S_{\beta\alpha} = \sum_i U_{\alpha i}^* U_{\beta i}\, e^{-i\frac{m_i^2 x}{2E}}$ for any $\alpha, \beta \in (e, \mu, \tau)$

- Flavor oscillation probability is obtained as $P = |S|^2$ by reorganizing terms:

$$
\begin{aligned}
P(\nu_\alpha \to \nu_\beta) &= |S_{\beta\alpha}|^2 \\
&= \left| \sum_i U_{\alpha i}^* U_{\beta i}\, e^{-i\frac{m_i^2}{2E}x} \right|^2 \\
&= \sum_{ij} U_{\alpha i}^* U_{\beta i} U_{\alpha j} U_{\beta j}^*\, e^{-i\frac{m_i^2}{2E}x}\, e^{+i\frac{m_j^2}{2E}x} \\
&= \sum_{ij} U_{\alpha i}^* U_{\beta i} U_{\alpha j} U_{\beta j}^* \left(e^{i\frac{m_j^2 - m_i^2}{2E}x} - 1 + 1\right) \\
&= \sum_{ij} U_{\alpha i}^* U_{\beta i} U_{\alpha j} U_{\beta j}^* \left(e^{i\frac{m_j^2 - m_i^2}{2E}x} - 1\right) + \sum_{ij} U_{\alpha i}^* U_{\beta i} U_{\alpha j} U_{\beta j}^*
\end{aligned}
$$

$\xrightarrow{\hspace{1cm}}$ cont'd

$$= \left(\sum_{i<j} + \sum_{i>j}\right) U^*_{\alpha i} U_{\beta i} U_{\alpha j} U^*_{\beta j} \left(e^{i\frac{m^2_j - m^2_i}{2E}x} - 1\right) + \sum_i U^*_{\alpha i} U_{\beta i} \sum_j U_{\alpha j} U^*_{\beta j}$$

$$= \sum_{i>j} U^*_{\alpha i} U_{\beta i} U_{\alpha j} U^*_{\beta j} \left(e^{i\frac{m^2_j - m^2_i}{2E}x} - 1\right) + \sum_{i>j} U_{\alpha i} U^*_{\beta i} U^*_{\alpha j} U_{\beta j} \left(e^{-i\frac{m^2_j - m^2_i}{2E}x} - 1\right) + \delta_{\alpha\beta} \cdot \delta_{\alpha\beta}$$

$$= \sum_{i>j} \left(U^*_{\alpha i} U_{\beta i} U_{\alpha j} U^*_{\beta j} + U_{\alpha i} U^*_{\beta i} U^*_{\alpha j} U_{\beta j}\right) \left[\cos\left(\frac{m^2_j - m^2_i}{2E}x\right) - 1\right]$$

$$+ \sum_{i>j} \left(U^*_{\alpha i} U_{\beta i} U_{\alpha j} U^*_{\beta j} - U_{\alpha i} U^*_{\beta i} U^*_{\alpha j} U_{\beta j}\right) \left[i \sin\left(\frac{m^2_j - m^2_i}{2E}x\right)\right] + \delta_{\alpha\beta}$$

$$= \delta_{\alpha\beta} - \sum_{i>j} 2\,\mathrm{Re}\left(U^*_{\alpha i} U_{\beta i} U_{\alpha j} U^*_{\beta j}\right) \left[\cos\left(\frac{m^2_j - m^2_i}{2E}x\right) - 1\right] - \sum_{i>j} 2\,\mathrm{Im}\left(U^*_{\alpha i} U_{\beta i} U_{\alpha j} U^*_{\beta j}\right) \sin\left(\frac{m^2_j - m^2_i}{2E}x\right)$$

$$= \delta_{\alpha\beta} - 4\sum_{i>j} \mathrm{Re}\left(U^*_{\alpha i} U_{\beta i} U_{\alpha j} U^*_{\beta j}\right) \sin^2\left(\frac{m^2_j - m^2_i}{4E}x\right) - 2\sum_{i>j} \mathrm{Im}\left(U^*_{\alpha i} U_{\beta i} U_{\alpha j} U^*_{\beta j}\right) \sin\left(\frac{m^2_j - m^2_i}{2E}x\right)$$

$$\overset{i \leftrightarrow j}{=} \delta_{\alpha\beta} - 4\sum_{i<j} \mathrm{Re}\, J^{ij}_{\alpha\beta} \sin^2\left(\frac{\Delta m^2_{ij}}{4E}x\right) - 2\sum_{i<j} \mathrm{Im}\, J^{ij}_{\alpha\beta} \sin\left(\frac{\Delta m^2_{ij}}{2E}x\right)$$

where $\begin{cases} \Delta m^2_{ij} = m^2_i - m^2_j \\ J^{ij}_{\alpha\beta} = U_{\alpha i} U^*_{\beta i} U^*_{\alpha j} U_{\beta j} \end{cases}$

Exercise: J = Jarlskog invariant

- Define $J \equiv \mathrm{Im}(J^{12}_{e\mu})$
- For any $\alpha \neq \beta$, it turns out that $\mathrm{Im}(J^{ij}_{\alpha\beta}) = \pm J$ where $\begin{cases} +J & (\alpha,\beta) \text{ cyclic over } (e,\mu,\tau) \\ +J & (i,j) \quad \text{"} \quad \text{"} \quad (1,2,3) \\ -J & \text{otherwise} \end{cases}$

- This can be checked by inspection of cases. For instance:
- Let's work out $J^{12}_{e\tau}$ as:

$$\begin{aligned} \mathrm{Im}(J^{12}_{e\tau}) &= \mathrm{Im}(U_{e1} U^*_{\tau 1} U^*_{e2} U_{\tau 2}) \\ &= \mathrm{Im}(U_{e1} U^*_{e2} (-U^*_{e1} U_{e2} - U^*_{\mu 1} U_{\mu 2})) \qquad \leftarrow \text{since } UU^\dagger = 1 \\ &= \mathrm{Im}(-U_{e1} U^*_{e2} U^*_{\mu 1} U_{\mu 2}) = -\mathrm{Im}(J^{12}_{e\mu}) = -J \quad . \qquad (\text{similarly for other } (\alpha,\beta)) \end{aligned}$$

- Note that $\mathrm{Im}(J^{21}_{e\tau}) = +J = (-1)(-1)J$ since both $(e\tau)$ and (21) are anticyclic.

- One can also write: $\mathrm{Im}(J^{ij}_{\alpha\beta}) = J \cdot \sum_\gamma \epsilon_{\alpha\beta\gamma} \sum_k \epsilon_{ijk}$

where ϵ is the totally antisymmetric tensor of rank 3.

Exercise : P^{CPV} in product form

- To transform a sum into a product, we use the following identity:

  If $x+y+z=0$, then $\sin 2x + \sin 2y + \sin 2z = -4 \sin x \sin y \sin z$ (from trigonometry)

- The identity is applied to $\Delta m^2_{12} + \Delta m^2_{23} + \Delta m^2_{31} = 0$.

- $$P^{CPV}(\nu_\alpha \to \nu_\beta) = -2 \sum_{i<j} \mathrm{Im}\, J^{ij}_{\alpha\beta} \sin\left(\frac{\Delta m^2_{ij}}{2E} x\right)$$

$$= -2\left[\mathrm{Im}\, J^{12}_{\alpha\beta} \sin\left(\frac{\Delta m^2_{12} x}{2E}\right) + \mathrm{Im}\, J^{23}_{\alpha\beta} \sin\left(\frac{\Delta m^2_{23}}{2E} x\right) + \mathrm{Im}\, J^{13}_{\alpha\beta} \sin\left(\frac{\Delta m^2_{13} x}{2E}\right)\right]$$

$$= -2\, \mathrm{Im}\, J^{12}_{\alpha\beta}\left[\sin\left(\frac{\Delta m^2_{12} x}{2E}\right) + \sin\left(\frac{\Delta m^2_{23}}{2E} x\right) - \sin\left(\frac{\Delta m^2_{13} x}{2E}\right)\right] \qquad = + \sin\left(\frac{\Delta m^2_{31} x}{2E}\right)$$

$$= +8\, \mathrm{Im}\, J^{12}_{\alpha\beta} \sin\left(\frac{\Delta m^2_{12} x}{4E}\right) \sin\left(\frac{\Delta m^2_{23} x}{4E}\right) \sin\left(\frac{\Delta m^2_{31} x}{4E}\right)$$

$$= +8\, \mathrm{Im}\, J^{12}_{\alpha\beta} \prod_{(ij)}^{\text{cyclic}} \sin\left(\frac{\Delta m^2_{ij} x}{4E}\right)$$

Exercise: Useful units for matter effects

- Remember that:
$$\begin{cases} 1 \text{ mol} = N_A \text{ particles} = 6.022\times 10^{23} \text{ particles} \\ 1\,\text{MeV}\cdot 1\,\text{m} = 5.0677\times 10^{+12} \\ G_F = 1.1664\times 10^{-5}\,\text{GeV}^{-2} = 1.1664\times 10^{-11}\,\text{MeV}^{-2} \end{cases}$$

- $$1\,\frac{\text{mol}}{\text{cm}^3} = \frac{6.022\times 10^{23}}{10^{-6}\,\text{m}^3}\left(\frac{\text{MeV}^3}{\text{MeV}^3}\right) = 6.022\times 10^{29}\,\frac{\text{MeV}^3}{(\text{m}\cdot\text{MeV})^3} = \frac{6.022\times 10^{29}}{(5.0677\times 10^{12})^3}\,\text{MeV}^3 = 4.627\times 10^{-9}\,\text{MeV}^3$$

- $$\frac{A}{\Delta m^2_{ij}} = \frac{2\sqrt{2}G_F N_e E}{\Delta m^2_{ij}} = 2\sqrt{2}\left(1.1664\times 10^{-11}\,\text{MeV}^{-2}\right)\left(\frac{N_e}{\text{mol/cm}^3}\,\text{mol/cm}^3\right)\left(\frac{E}{\text{MeV}}\cdot\text{MeV}\right)\left(\frac{\text{eV}^2}{\Delta m^2_{ij}}\,\frac{1}{\text{eV}^2}\right)$$

$$= 3.299\times 10^{-11}\,\frac{\text{MeV}^{-2}\,\text{MeV}}{\text{eV}^2}\,\frac{\text{mol}}{\text{cm}^3}\left(\frac{N_e}{\text{mol/cm}^3}\right)\left(\frac{E}{\text{MeV}}\right)\left(\frac{\text{eV}^2}{\Delta m^2_{ij}}\right)$$

$$= 3.299\times 10^{-11}\,\frac{10^{12}}{\text{MeV}^3}\times 4.627\times 10^{-9}\,\text{MeV}^3\left(\frac{N_e}{\text{mol/cm}^3}\right)\left(\frac{E}{\text{MeV}}\right)\left(\frac{\text{eV}^2}{\Delta m^2_{ij}}\right)$$

$$= 1.526\times 10^{-7}\left(\frac{N_e}{\text{mol/cm}^3}\right)\left(\frac{E}{\text{MeV}}\right)\left(\frac{\text{eV}^2}{\Delta m^2_{ij}}\right)$$

Exercise: Adiabatic evolution

- In flavor basis one can write always:

$$i\frac{d}{dx}\begin{pmatrix}\nu_e\\ \nu_\mu\\ \nu_\tau\end{pmatrix} = H_f\begin{pmatrix}\nu_e\\ \nu_\mu\\ \nu_\tau\end{pmatrix} \quad \text{where} \quad H_f = \frac{1}{2E}\tilde{U}\begin{pmatrix}\tilde{m}_1^2 & & \\ & \tilde{m}_2^2 & \\ & & \tilde{m}_3^2\end{pmatrix}U \quad \text{and} \quad \begin{pmatrix}\nu_e\\ \nu_\mu\\ \nu_\tau\end{pmatrix} = \tilde{U}\begin{pmatrix}\tilde{\nu}_1\\ \tilde{\nu}_2\\ \tilde{\nu}_3\end{pmatrix}$$

- Therefore:

$$i\frac{d}{dx}\tilde{U}\begin{pmatrix}\tilde{\nu}_1\\ \tilde{\nu}_2\\ \tilde{\nu}_3\end{pmatrix} = \left[\frac{1}{2E}\tilde{U}\begin{pmatrix}\tilde{m}_1^2 & & \\ & \tilde{m}_2^2 & \\ & & \tilde{m}_3^2\end{pmatrix}\tilde{U}^+\right]U\begin{pmatrix}\tilde{\nu}_1\\ \tilde{\nu}_2\\ \tilde{\nu}_3\end{pmatrix} = \frac{1}{2E}\tilde{U}\begin{pmatrix}\tilde{m}_1^2 & & \\ & \tilde{m}_2^2 & \\ & & \tilde{m}_3^2\end{pmatrix}\begin{pmatrix}\tilde{\nu}_1\\ \tilde{\nu}_2\\ \tilde{\nu}_3\end{pmatrix}$$

- If $d\tilde{U}/dx \simeq 0$:

$$i\frac{d}{dx}\tilde{U}\begin{pmatrix}\tilde{\nu}_1\\ \tilde{\nu}_2\\ \tilde{\nu}_3\end{pmatrix} \simeq \tilde{U}\, i\frac{d}{dx}\begin{pmatrix}\tilde{\nu}_1\\ \tilde{\nu}_2\\ \tilde{\nu}_3\end{pmatrix}$$

- Multiply by $\tilde{U}^+$ and get:

$$i\frac{d}{dx}\begin{pmatrix}\tilde{\nu}_1\\ \tilde{\nu}_2\\ \tilde{\nu}_3\end{pmatrix} = \frac{1}{2E}\begin{pmatrix}\tilde{m}_1^2 & & \\ & \tilde{m}_2^2 & \\ & & \tilde{m}_3^2\end{pmatrix}\begin{pmatrix}\tilde{\nu}_1\\ \tilde{\nu}_2\\ \tilde{\nu}_3\end{pmatrix} \longrightarrow \begin{cases}\text{no off-diagonal terms,}\\ \text{three decoupled equations,}\\ \text{the } \tilde{\nu}_i \text{ evolve independently.}\end{cases}$$